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Derivation algebra of direct sum of lie algebras

Mohammad Reza Alemi[1] and Farshid Saeedi[1]*

Abstract: Let L_1 and L_2 be two finite dimensional Lie algebras on arbitrary field F with no common direct factor and $L = L_1 \oplus L_2$. In this article, we express the structure and dimension of derivation algebra of L, $Der(L)$, and some of their subalgebras in terms of $Der(L_1)$, $Der(L_2)$, $Hom(L_1, Z(L_2))$, and $Hom(L_2, Z(L_1))$.

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*Corresponding author: Farshid Saeedi, Mathematics, Mashhad Branch, Islamic Azad University, Iran, Islamic Republic of
E-mail: fsaeedi94@yahoo.co.uk

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Hari M. Srivastava, Department of Mathematics and Statistics, Department of Mathematics and Statistics, University of Victoria, Victoria Canada

Additional information is available at the end of the article

1. Introduction

Lie algebras were first discovered by Sophus Lie (1842–1899) when he attempted to classify certain ”smooth” subgroups of general linear groups. The groups he considered are now called Lie groups. By taking the tangent space at the identity element of such a group, he obtained the Lie algebra and hence the problems on groups can be reduced to problems on Lie algebras so that it becomes more tractable. Lie algebra is applied in different domains of physics and mathematics, such as spectroscopy of molecules, atoms, nuclei, hadrons, hyperbolic, and stochastic differential equations. After the introduction of fuzzy sets by L. Zadeh (Zadeh, 1965), various notions of higher-order fuzzy sets have been proposed. Fuzzy and anti fuzzy Lie ideals in Lie algebras have been studied in (Akram, 2006; Davvaz, 2001; Keyun, Quanxi, & Chaoping, 2001; Kim & Lee, 1998).

Throughout this article, Lie algebras are considered finite dimensional.

Farshid Saeedi

ABOUT THE AUTHORS

Dr. Farshid Saeedi is a Mathematician and a faculty in the Department of Mathematics, Mashhd Branch, Islamic Azad University, Iran. He has Bachelor of Science (B.Sc.) and Master of Science (M.Sc.) degrees in pure Mathematics in 1992 and 1996 from University of Birjand and University of Amir Kabir, respectively. He earned his Ph.D. degree in pure Mathematics (group theory) in 2006 from Islamic Azad University, Iran. His area of specialization are group theory and Lie algebra.

Mohammad Reza Alemi is Ph.D. student atIslamic Azad University, Mashhad Branch.

PUBLIC INTEREST STATEMENT

There are many results on the automorphism of some of algebraic structures such as groups, Lie rings, and Lie algebras. The theory of Lie algebras is one of the most important parts of algebras. There are many other connections between the group theory and the theory of Lie algebras. Many papers make an attempt to generalize the results on finite p-groups to the theory of Lie algebras. Although there are some sporadic results for the Lie algebra that do not coincide with the results for groups. However, there are analogies between groups and Lie algebras, but these analogies are not completely identical and most of them should be checked carefully.

The study of the set of all commuting derivations, for this reason, is interesting that in spite of this in groups is not generally a subgroup from the automorphisms group but in Lie algebras, it is always a subalgebra of the derivations algebra.

Let L be a Lie algebra over field F with bracket [–,–]. A *derivation* of L is an F-linear transformation $\alpha : L \to L$ such that $\alpha([x,y]) = [\alpha(x),y] + [x,\alpha(y)]$ for all $x,y \in L$. Set of all derivations L, by given bracket $[\alpha,\beta] = \alpha\beta - \beta\alpha$, where α and β are derivations of L, forms a Lie algebra that we denote it by $Der(L)$.

Suppose that $L = L_1 \oplus L_2$ is a direct sum of Lie algebras L_1 and L_2. First, we provide some of the symbols which we need. We denote inclusion maps by i_{L_1} and i_{L_2} from L_1 and L_2 into L, respectively, and projection maps by π_{L_1} and π_{L_2} from L into L_1 and L_2, respectively.

For $\theta \in gl(L) = T(L,L)$ we put $\alpha = \pi_{L_1}\theta i_{L_1}$, $\beta = \pi_{L_1}\theta i_{L_2}$, $\gamma = \pi_{L_2}\theta i_{L_1}$, and $\delta = \pi_{L_2}\theta i_{L_2}$. Then $\alpha \in gl(L_1)$, $\beta \in T(L_2,L_1)$, $\gamma \in T(L_1,L_2)$, and $\delta \in gl(L_2)$, where $T(L_1,L_2)$ is the set of all F-linear transformations from L_1 to L_2.

