Existence of solutions for the infinite systems of integral equations in the space $L^\infty(\mathbb{R}^n)$

Ayub Samadi and Shahram Banaei

Abstract: In this paper, by applying the behavior of measure of noncompactness in $L^\infty(\mathbb{R}^n)$, we study the existence of solutions of an infinite system of integral equations in the space $L^\infty(\mathbb{R}^n)$. Finally, an example is included to show the usefulness of the outcome.

Subjects: 47H09; 47H10

Keywords: Darbo's fixed point theorem; measure of noncompactness; integral equations

1. Introduction

The concept measure of noncompactness (M.N.C) was introduced by the essential article of Kuratowski (Kuratowski, 1930). (Darbo, 1955) applied this measure to generalize both the Banach contraction principle and the Schauder fixed point theorem. Measures of noncompactness are very suitable tools which are widely applied for proving solvability of nonlinear differential and integral equations in Banach spaces (Aghajani, Allahyari, & Mursaleen, 2014; Aghajani & Shole Haghighi, 2014; Banaei & Ghaemi, 2017; Banaei, Ghaemi, & Saadati, 2017; Banás, Jileš, Mursaleen, & Samet, 2017; Hazarika, Srivastava, Arab, & Rabbani, 2018; Srivastava, Das, Hazarika, & Mohiuddine, 2018). Also, many authors extended Darbo fixed point theorem and applied it to investigate the solvability of integral equations in two variables (Aghajani & Shole Haghighi, 2014; Arab, Allahyari, & Shole Haghighi, 2014; Das, Hazarika, Arab, & Mursaleen, 2017; Srivastava, Das, Hazarika, & Mohiuddine, 2019). Srivastava et al. (Srivastava et al., 2019) studied existence of solution for non-linear functional integral equations of two variables in Banach Algebra. Moreover, they investigated existence of solutions of infinite systems of differential equations of general order with boundary conditions in the spaces $c_0$ and $l_1$ via the measure of noncompactness in (Srivastava et al., 2018). (Das et al., 2017) studied solvability of infinite system of integral equations in the sequence spaces $c_0$ and $l_1$. (Arab et al., 2014) investigated the existence of solutions of infinite systems of integral equations in the Fréchet space. On the other hand Allahyari (Allahyari, 2017) introduced the construction of (M.N.C) in $L^\infty(\mathbb{R}^n)$. In this paper, by applying (M.N.C) in $L^\infty(\mathbb{R}^n)$ (Allahyari, 2017), we give a fixed point theorem and study the existence of solutions of infinite systems of nonlinear functional integral equations of Urysohn type in two variables.

\begin{equation}
\begin{aligned}
\chi_i(t,s) &= f_i(t,s,x_1(t,s), \ldots, x_n(t,s), \int_{\Lambda} \int_{X} k_j(t,s,u,v,x_j(u,v)^{\infty}_{j=1}) du dv) \\
\end{aligned}
\end{equation}
where \( \Lambda \subseteq \mathbb{R}^n \), \( t, s \in \mathbb{R} \), \( x_i \in (\mathbb{R} \times \mathbb{R}) \) and \( (i \in \mathbb{N}) \). The importance of the space \( L^\infty(\mathbb{R}^n) \) is that the functions in this space do not need to be continuous. The structure of this paper is as follows. In Section 2, some definitions and concepts are recalled. Sections 3 is devoted to prove a fixed point theorem. In section 4, as an application for the obtained results, we present an existence theorem. Finally, in section 5 an example is given to illustrate the effectiveness of our results.

2. Preliminaries

Here, we recall some facts which will be used in our main results. Throughout this article, let \( \mathbb{R} \) denote the set of real numbers, \( \mathbb{R}_+ = [0, +\infty) \) and put \( \mathbb{R}^\infty \) countable cartesian product of \( \mathbb{R} \) with itself. Let \( (E, \| \cdot \|) \) be a real Banach space. Moreover, \( \mathcal{B}(x, r) \) denotes the closed ball centered at \( x \) with radius \( r \). The symbol \( \overline{B} \) stands for the ball \( \overline{B}(0, r) \). For \( X \), a nonempty subset of \( E \), we denote by \( \mathcal{C} \) and \( \text{Conv}X \) the closure and the closed convex hull of \( X \), respectively. Furthermore, let us denote by \( m \) the family of nonempty bounded subsets of \( E \) and by \( m_c \) its subfamily consisting of all relatively compact subsets of \( E \).