Now assume that

$$\mathcal{M} = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \middle| \begin{matrix} \alpha \in gl(L_1) & \beta \in T(L_2, Z(L_1)) \\ \gamma \in T(L_1, Z(L_2)) & \delta \in gl(L_2) \end{matrix} \right\}.$$

It is easy to see that $\mathcal{M}$ forms a Lie algebra over field F and $\begin{pmatrix} \pi_{L_1}\theta i_{L_1} & \pi_{L_1}\theta i_{L_2} \\ \pi_{L_2}\theta i_{L_1} & \pi_{L_2}\theta i_{L_2} \end{pmatrix} \in \mathcal{M}$. Conversely, suppose that $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathcal{M}$. We define the map $\theta : L \to L$ by $\theta(x,y) = (\alpha(x) + \beta(y), \gamma(x) + \delta(y))$, where $x \in L_1$ and $y \in L_2$; then $\theta \in gl(L)$. It is easy to check that f: $gl(L) \to \mathcal{M}$, which defined by $f(\theta) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is a Lie isomorphism. Thus, we have the following result.

Proposition 1.1. *If $L = L_1 \oplus L_2$, then $gl(L) \cong \mathcal{M}$.*

Consider an arbitrary Lie algebra L and an arbitrary abelian Lie algebra A. Notations $Z(L)$, L^2, $Hom(L,A)$, and $T(L,A)$ denote the center L, derived subalgebra L, set of all lie homomorphism from L into A, and set of all linear transformations from L into A, respectively.

Let L_1 and L_2 be two finite dimensional Lie algebras with no common direct factor, and let $L = L_1 \oplus L_2$. The main aim of this article is to express the structure and dimensional $Der(L)$ in terms of $Der(L_1)$, $Der(L_2)$, and Lie algebras of central homomorphisms $Hom(L_1, Z(L_2))$ and $Hom(L_2, Z(L_1))$. Note that, before in groups theory, the structure and order automorphisms group of G had been investigated, which G is a group in the form of direct product of two finite groups; see (Bidwell, Curran, & McCaughan, 2006). Therefore, this is our main theorem.

Theorem 1.2. *Let $L = L_1 \oplus L_2$ be direct sum of two Lie algebras of finite dimensional that L_1 and L_2 do not have nontrivial common direct factor, and let*

$$\mathcal{A} = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \middle| \begin{matrix} \alpha \in Der(L_1) & \beta \in Hom(L_2, Z(L_1)) \\ \gamma \in Hom(L_1, Z(L_2)) & \delta \in Der(L_2) \end{matrix} \right\}.$$

Then $Der(L) \cong \mathcal{A}$; furthermore

$$dimDer(L) = dimDer(L_1) + dimDer(L_2) + dimHom(L_1, Z(L_2)) + dimHom(L_2, Z(L_1)).$$

In Section 2, among the proof of Theorem 1.2, we get the structure and dimension of some subalgebras $Der(L)$. In the last section, we create conditions that under it $Der(L_1 \oplus L_2) \cong Der(L_1) \oplus Der(L_2)$.

2. Main result

Let L be a Lie algebra, and let $Der_c(L)$ be the set all of derivations α from $Der(L)$ that $\alpha(x) \in [x,L]$ for all $x \in L$, which is subalgebra of derivation algebra L. A derivation of α from L is called ID-derivation when $\alpha(x) \in L^2$ for all $x \in L$. Set all of ID-derivations are denoted by *ID(L)*. Also the set all of $\alpha \in ID(L)$ that sends elements of $Z(L)$ to zero is denoted by $ID^*(L)$.

A derivation α from L is central if α commutes with all of inner derivations from L, or equivalently if $\alpha(x)$ lies in center of L for all $x \in L$. Central derivations of L are ideals of $Der(L)$that we denote it by $Der_z(L)$.

It is obviously that always

$$ad(L) \subseteq Der_c(L) \subseteq ID^*(L) \subseteq ID(L) \subseteq Der(L) \subseteq gl(L) \quad (1)$$

wherein $ad(L)$ the ideal of $Der(L)$ consists the set all of inner derivations.

Many authors investigated characterizing Lie algebras with the help of the above subalgebras of $Der(L)$. For example, Tôgô (Tôgô, 1964) characterized Lie algebras of L over fields with $char(L) = 0$ and $Z(L) \neq \{0\}$ that $ad(L) = Der(L)$; he also proved $Der_z(L) = Der(L)$ if only if L is abelian. Sheikh-Mohseni et al. (Saeedi & Sheikh-Mohseni, 2015; Sheikh-Mohseni, Saeedi, & Badrkhani Asl, 2015) found conditions for Lie algebra of L that $ad(L) = Der_c(L)$ or $Der_c(L) = ID(L)$. For more information about this subalgebras, refer to (Saeedi & Sheikh-Mohseni, 2018; Tôgô, 1955, 1961, 1964, 1967). Following lemmas are helpful to prove Theorem 1.2. The proof of the following lemma is straightforward.

Lemma 2.1 *Let L be a Lie algebra, and let A be an abelian Lie algebra . Then*

(i) $Hom(L, A)$ *is an abelian Lie algebra with the following bracket*:

$$[\alpha, \beta](x) = [\alpha(x), \beta(x)] \qquad \forall \alpha, \beta \in Hom(L, A), x \in L.$$

(ii) $T(L/L^2, A) = Hom(L/L^2, A) \cong Hom(L, A)$.

Lemma 2.2. *Let* $L = L_1 \oplus L_2$ *be such that* L_1 *and* L_2 *with no nontrivial common direct factor. Then*

(i) *If* θ *belongs to* $Der(L)$, *then* $\alpha = \pi_{L_1}\theta i_{L_1} \in Der(L_1)$, $\beta = \pi_{L_1}\theta i_{L_2} \in Hom(L_2, Z(L_1))$, $\gamma = \pi_{L_2}\theta i_{L_1} \in Hom(L_1, Z(L_2))$, *and* $\delta = \pi_{L_2}\theta i_{L_2} \in Der(L_2)$.

(ii) *If* θ *belongs to* $Der_c(L)$, *then* $\alpha \in Der_c(L_1)$, $\delta \in Der_c(L_2)$ *and* $\beta = \gamma = 0$.

(iii) *If* θ *belongs to* $ID(L)$, *then* $\alpha \in ID(L_1)$, $\beta \in Hom(L_2, Z(L_1) \cap L_1^2)$, $\gamma \in Hom(L_1, Z(L_2) \cap L_2^2)$, *and* $\delta \in ID(L_2)$.

(iv) *If* θ *belongs to* $ID^*(L)$, *then* $\alpha \in ID^*(L_1)$, $\beta \in Hom(L_2, Z(L_1) \cap L_1^2)$, $\gamma \in Hom(L_1, Z(L_2) \cap L_2^2)$, *and* $\delta \in ID^*(L_2)$.

(v) *If* θ *belongs to* $Der_z(L)$, *then* $\alpha \in Der_z(L_1)$, $\beta \in Hom(L_2, Z(L_1))$, $\gamma \in Hom(L_1, Z(L_2))$, *and* $\delta \in Der_z(L_2)$.

Proof. Since the proofs are similar, only we give the proof of (i)

(i) Let $x_1, x_2 \in L_1$; then

$$\alpha([x_1, x_2]) = \pi_{L_1}\theta[(x_1, 0), (x_2, 0)] = \pi_{L_1}[\theta(x_1, 0), (x_2, 0)] + \pi_{L_1}[(x_1, 0), \theta(x_2, 0)].$$

Now suppose that $\theta(x_1, 0) = (x_1', y_1')$ and $\theta(x_2, 0) = (x_2', y_2')$; then

$$\pi_{L_1}[(x_1', y_1'), (x_2, 0)] + \pi_{L_1}[(x_1, 0), (x_2', y_2')] = [x_1', x_2] + [x_1, x_2'].$$

On the other hand,

$$[\alpha(x_1), x_2] = [\pi_{L_1}\theta(x_1, 0), x_2] = [\pi_{L_1}(x_1', y_1'), x_2] = [x_1', x_2],$$

$$[x_1, \alpha(x_2)] = [x_1, \pi_{L_2}\theta(x_2, 0)] = [x_1, \pi_{L_1}(x_2', y_2')] = [x_1, x_2'].$$

Therefore $\alpha = \pi_{L_1}\theta i_{L_1} \in Der(L_1)$. Similarly $\delta = \pi_{L_2}\theta i_{L_2} \in Der(L_2)$.

Now, Let $(x,0)$, $(0,y) \in L_1 \oplus L_2$; we have

$\theta([(x,0),(0,y)]) = [\theta(x,0),(0,y)] + [(x,0)+\theta(0,y)]$.