Definition 2.1 (Banás & Goebel, 1980). A mapping \( \mu : m_E \rightarrow \mathbb{R}_+ \) is said to be a measure of noncompactness in \( E \) if it satisfies the following conditions:

1° The family \( \ker \mu = \{ X \in m_E : \mu(X) = 0 \} \) is nonempty and \( \ker \mu \subseteq m_c \).
2° \( X \subseteq Y \Rightarrow \mu(X) \leq \mu(Y) \).
3° \( \mu(\mathcal{C}) = \mu(X) \).
4° \( \mu(\text{Conv}X) = \mu(X) \).
5° \( \mu(\lambda X + (1 - \lambda) Y) \leq \lambda \mu(X) + (1 - \lambda) \mu(Y) \) for \( \lambda \in [0, 1] \).
6° If \( \{ X_n \} \) is a sequence of closed sets from \( m_E \) such that \( X_{n+1} \subseteq X_n \) for \( n = 1, 2, \ldots \) and if \( \lim_{n \to \infty} \mu(X_n) = 0 \) then \( X_\infty = \cap_{n=1}^{\infty} X_n \neq \emptyset \).

The following theorems are basic for our main results.

Theorem 2.2 (Darbo, 1955). Let \( C \) be a nonempty, bounded, closed and convex subset of a Banach space \( E \) and \( T : C \rightarrow C \) be a continuous mapping. Assume that there exists a constant \( K \in \mathbb{R} \) such that \( \mu(TX) \leq K \mu(X) \) for any nonempty subset \( X \) of \( C \), where \( \mu \) is a \( (M.N.C) \) defined in \( E \). Then \( T \) has at least a fixed point in \( C \).

Theorem 2.3 (Aghajani et al., 2014). Suppose \( \mu_1, \mu_2, \ldots, \mu_n \) are measures of noncompactness in Banach spaces \( E_1, E_2, \ldots, E_n \), respectively. Moreover assume that the function \( F : \mathbb{R}_+^n \rightarrow \mathbb{R}_+ \) is convex and \( F(x_1, \ldots, x_n) = 0 \) if and only if \( x_i = 0 \) for \( i = 1, 2, \ldots, n \). Then

\[
\tilde{\mu}(X) = F(\mu_1(X_1), \mu_2(X_2), \ldots, \mu_n(X_n))
\]

defines a measure of noncompactness in \( E_1 \times E_2 \times \cdots \times E_n \) where \( X_i \) denotes the natural projection of \( X \) into \( E_i \), for \( i = 1, 2, \ldots, n \).

Theorem 2.4 (Tychonoff fixed point theorem (Banás et al., 2017)). Let \( E \) be a Hausdorff locally convex linear topological space, \( C \) a convex subset of \( E \) and \( F : C \rightarrow E \) a continuous mapping such that

\( F(C) \subseteq A \subseteq C \)

with \( A \) compact. Then \( F \) has at least one fixed point.

3. Main results

In this section, we recall the definition of a measure of noncompactness in \( L^\infty(\mathbb{R}^n) \) which has been presented in (Allahyari, 2017). First, we recall the compact subsets of \( L^\infty(\mathbb{R}^n) \).
Theorem 3.1. Let $B$ be a bounded set in $L^\infty(\mathbb{R}^n)$. Then $B$ is relatively compact if the following conditions are satisfied:

(i) $\lim_{h \to 0} \| \tau_h f - f \|_{L^\infty(B_T)} = 0$ uniformly with respect to $f \in B$ for any $T > 0$, where $\tau_h f(x) = f(x + h)$.

(ii) For $\varepsilon > 0$ there is some $T > 0$ so that for every $f, g \in B$

$\| f - g \|_{L^\infty(\mathbb{R}^n; B_T)} < \varepsilon$.

We recall the Euclidean norm on the space $\mathbb{R}^n$ to be

$\| x \| = (\sum_{i=1}^{n} (x_i^2))^\frac{1}{2},$

for $x = (x_1, x_2, \ldots, x_n)$. Let $L^\infty(\mathbb{R}^n)$ denote the space of all Lebesgue measurable functions on $\mathbb{R}^n$ with the standard norm

$\| f \|_\infty = \inf \{ C > 0 : |f(t)| \leq C \text{ a.e. on } \mathbb{R}^n \}$.

Theorem 3.2 (Allahyari, 2017). Let $B$ be a bounded set in $L^\infty(\mathbb{B}_T)$. Then $B$ is relatively compact if and only if

$\lim_{h \to 0} \| \tau_h f - f \|_{L^\infty(B_T)} = 0$ \hspace{1cm} (3.2)

uniformly with respect to $f \in B$, where $\tau_h f(x) = f(x + h)$.