The left side of equality is $(0,0)$, which implies $[(x,0),\theta(0,y)] = -[\theta(x,0),(0,y)]$. Notice that projection mappings π_{L_1} and π_{L_2} are lie epimorphism. First suppose $y \in L_2^2$. Then, there exist $y_1, y_2 \in L_2$ such that $y = [y_1, y_2]$. we have

$$\begin{aligned}\beta(y) = \beta([y_1,y_2]) &= \pi_{L_1}\theta(0,[y_1,y_2]) = \pi_{L_1}\theta([(0,y_1),(0,y_2)])\\ &= \pi_{L_1}[\theta(0,y_1),(0,y_2)] + \pi_{L_1}[(0,y_1),\theta(0,y_2)]\end{aligned}$$

Now, if $\theta(0,y_1) = (x_1', y_1')$ and $\theta(0,y_2) = (x_2', y_2')$, Then

$$\pi_{L_1}[(x_1',y_1'),(0,y_2)] + \pi_{L_1}[(0,y_1),(x_2',y_2')] = \pi_{L_1}(0,[y_1',y_2]) + \pi_{L_1}(0,[y_1,y_2']) = 0.$$

That is, β maps L_2^2 to zero.

Now suppose $x \in L$. Then, we have

$$\begin{aligned}[x,\beta(y)] &= [x,\pi_{L_1}\theta i_{L_2}(y)] = [\pi_{L_1}(x,0),\pi_{L_1}\theta(0,y)] = \pi_{L_1}[(x,0),\theta(0,y)]\\ &= -\pi_{L_1}[\theta(x,0),(0,y)]\end{aligned}$$

Suppose $\theta(x,0) = (x', y')$. Then, we have $-\pi_{L_1}[(x',y'),(0,y)] = -\pi_{L_1}(0,[y',y]) = 0$, which implies $\beta(y) \in Z(L_1)$. Thus $\beta = \pi_{L_1}\theta i_{L_2} \in Hom(L_2, Z(L_1))$. Similarly $\gamma = \pi_{L_2}\theta i_{L_1} \in Hom(L_1, Z(L_1))$.

Now we are ready to prove Theorem 1.2. □

Proof of Theorem 1.2. Let $g = f_{|Der(L)}$, where f is the same of lie homomorphism of Proposition 1.1; thus, by Lemma 2.2(i), $Im(g) \subseteq \mathcal{A}$.

Now let $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \mathcal{A}$. So f is onto; then there exists $\theta \in gl(L)$ such that $g(\theta) = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$. We prove $\theta \in Der(L)$. Let $(x_1,y_1),(x_2,y_2) \in L_1 \oplus L_2$; then

$$\begin{aligned}\theta([(x_1,y_1),(x_2,y_2)]) &= \left[\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}x_1\\ y_1\end{pmatrix},\begin{pmatrix}x_2\\ y_2\end{pmatrix}\right] = \left[\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}[x_1,x_2]\\ [y_1,y_2]\end{pmatrix}\right]\\ &= \begin{pmatrix}\alpha([x_1,x_2]) + \beta([y_1,y_2])\\ \gamma([x_1,x_2]) + \delta([y_1,y_2])\end{pmatrix} = \begin{pmatrix}\alpha([x_1,x_2])\\ \delta([y_1,y_2])\end{pmatrix}.\end{aligned}$$

On the other hand,

$$\begin{aligned}[\theta(x_1,y_1),(x_2,y_2)] &= \left[\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}x_1\\ y_1\end{pmatrix},\begin{pmatrix}x_2\\ y_2\end{pmatrix}\right] = \left[\begin{pmatrix}\alpha(x_1)+\beta(y_1)\\ \gamma(x_1)+\delta(y_1)\end{pmatrix},\begin{pmatrix}x_2\\ y_2\end{pmatrix}\right]\\ &= \begin{pmatrix}[\alpha(x_1)+\beta(y_1),x_2]\\ [\gamma(x_2)+\delta(y_1),y_2]\end{pmatrix} = \begin{pmatrix}[\alpha(x_1),x_2]\\ [\delta(y_1),y_2]\end{pmatrix}.\end{aligned}$$

Similarly,

$$[(x_1,y_1),\theta(x_2,y_2)] = \begin{pmatrix}[x_1,\alpha(x_2)]\\ [y_1,\delta(y_2)]\end{pmatrix}.$$

Therefore, the favorable result obtains. □

By the above theorem, we can obtain the structure and dimension subalgebras of $Der(L)$ that is in sequence (1).

Consider the following sets:

$$\mathcal{J}_z = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \middle| \begin{matrix} \alpha \in ID(L_1) & \beta \in Hom(L_2, Z(L_1) \cap L_1^2) \\ \gamma \in Hom(L_1, Z(L_2) \cap L_2^2) & \delta \in ID(L_2) \end{matrix} \right\} \subseteq = \mathcal{A}.$$

$$\mathcal{J}^* = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \middle| \begin{matrix} \alpha \in ID^*(L_1) & \beta \in Hom(L_2, Z(L_1) \cap L_1^2) \\ \gamma \in Hom(L_1, Z(L_2) \cap L_2^2) & \delta \in ID^*(L_2) \end{matrix} \right\} \subseteq = \mathcal{J}.$$

$$\mathcal{A}_c = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \middle| \begin{matrix} \alpha \in Der_c(L_1) \\ \delta \in Der_c(L_2) \end{matrix} \right\} \subseteq = \mathcal{J}^*,$$

$$\mathcal{A}_z = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \middle| \begin{matrix} \alpha \in Der_z(L_1) & \beta \in Hom(L_2, Z(L_1)) \\ \gamma \in Hom(L_1, Z(L_2)) & \delta \in Der_z(L_2) \end{matrix} \right\} \subseteq = \mathcal{A}.$$