Suppose $X$ is a bounded subset of the space $L^\infty(\mathbb{R}^n)$. For $x \in X$ and $\varepsilon > 0$, let us denote

$\omega^T(f, \varepsilon) = \sup \{ \| \tau_h f - f \|_{L^\infty(B_T)} : \| h \| < \varepsilon \},$

$\omega^T(X, \varepsilon) = \sup \{ \omega^T(f, \varepsilon) : f \in X \},$

$\omega^T(X) = \lim_{\varepsilon \to 0} \omega^T(X, \varepsilon),$

$\omega(X) = \lim_{T \to \infty} \omega^T(X),$

and

$d(X) = \lim_{T \to \infty} \sup \{ \| f - g \|_{L^\infty(\mathbb{R}^n; B_T)} : f, g \in X \},$

$\omega_0(X) = \omega(X) + d(X).$

We know that the function $\omega_0$ is a (M.N.C) in $L^\infty(\mathbb{R}^n)$ (Allahyari, 2017).

Theorem 3.3. Let $C_i$ $(i \in \mathbb{N})$ be nonempty, closed and convex subsets of $L^\infty(\mathbb{R}^n)$, $\omega_0(X)$ an arbitrary (M.N.C) in $L^\infty(\mathbb{R}^n)$ and sup$_i \{ \omega_0(C_i) \} < \infty$. Let $F_i : C^0 \to C$ $(i \in \mathbb{N})$ be a continuous operator such that

$\omega_0(F_i(X_1 \times X_2 \times \ldots \times X_n)) \leq \lambda \sup_{i} \omega_0(X_i),$ \hspace{1cm} (3.3)

where $X_i \in C_i$ and $\lambda \in [0, 1)$. Then there exists $(x^*_i)_{i=1}^\infty \in C^w$ (countable cartesian product of $C$ with itself) such that

$F_i(x^*_i)_{i=1}^\infty = x^*_i$ \hspace{1cm} (3.4)

for all $(i \in \mathbb{N})$. 

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Proof. Consider the operator \( \tilde{F} : C^\omega \to C^\omega \) defined by
\[
\tilde{F}(x_k)_{k=1}^\infty = (F_1((x_k)_{k=1}^\infty), F_2((x_k)_{k=1}^\infty), \ldots, (F_i((x_k)_{k=1}^\infty), \ldots)
\]
for all \((x_k)_{k=1}^\infty \in C^\omega\). Also, \( \omega_0^\omega(X) := \sup_i \omega_0(X_i) \) is a measure of noncompactness in the space \( C^\omega \) where \( X_i \) denote the natural projections of \( X \) and \( i \in \mathbb{N} \). Now, by induction, we define a sequence \( \{C_m\} \) such that \( C_0 = C^\omega \) and \( C_m = \text{Conv}(\tilde{F}(C_{m-1})) \), \( m \geq 1 \). Then we have \( \tilde{F}C_0 = \tilde{F}C^\omega \subseteq C^\omega = C_0, C_1 = \text{Conv}(\tilde{F}C_0) \subseteq C^\omega = C_0 \), and by continuing this process we obtain \( C_0 \supseteq C_1 \supseteq C_2 \supseteq \ldots \).

If there exists an integer \( N \geq 0 \) such that \( \omega_0^\omega(C_N) = 0 \) a.e. on \( \mathbb{R}^n \), then \( C_N \) is relatively compact and since \( \tilde{F}C_N \subseteq \text{Conv}(\tilde{F}C_N) = C_{N+1} \subseteq C_N \), therefore, Theorem 2.4 implies that \( \tilde{F} \) has a fixed point. Thus, there exists \( n > 0 \) such that \( \omega_0^\omega(C_n) \neq 0 \) for \( n \geq 0 \). By our assumptions, we have
\[
\omega_0^\omega(C_{n+1}) = \omega_0^\omega(\text{Conv}(\tilde{F}C_n)) = \omega_0^\omega(\tilde{F}C_n) \leq \omega_0^\omega(C_n).
\]
Since \( \lambda \in [0,1) \), so \( \omega_0^\omega(C_n) \) is a positive decreasing sequence of real numbers. Therefore, there is a \( r \geq 0 \) such that \( \omega_0^\omega(C_n) \to r \) as \( n \to \infty \). On the other hand, from the inequality (5) we have
\[
limit_{n \to \infty} \sup \omega_0^\omega(C_{n+1}) \leq \lim \sup \omega_0^\omega(C_n).
\]
This shows that \( r \leq \lambda r \). Consequently \( r = 0 \). Hence, we derive that \( \omega_0^\omega(C_n) \to 0 \) as \( n \to \infty \). Since the sequence \( \{C_n\} \) is nested, from (6) of Definition 2.1 we infer that the set \( C_\infty = \bigcap_{n=1}^\infty C_n \neq \emptyset \) is closed and convex subset of the set \( C^\omega \). Now, using Theorem 2.4 implies that \( \tilde{F} \) has a fixed point.

4. Application

In this section, we present an existence result for the system of integral equations of Urysohn type in two variables in the spaces \( L^\omega(\Lambda) \), where \( \Lambda \subseteq \mathbb{R}^n \).