Corollary 2.3. *Let L_1 and L_2 be two Lie algebras finite dimension over field F with no nontrivial common direct factor, and let $L = L_1 \oplus L_2$. Then*

(i) ID(L) $\cong \mathcal{I}$ in Lie algebra. Furthermore,

$$dimID(L) = dimID(L_1) + dimID(L_2) + dimHom(L_1, Z(L_2) \cap L_2^2) + dimHom(L_2, Z(L_1) \cap L_1^2).$$

(ii) $ID^*(L) \cong \mathcal{I}^*$ in Lie algebra. Moreover,

$$dimID^*(L) = dimID^*(L_1) + dimID^*(L_2) + dimHom(L_1, Z(L_2) \cap L_2^2) + dimHom(L_2, Z(L_1) \cap L_1^2).$$

(iii) $Der_c(L) \cong \mathcal{A}_c$ in Lie algebra. Hence,

$$dimDer_c(L) = dimDer_c(L_1) + dimDer_c(L_2).$$

(iv) $Der_z(L) \cong \mathcal{A}_z$ in Lie algebra. Hence,

$$dimDer_z(L) = dimDer_z(L_1) + dimDer_z(L_2) + dimHom(L_1, Z(L_2)) + dimHom(L_2, Z(L_1)).$$

Proof. (i) We have $ID(L) \le Der(L)$; suppose that $h = g_{|ID(L)}$, where g is the same of lie homomorphism of Theorem 1.2. Thus, by Lemma 2.2(iii), $Im(h) \subseteq \mathcal{J}$. Now suppose that $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathcal{J} \subseteq \mathcal{A}$. By Theorem 1.2, there exists $\theta \in Der(L)$ such that $h(\theta) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Now let $(x, y) \in L_1 \oplus L_2$; then we have

$$\theta(x, y) = (\alpha(x) + \beta(y), \gamma(x) + \delta(y)) \in L_1^2 \oplus L_2^2 = (L_1 \oplus L_2)^2$$

Therefore $\theta \in ID(L)$ as required.

Next, parts are similar to part (i). □

Let L_1 and L_2 be two Lie algebras over an arbitrary field F, and let $\beta = \{x_1, \ldots, x_n\}$ and $\beta' = \{y_1, \ldots, y_m\}$ be, respectively, two ordered basis of L_1 and L_2. Suppose that $\gamma \in T(L_1, L_2)$. Using this two basis, we may define scalars $\gamma_{i,j}$ by $\gamma(x_j) = \sum_{i=1}^{m} \gamma_{i,j} y_i$; then the matrix of γ is such as follows:

$$\begin{bmatrix} \gamma_{1,1} & \gamma_{1,2} & \cdots & \gamma_{1,n} \\ \gamma_{2,1} & \gamma_{2,2} & \cdots & \gamma_{2,n} \\ \vdots & \vdots & & \vdots \\ \gamma_{m,1} & \gamma_{m,2} & \cdots & \gamma_{m,n} \end{bmatrix}$$

Also, we recall that a Lie algebra L is called Heisenberg if $L^2 = Z(L)$ and dimL^2= 1. Heisenberg Lie algebras of finite dimension are from odd dimension with basis of $\{x_1, x_2, \ldots, x_{2k}, x_{2k+1}\}$, that is the nonzero bracket between the basis elements in the form $[x_{2i-1}, x_{2i}] = x_{2k+1}$ for all $i = 1, 2, \ldots, k$. Symbol of $H(k)$ is a Heisenberg Lie algebra of dimension 2k+1. A Lie algebra L is called purely nonabelian, when L has no nontrivial abelian direct factor.

The following example illustrates dimension $Der(A(m) \oplus H(1))$ and some subalgebras of derivation algebra it, by using Theorem1.2 and Corollary 2.3.