Definition 4.1 ([Bans et al., 2017]). A function \( f : \Lambda \times \mathbb{R}^n \to \mathbb{R} \) is said to have the Carathéodory property if
(i) For all \( u \in \mathbb{R}^n \) the function \( x \to f(x, u) \) is measurable on \( \Lambda \).

(ii) For almost all \( x \in \mathbb{R}^n \) the function \( u \to f(x, u) \) is continuous on \( \mathbb{R}^n \).

We will consider the Equation (1.1) under the following hypotheses:

\( (A_1) \) \( (x_j(u,v))_{j=1}^\infty : \mathbb{R}^2 \to \mathbb{R}^2 \) are measurable functions.

\( (A_2) \) \( f_i : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) \( (i \in \mathbb{N}) \) satisfies the Carathéodory conditions and
\[
f_i(\ldots, 0, 0, \ldots, 0) \in L^\infty(\mathbb{R}^n), \text{Furthermore, there exists } \lambda \in [0,1) \text{ such that}
\]
\[
|f(t, s, x_1, x_2, \ldots, x_{n+1}) - f(t, s, y_1, y_2, \ldots, y_{n+1})| \leq \lambda \max \{|x_k - y_k|\}
\]
\[
+ |x_{k+1} - y_{k+1}|, \text{a.e. on } t, s \in \mathbb{R}
\]

\( (A_3) \) \( k_i : \mathbb{R}^4 \to \mathbb{R} \) \( (i \in \mathbb{N}) \) satisfies the Carathéodory conditions, \( k_i \in L^\infty(\mathbb{R}^4 \times \mathbb{R}^m) \) and there exists nondecreasing function \( b : \mathbb{R} \to \mathbb{R} \) such that for all \( r > 0 \) and \( x_j \in L^\infty(\Lambda) \) with \( \|x_j\|_{\infty} \leq r \) we have
\[
\text{ess sup}_{t,s \in \mathbb{R}} \left| \int_{\mathbb{R} \times \mathbb{R}} k_i(t, s, u, v, x_j(u,v))_{j=1}^\infty du dv \right| \leq b(r).
\]
Moreover, for any $r \in \mathbb{R}_+$

\[
\lim \text{ess sup} \sup_{t \leq r} \left| \int_{A(t)} \int_A k_i(t, s, u, v, x_j(u, v)) dudv - k_i(t, s, u, v, y_j(u, v)) dudv \right| = 0
\]

uniformly with respect to $(x_j)^{\infty}_{j=1}, (y_j)^{\infty}_{j=1} \in L^\infty(\Lambda)$ such that

\[
\|x_j\|_{\infty}, \|y_j\|_{\infty} \leq r
\]

(A4) The following equality holds:

\[
\lim \text{ess sup}_{t \in [0, \infty)} \int_{A(t)} \int_A |k_i(t, s, u, v, x_j(u, v))| dudv = 0.
\]

(A5) The inequality

\[
\lambda(r) + \sup_{i \in \mathbb{N}} \{\|f_i(\ldots, 0, 0, \ldots 0)\|_{\infty}\} + b(r) \leq r
\]

has a positive solution $r_0$.

**Theorem 4.2.** Under assumptions (A1)-(A5), the Equation (1.1) has at least one solution in the space $(L^\infty(\Lambda))^w$, where $\Lambda \subseteq \mathbb{R}^n$.

**Proof.** First we fix arbitrary $i \in \mathbb{N}$. Define $F_i : \{L^\infty(\Lambda)\}^w \rightarrow L^\infty(\Lambda)$ by

\[
F_i((x_j)^{\infty}_{j=1})(t, s) = f_i(t, s, x_1(t, s), \ldots, x_n(t, s),
\int_{A(t)} \int_A k_i(t, s, u, v, x_j(u, v))^{\infty}_{j=1} |dudv|.
\]

In view of the Carathéodory conditions, we infer that $F_i((x_j)^{\infty}_{j=1})$ is measurable for any $(x_j)^{\infty}_{j=1} \in L^\infty(\Lambda)^w$.

Now, we show that $F_i((x_j)^{\infty}_{j=1}) \in L^\infty(\Lambda)$. To this end, from conditions (A1) -(A5) we have

\[
|F_i((x_j)^{\infty}_{j=1})(t, s)| \leq |f_i(t, s, x_1(t, s), \ldots, x_n(t, s),
\int_{A(t)} \int_A k_i(t, s, u, v, x_j(u, v))^{\infty}_{j=1} |dudv) - f_i(t, s, 0, 0, \ldots, 0)|
\]

\[
\leq \lambda \max_{1 \leq k \leq n} \{\|x_k(t, s)\|_{\infty}\} + \|f_i(t, s, 0, 0, \ldots, 0)\|_{\infty} + b(\|x_j\|_{\infty})
\]

a.e. on $t, s \in \mathbb{R}$

Therefore, we obtain

\[
\|F_i((x_j)^{\infty}_{j=1})(t, s)\|_{\infty} \leq \lambda \max_{1 \leq k \leq n} \{\|x_k(t, s)\|_{\infty}\} + \|f_i(t, s, 0, 0, \ldots, 0)\|_{\infty} + b(\|x_j\|_{\infty})
\]