Example 1. Let $L = A(m) \oplus H(1)$ be direct sum of abelian Lie algebra of dimension m, $A(m)$, and Heisenberg Lie algebra of dimension three, $H(1)$. It is easy to see that the matrix representation of each element in $Der(H(1))$ is such as follows:

$$\begin{bmatrix} \gamma_{1,1} & \gamma_{1,2} & 0 \\ \gamma_{2,1} & \gamma_{2,2} & 0 \\ \gamma_{3,1} & \gamma_{3,2} & \gamma_{1,1} + \gamma_{2,2} \end{bmatrix}$$

Because $H(1)$ is purely nonabelian, thus, by Theorem1.2 and Corollary 2.2, we have

$$\begin{aligned} dimDer(L) &= dimDer(A(m)) + dimDer(H(1)) + dimHom(A(m), Z(H(1))) \\ &+ dimHom(H(1), Z(A(m))) = m^2 + 6 + m + 2m = m^2 + 3m + 6 \end{aligned}$$

$$\begin{aligned} dimID(L) &= dimID(A(m)) + dimID(H(1)) + dimHom(A(m), Z(H(1)) \cap H(1)^2) \\ &+ dimHom(H(1), Z(A(m)) \cap A(m)^2) = m + 2 \end{aligned}$$

$$\begin{aligned} dimID^*(L) &= dimID^*(A(m)) + dimID^*(H(1)) + dimHom(A(m), Z(H(1)) \cap H(1)^2) \\ &+ dimHom(H(1), Z(A(m)) \cap A(m)^2) = m + 2 \end{aligned}$$

$$dimDer_c(L) = dimDer_c(A(m)) + dimDer_c(H(1)) = 2$$

$$\begin{aligned} dimDer_z(L) &= dimDer_z(A(m)) + dimDer_z(H(1)) + dimHom(A(m), Z(H(1))) \\ &+ dimHom(H(1), Z(A(m))) = m^2 + 2 + m + 2m = m^2 + 3m + 2 \end{aligned}$$

In the following example, by using Theorem 1.2, Corollary 2.3, and the induction, we obtain dimension of derivation n-copy of direct sum of Lie algebras $H(1)$.

Example 2. Let $L = H(1) \oplus H(1) \oplus \cdots \oplus H(1)$ be direct sum n-copy of Lie algebras $H(1)$. Then using induction on n, for n $\geq$ 1, we have

$$dimDer(L) = 2n(n+2).$$

In special case, if n = 2, then dimDer(L) = 16.

In Theorem 1.2, it is necessary that L_1 and L_2 has no nontrivial common direct factor. Consider the following example.

Example 3. *Let* $L_1 = A(1) \oplus \langle x_1, x_2, x_3 \mid [x_1, x_2] = [x_1, x_3] = x_2 \rangle$ *and* $L_2 = A(1) \oplus \langle y_1, y_2 \mid [y_1, y_2] = y_1 \rangle$. *Put* $L = L_1 \oplus L_2$; *then*

$$\begin{aligned} & dimDer(L_1) + dimDer(L_2) + dimHom(L_1, Z(L_2)) + dimHom(L_2, Z(L_1)) \\ & = 9 + 5 + 3 + 2 = 19. \end{aligned}$$

While $Der(L_1 \oplus L_2)$ is the form of the following matrix:

$$\begin{bmatrix} \alpha_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_{21} & \alpha_{11} + \alpha_{22} & \alpha_{23} & \alpha_{24} & 0 & \alpha_{26} & \alpha_{27} \\ \alpha_{31} & 0 & \alpha_{22} - \alpha_{23} & -\alpha_{24} & 0 & -\alpha_{26} & -\alpha_{27} \\ \alpha_{41} & 0 & \alpha_{43} & \alpha_{44} & 0 & \alpha_{46} & \alpha_{47} \\ 0 & 0 & 0 & 0 & \alpha_{55} + \alpha_{66} & \alpha_{56} & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_{66} & 0 \\ \alpha_{71} & 0 & \alpha_{73} & \alpha_{74} & 0 & \alpha_{76} & \alpha_{77} \end{bmatrix}$$

Therefore $dimDer(L_1 \oplus L_2)$ = 21; so the equality is not established.

Conversely, Theorem 1.2 is not correct generally. See the following example.

Example 4. Let $L_1 = \langle x_1, \cdots, x_6 \,|\, [x_1, x_2] = x_3, [x_4, x_5] = x_6 \rangle \cong H(1) \oplus H(1)$ and $L_2 = \langle y_1, y_2, y_3, y_4 | [y_1, y_2] = y_3 \rangle \cong H(1) \oplus A(1)$. Put $L = L_1 \oplus L_2$; then representation of matrix each element of $Der(L)$ is as follows:

$$\begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_{2,1} & \alpha_{2,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_{3,1} & \alpha_{3,2} & \alpha_{1,1}+\alpha_{2,2} & \alpha_{3,4} & \alpha_{3,5} & 0 & \alpha_{3,7} & \alpha_{3,8} & 0 & \alpha_{3,10} \\ 0 & 0 & 0 & \alpha_{4,4} & \alpha_{4,5} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_{5,4} & \alpha_{5,5} & 0 & 0 & 0 & 0 & 0 \\ \alpha_{6,1} & \alpha_{6,2} & 0 & \alpha_{6,4} & \alpha_{6,5} & \alpha_{4,4}+\alpha_{5,5} & \alpha_{6,7} & \alpha_{6,8} & 0 & \alpha_{6,10} \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha_{7,7} & \alpha_{7,8} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha_{8,7} & \alpha_{8,8} & 0 & 0 \\ \alpha_{9,1} & \alpha_{9,2} & 0 & \alpha_{9,4} & \alpha_{9,5} & 0 & \alpha_{9,7} & \alpha_{9,8} & \alpha_{7,7}+\alpha_{8,8} & \alpha_{9,10} \\ \alpha_{10,1} & \alpha_{10,2} & 0 & \alpha_{10,4} & \alpha_{10,5} & 0 & \alpha_{10,7} & \alpha_{10,8} & 0 & \alpha_{10,10} \end{bmatrix}$$

Therefore dimDer(L) = 40, also

$$\begin{aligned} &dimDer(L_1) + dimDer(L_2) + dimHom(L_1, Z(L_2)) + dimHom(L_2, Z(L_1)) \\ &\qquad = 16 + 10 + 8 + 6 = 40. \end{aligned}$$

As seen on, equality established while L_1 and L_2 have nontrivial common direct factor.

3. Relation between derivation of direct sum and direct sum of derivations

First, we recall that Lie algebra L is called *stem*, when $Z(L) \subseteq L^2$, and it is called *perfect* if $L = L^2$. Suppose that L is a Lie algebra and that H is a subalgebra of L and K is an ideal of it such that $L = H + K$ and $H \cap K = \{0\}$; then we say L is the semidirect sum of H with K and we write $L = HK$.

Now we turn to the structure $Der(L)$, where $L = L_1 \oplus L_2$. As we saw in Example 2, if $L = L_1 \oplus L_2$ is the direct sum of two Lie algebras with no nontrivial common direct factor, then it could $Der(L) \ncong Der(L_1) \oplus Der(L_2)$. In this section, we give conditions that create this Isomorphism.

Let

$$A = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \middle| \begin{matrix} \alpha \in Der(L_1) \\ \delta \in Der(L_2) \end{matrix} \right\} \qquad B = \left\{ \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} \middle| \begin{matrix} \beta \in Hom(L_2, Z(L_1)) \\ \gamma \in Hom(L_1, Z(L_2)) \end{matrix} \right\}$$

It is obvious that A is a subalgebra of $\mathcal{A}$, and $A \cong Der(L_1) \oplus Der(L_2)$. Now if L_1 and L_2 be *stem*, then B is not only a subalgebra of $\mathcal{A}$ but also it is an ideal of it; thus, by Theorem 1.2, we have

Corollary 3.1. *Let $L = L_1 \oplus L_2$ be such that L_1 and L_2 are stem Lie algebras with no nontrivial common direct factor; then*

$$Der(L) \cong (Der(L_1) \oplus Der(L_2))(Hom(L_1, Z(L_2)) \oplus Hom(L_2, Z(L_1))).$$

Proof. It is easy to check that $B \cong (Hom(L_1, Z(L_2)) \oplus Hom(L_2, Z(L_1)))$ and $\mathcal{A} \cong A \times B$; then, by Theorem 1.2, the assert is valid. □

Corollary 3.2. *Let L_1 and L_2 be two Lie algebras with no nontrivial common direct factor and $L = L_1 \oplus L_2$; then*

$$ID(L) \cong (ID(L_1) \oplus ID(L_2))\left(Hom(L_1, Z(L_2) \cap_2^L 2) \oplus Hom(L_2, Z(L_1) \cap_1^L 2)\right).$$

Proof. By Theorem 2.3(i), we have

$$A \cong (ID(L_1) \oplus ID(L_2))$$

and

$B \cong \left(Hom(L_1, Z(L_2) \cap L_2^2) \oplus Hom(L_2, Z(L_1) \cap L_1^2)\right)$

Also B is an ideal of $\mathcal{I}$; thus, $\mathcal{I} \cong A \times B$. □

Corollary 3.3 *Let L_1 and L_2 be two Lie algebras with no nontrivial common direct factor and $L = L_1 \oplus L_2$; then*

$ID^*(L) \cong (ID^*(L_1) \oplus ID^*(L_2))\left(Hom(L_1, Z(L_2) \cap_2^L 2) \oplus Hom(L_2, Z(L_1) \cap_1^L 2)\right).$

Proof. It is similar to previous corollary.