Thus, $F_i((x_j)^{\infty}_{j=1}) \in L^\infty(\Lambda)$ and $F_i$ is well defined. From (4.9) and using (A5), we infer the function $F_i$ maps $B_{r_0}$ into $B_{r_0}$.

Now we show that $F_i$ is continuous function. Let us fix $\epsilon > 0$ and consider $(x_j)^{\infty}_{j=1}, (y_j)^{\infty}_{j=1} \in (L^\infty(\Lambda))^w$ such that $\|x_j - y_j\|_{\infty} < \epsilon$. Then, we have
By applying condition \((A_3)\) and \((A_4)\) we have \(T_1 > 0, T_2 > 0\) such that for \(T = \max\{T_1, T_2\}\) the following inequalities hold

\[
\text{ess sup}_{\|x\|, \|v\| \leq r} \left| \int_{\Lambda} \int_{\Lambda'} \kappa_i(t, s, u, v, x_i(u, v), y_j(u, v)) \, du \, dv \right| < \epsilon, \tag{4.10}
\]

\[
\text{ess sup}_{\|x\|, \|v\| \leq r} \left| \int_{\Lambda} \int_{\Lambda'} \left| \kappa_i(t, s, u, v, x_i(u, v)) - \kappa_i(t, s, u, v, y_j(u, v)) \right| \, du \, dv \right| < \epsilon. \tag{4.11}
\]

From (4.10), we have

\[
\text{ess sup}_{\|x\|, \|v\| \leq r} \left| \int_{\Lambda} \int_{\Lambda'} \left| \kappa_i(t, s, u, v, x_i(u, v)) - \kappa_i(t, s, u, v, y_j(u, v)) \right| \, du \, dv \right| < \epsilon. \tag{4.12}
\]

Now, for almost all \(t, s \in \bar{B}_T \cap \mathbb{R}\), we have

\[
\left| F_i((x_j)^{\infty}_{j=1})(t, s) - F_i((y_j)^{\infty}_{j=1})(t, s) \right| \leq \lambda \max_{1 \leq i \leq n} \left| x_i(t, s) - y_i(t, s) \right| \tag{4.13}
\]

where

\[
\vartheta(\epsilon) = \inf\{M \geq 0 : |\kappa_i(t, s, u, v, x_j) - \kappa_i(t, s, u, v, y_j)| \leq M \text{ a.e. on } t, s, u, v \in \bar{B}_T \subset \Lambda', \ x_j, y_j \in [-r_0, r_0], |x_j - y_j| \leq \epsilon\}.
\]

In view of the Carathéodory conditions for \(\kappa_i\) on the compact set \(\bar{B}_T \times \bar{B}_T \times \bar{B}_T \times \bar{B}_T \times [-r_0, r_0]^n\), we have \(\vartheta(\epsilon) \to 0\) as \(\epsilon \to 0\). Thus from (4.11), (4.12) and (4.13) we infer that \(F\) is a continuous function on \(L^\infty(\Lambda)\). In order to finish the proof, we show that \(F\) satisfies assumptions imposed in Theorem 3.3. Let \(x_j (j \in \mathbb{N})\) be nonempty and bounded subset of \(\bar{B}_r\). Suppose that \(T > 0\) and \(\epsilon > 0\) are arbitrary constants. For almost all \(t_1, t_2, s_1, s_2 \in [0, T]\) and \(\|t_2 - t_1\| \leq \epsilon, \|s_2 - s_1\| \leq \epsilon\) we have
\[ |F_i((x_j)_{j=1}^m)(t_2, s_2) - F_i((x_j)_{j=1}^m)(t_1, s_1)| \leq |f_i(t_2, s_2, x_1(t_2, s_2), \ldots, x_n(t_2, s_2),
\int_{\mathcal{X}} \int_{\mathcal{X}} k_i(t_2, s_2, u, v, x_j(u, v)_{j=1}^m) dudv
- f_i(t_1, s_1, x_1(t_2, s_2), \ldots, x_n(t_2, s_2),
\int_{\mathcal{X}} \int_{\mathcal{X}} k_i(t_2, s_2, u, v, x_j(u, v)_{j=1}^m) dudv) |
+ |f_i(t_1, s_1, x_1(t_2, s_2), \ldots, x_n(t_2, s_2),
\int_{\mathcal{X}} \int_{\mathcal{X}} k_i(t_2, s_2, u, v, x_j(u, v)_{j=1}^m) dudv
- f_i(t_1, s_1, x_1(t_1, s_1), \ldots, x_n(t_1, s_1),
\int_{\mathcal{X}} \int_{\mathcal{X}} k_i(t_1, s_1, u, v, x_j(u, v)_{j=1}^m) dudv) |
\leq \omega_{\mathcal{B}}^T(f_i, \varepsilon) + \lambda \max_{1 \leq i \leq n} \{ |x_i(t_2, s_2) - x_i(t_1, s_1)| \}
+ |f_i(t_2, s_2, u, v, x_j(u, v)_{j=1}^m)
- k_i(t_1, s_1, u, v, x_j(u, v)_{j=1}^m) dudv| \]
(4.14)
\[
+ |f_i(t_1, s_1, u, v, x_j(u, v)_{j=1}^m)
- k_i(t_1, s_1, u, v, x_j(u, v)_{j=1}^m) dudv| \\
\leq \omega_{\mathcal{B}}^T(f_i, \varepsilon) + \lambda \max_{1 \leq i \leq n} \{ a^T(x_i, \varepsilon) \} + m(\mathcal{B}) \omega_{\mathcal{B}}^T(k_i, \varepsilon) + 2 \text{ess sup}_{t \in \mathcal{B}} \int_{\mathcal{X} \times \mathcal{X}} |k_i(t, s, u, v, x_j(u, v)_{j=1}^m)| dudv
\]

where

\[
 \omega_{\mathcal{B}}^T(f_i, \varepsilon) = \inf \{ M \geq 0 : |f_i(t_1, s_1, x_1, \ldots, x_n, y) - f_i(t_2, s_2, x_1, \ldots, x_n, y)| \leq M
\]
a.e. on \( t_1, t_2, s_1, s_2 \in \tilde{\mathcal{B}}, \| t_2 - t_1 \| \leq \varepsilon, \| s_2 - s_1 \| \leq \varepsilon, |x_i| \leq r_0, |y| < b(r_0) \)

\[
 \omega_{\mathcal{B}}^T(k_i, \varepsilon) = \inf \{ M \geq 0 : |k_i(t_1, s_1, u, v, x_j(u, v)_{j=1}^m) - k_i(t_1, s_1, u, v, x_j(u, v)_{j=1}^m)| \leq M. \\
a.e. on \( t_1, t_2, s_1, s_2 \in \tilde{\mathcal{B}} \subset \Lambda, u, v \in \tilde{\mathcal{B}} \subset \Lambda, |x_i| \leq r_0 \}.
\]

Since \( x_i \) is arbitrary element of \( X_i, i \in \mathbb{N} \) in (4.14), we have

\[
 \omega^T(F_i(X_1 \times \ldots \times X_i)_{i=1}^m, \varepsilon) \leq \omega_{\mathcal{B}}^T(f_i, \varepsilon) + \lambda \max_{1 \leq i \leq n} \{ a^T(x_i, \varepsilon) \} + m(\mathcal{B}) \omega_{\mathcal{B}}^T(k_i, \varepsilon) + 2 \text{ess sup}_{t \in \mathcal{B}} \int_{\mathcal{X} \times \mathcal{X}} |k_i(t, s, u, v, x_j(u, v)_{j=1}^m)| dudv.
\]

In view of the Carathéodory conditions for \( f_i \) and \( k_i \) on the compact set \( \tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times [-b(r_0), b(b_0)] \) and \( \mathcal{B} \times \mathcal{B} \times \mathcal{B} \times [-b(r_0), b(b_0)] \), respectively and Corollary 3.2, we infer that \( \omega_{\mathcal{B}}^T(f_i, \varepsilon) \to 0 \) and \( \omega_{\mathcal{B}}^T(k_i, \varepsilon) \to 0 \) as \( \varepsilon \to 0 \). Therefore, we have
\[ \omega^T(F_i(X_1 \times \ldots \times X_i)_{t-1}^\infty) \leq \lambda \max_{1 \leq i \leq n} \{ \omega^T(x_i) \} \\
+ 2 \esssup_{t : s \in \mathcal{B}_T} \int_{X} \int_{X} |k_i(t, s, u, v, x_j(u, v)_{j-1})| \text{dudv}. \]

Now taking \( T \to \infty \) and by applying assumption (A4) we obtain
\[ \omega(F_i(X_1 \times \ldots \times X_i)_{t-1}^\infty) \leq \lambda \max_{1 \leq i \leq n} \omega(x_i). \] (4.15)

On the other hand, for all \( x_i, y_i \in X, (i \in \mathbb{N}) \) and \( t, s \in \mathbb{R} \), we get
\[ \esssup_{t : s \in \mathcal{B}_T} \| F_i(x_i)_{t-1}^\infty(t, s) - F_i(y_i)_{t-1}^\infty(t, s) \| \leq \lambda (\esssup_{t : s \in \mathcal{B}_T} |x_i(t, s) - y_i(t, s)|) \]
\[ + \esssup_{t : s \in \mathcal{B}_T} \int_{X} \int_{X} |k_i(t, s, u, v, x_j(u, v)_{j-1})| \text{dudv}. \]

Thus, we have
\[ \| F_i(x_i)_{t-1}^\infty - F_i(y_i)_{t-1}^\infty \|_{L^1(\mathcal{B}_T \times \mathcal{B}_T)} \leq \lambda (\| x_i - y_i \|_{L^1(\mathcal{B}_T \times \mathcal{B}_T)}) \]
\[ + \esssup_{t : s \in \mathcal{B}_T} \int_{X} \int_{X} |k_i(t, s, u, v, x_j(u, v)_{j-1})| \text{dudv}. \] (4.16)

If we take \( T \to \infty \) in the inequality (4.16), then using (A3) we have
\[ d(F_i(X_1 \times \ldots \times X_i)_{t-1}^\infty)(t, s) \leq \lambda (\max_{1 \leq i \leq n} d(X_i(t, s))). \] (4.17)

If we consider \( \max(t, s) \to \infty \) in the inequality (4.17), then
\[ \lim_{\max(t, s) \to \infty} d(F_i(X_1 \times \ldots \times X_i)_{t-1}^\infty)(t, s) \leq \lambda (\max_{1 \leq i \leq n} \lim_{\max(t, s) \to \infty} d(X_i(t, s))). \] (4.18)

Moreover, combining (4.15) and (4.18) imply that
\[ \lim_{\max(t, s) \to \infty} d(F_i(X_1 \times \ldots \times X_i)_{t-1}^\infty)(t, s) + \omega(F_i(X_1 \times \ldots \times X_i)_{t-1}^\infty) \]
\[ \leq \lambda (\max_{1 \leq i \leq n} \omega(x_i)) + \lambda (\max_{1 \leq i \leq n} \lim_{\max(t, s) \to \infty} d(X_i(t, s))). \] (4.19)

Since \( \lambda \in [0, 1) \) is a constant, (4.19) shows that
\[ \frac{1}{2} (\omega(F_i(X_1 \times \ldots \times X_i)_{t-1}^\infty) + \lim_{\max(t, s) \to \infty} d(F_i(X_1 \times \ldots \times X_i)_{t-1}^\infty)) \]
\[ \leq \frac{1}{2} \left[ \lambda \max_{1 \leq i \leq n} \omega(x_i) + \lambda \lim_{\max(t, s) \to \infty} d(X_i(t, s)) \right] \leq \frac{1}{2} \lambda \max_{1 \leq i \leq n} \omega(x_i) + \frac{1}{2} \lim_{\max(t, s) \to \infty} d(X_i(t, s)) \]
and we get
\[ \frac{1}{2} \omega_0(F_i(X_1 \times \ldots \times X_i)_{t-1}^\infty) \leq \lambda \left( \frac{1}{2} \sup_i \omega_0(x_i) \right). \]

Taking \( \omega_0 = \frac{1}{2} \omega_0 \), we have:
\[ \omega_0(F_i(X_1 \times \ldots \times X_i)_{t-1}^\infty) \leq \lambda \sup_i \omega_0(x_i). \]
Now by applying Theorem 3.3, there exists \((x_1, x_2, \ldots, x_n)\) that is solution of the system of integral Equation (1.1) and this completes the proof. 

Now, we study the following example to show the usefulness of the Theorem 4.2.

**Example 4.3.** Consider the system of integral equations

\[
x_i(t, s) = \frac{1}{2ts + 2n} + \frac{1}{2n} \sum_{j=1}^{n} (1 + |x_j(t, s)|) + \frac{1}{2e} \cos \left( \int_{\mathbb{R}^2} \left[ \frac{\left| \cos(u^3) \sin(x_1) \right| + \left| \sin^3(v) \cos \left( \sum_{j=1}^{n} x_j^2 \right) \right|}{3 + \cos \left( \sum_{j=1}^{n} x_j \right)} \right] dudv \right),
\]

(4.20)

where the symbol \(|X|\) shows the integer part of \(X\) and \(i, n \in \mathbb{N}\).

Equation (4.20) is a special case of Equation (1.1) with

\[
f_i(t, s, x_1, \ldots, x_n, z) = \frac{1}{2ts + 2n} + \frac{1}{2n} \sum_{j=1}^{n} (1 + |x_j|) + \frac{1}{2e} \cos z
\]

\[
k_i(t, s, u, v, x_1, x_2, \ldots) = \frac{\left| \cos(u^3) \sin(x_1) \right| + \left| \sin^3(v) \cos \left( \sum_{j=1}^{n} x_j^2 \right) \right|}{3 + \cos \left( \sum_{j=1}^{n} x_j \right)}
\]

\[
b(t) = \ln(t), \quad \Lambda' = \mathbb{R}^2, \quad \lambda = \frac{1}{2n}
\]

Condition (A1) is satisfied. Suppose that \(t, s \in \mathbb{R}\) and \(|x| \geq |y|\). We have

\[
|f_i(t, s, x_1, \ldots, x_n, m)| = |f_i(t, s, y_1, \ldots, y_n, n)| \leq \frac{1}{n} \sum_{j=1}^{n} (1 + |x_j|)
\]

\[
\leq \frac{1}{2n} \sum_{j=1}^{n} (1 + |y_j|) + \frac{1}{2e} \cos m - \cos n
\]

\[
\leq \frac{1}{2n} \sum_{j=1}^{n} \ln \left( \frac{1 + |x_j|}{1 + |y_j|} \right) + \frac{1}{2e} |m - n|
\]

\[
\leq \frac{1}{2n} \sum_{j=1}^{n} \ln (1 + |x_j| - |y_j|) + \frac{1}{2e} |m - n|
\]

\[
\leq \frac{1}{2n} \ln \left( \max_{1 \leq i \leq n} |x_i - y_i| \right) + \frac{1}{2e} |m - n|
\]

\[
\leq J \left( \max_{1 \leq i \leq n} |x_i - y_i| \right) + |m - n| \quad \text{a.e. on } t, s \in \mathbb{R}.
\]

The case \(|x| \leq |y|\) can be treated in the same way. Also, \(f_i\) and \(k_i\) satisfies the Carathéodory conditions and \(f_i(\cdot, \ldots, 0, \ldots, 0) \in \mathcal{L}^\infty (\mathbb{R}^n)\). Thus, hypothesis (A2) holds.

\[
\text{ess sup} \left\{ \frac{1}{2e} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} k_i(t, s, u, v, x_j(u, v)) dudv \right\} = \text{ess sup} \left\{ \frac{1}{2e} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left[ \frac{\left| \cos(u^3) \sin(x_1) \right| + \left| \sin^3(v) \cos \left( \sum_{j=1}^{n} x_j^2 \right) \right|}{3 + \cos \left( \sum_{j=1}^{n} x_j \right)} \right] dudv \right\}
\]

\[
\leq \frac{1}{2e} \leq \ln(r),
\]

for any number \(r \geq 2\).

Also,

\[
\lim_{T \to \infty} \text{ess sup} \left\{ \frac{1}{2e} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left[ \frac{\left| \cos(u^3) \sin(x_1) \right| + \left| \sin^3(v) \cos \left( \sum_{j=1}^{n} x_j^2 \right) \right|}{3 + \cos \left( \sum_{j=1}^{n} x_j \right)} \right] dudv \right\}
\]
- \|\cos(u^3) \sin(y_0)\| + \|\sin^3(v) \cos(\sum_{i=1}^{n} u^6)\|}{(3 + \cos \sum_{i=1}^{n} u^6)} \, du \ d\nu = 0

uniformly with respect to \(x_j, y_j \in L^\infty(R^2)\) such that \(\|x\|_{\infty}, \|y\|_{\infty} \leq r\). Therefore, condition (A_3) is satisfied. Moreover,

\[
\lim_{l \to \infty} \operatorname{ess} \sup_{t \in [0, T]} \frac{1}{2e} \int_{R^2} \int_{R^2} \frac{\|\cos(u^3) \sin(x_1)\| + \|\sin^3(v) \cos(\sum_{i=1}^{n} x^6)\|}{(3 + \cos \sum_{i=1}^{n} x^6)} \, du \ d\nu = 0,
\]

which shows that condition (A_4) holds.

It is easy to check that condition (A_5) satisfies i.e.,

\[
\lambda(r) + \sup_{i} \|f_i(\ldots, 0, \ldots, 0)\|_{\infty} + b(r) = \frac{1}{2} \ln(1 + r) + \ln(r) \leq r.
\]

We take \(r_0 = 2\), which shows the inequality of condition (A_5) holds. Therefore, the system of integral Equation (4.20) has at least one solution.

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Author details
Ayub Samadi
E-mail: ayub.samadi@iau.ac.ir
Shahram Banaei
E-mail: math.sh.banaei@gmail.com

ORCID ID: http://orcid.org/0000-0003-0552-1836

1 Department of Mathematics, Miyaneh Branch, Islamic Azad University, Miyaneh, Iran.
2 Department of Mathematics, Bonab Branch, Islamic Azad University, Bonab, Iran.

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References