Corollary 3.4. *Let $L = L_1 \oplus L_2$ be such that L_1 and L_2 with no nontrivial common direct factor and $Z(L_2) = \{0\}$. Then*

$Der(L) \cong (Der(L_1) \oplus Der(L_2))Hom(L_2, Z(L_1)).$

Proof. Since $Z(L_2) = \{0\}$, then B is an ideal of $\mathcal{A}$ and $B \cong Hom(L_2, Z(L_1))$. Therefore $\mathcal{A} \cong AB$, and, by Theorem 1.2, the assert is valid.

Corollary 3.5. *Let L_1 and L_2 with no common direct factor, and let $L = L_1 \oplus L_2$. Then if L_1 and L_2 are perfect Lie algebras or $Z(L_1)$ and $Z(L_2)$ are trivial, then we have*

$Der(L) \cong Der(L_1) \oplus Der(L_2).$

Proof. Since $Hom(L_1, Z(L_2)) = Hom(L_2, Z(L_1)) = \{0\}$, thus, $B = \{0\}$ and $\mathcal{A} \cong A$.

Example 5 Let $L = \langle x_1, x_2, x_3 \,|\, [x_1, x_2] = x_3, [x_2, x_3] = x_1, [x_3, x_1] = x_2 \rangle$ be a three-dimensional Lie algebra such that $L^2 = L$. Then, $Der(L)$ and $Der(L \oplus L)$ have the following matrix form, respectively,

$$\begin{bmatrix} \alpha_{22}+\alpha_{33} & -\alpha_{21} & -\alpha_{31} \\ -\alpha_{12} & \alpha_{11}+\alpha_{22} & -\alpha_{32} \\ -\alpha_{13} & -\alpha_{23} & \alpha_{11}+\alpha_{33} \end{bmatrix}$$

$$\begin{bmatrix} \alpha_{22}+\alpha_{33} & -\alpha_{21} & -\alpha_{31} & 0 & 0 & 0 \\ -\alpha_{12} & \alpha_{11}+\alpha_{22} & -\alpha_{32} & 0 & 0 & 0 \\ -\alpha_{13} & -\alpha_{23} & \alpha_{11}+\alpha_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_{55}+\alpha_{66} & -\alpha_{54} & -\alpha_{64} \\ 0 & 0 & 0 & -\alpha_{45} & \alpha_{44}+\alpha_{66} & -\alpha_{65} \\ 0 & 0 & 0 & -\alpha_{46} & -\alpha_{56} & \alpha_{44}+\alpha_{55} \end{bmatrix}$$

It is obviously $Der(L \oplus L) \cong Der(L) \oplus Der(L)$.

Corollary 3.6 *Let L_1 and L_2 with no common direct factor, and let $L = L_1 \oplus L_2$. Then*

$Der_c(L) \cong Der_c(L_1) \oplus Der_c(L_2).$

Proof. By lemma 2.3(iii), the assert is valid.

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Author details
Mohammad Reza Alemi[1]
E-mail: reza_alemi1988@yahoo.com
Farshid Saeedi[1]
E-mail: fsaeedi94@yahoo.co.uk
[1] Department of Mathematics, Mashhad Branch, Islamic Azad University, Mashhad, Mashhad, Iran.

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References

Akram, M. (2006). Anti fuzzy Lie ideals of Lie algebras. *Quasigroups and Related Systems*, *14*, 123–132.

Bidwell, J. N. S., Curran, M. J., & McCaughan, D. J. (2006). Automorphisms of direct products of finite groups. *Archiv Der Mathematik*, *86*(6), 481–489. doi:10.1007/s00013-005-1547-z

Davvaz, B. (2001). Fuzzy Lie algebras. *International Journal of Applied Mathematics*, *6*, 449–461.

Keyun, Q., Quanxi, Q., & Chaoping, C. (2001). Some properties of fuzzy Lie algebras. *Journal Fuzzy Mathematics*, *9*, 985–989.

Kim, C. G., & Lee, D. S. (1998). Fuzzy Lie ideals and fuzzy Lie subalgebras. *Fuzzy Sets and Systems*, *94*, 101–107. doi:10.1016/S0165-0114(96)00230-8

Saeedi, F., & Sheikh-Mohseni, S. (2015). On semicomplete Lie algebras. *Rendiconti Circular Materiaux Palermo*, *65*(1), 111–122. doi:10.1007/s12215-015-0223-9

Saeedi, F., & Sheikh-Mohseni, S. (2018). On ID* (L)-derivations of Filippov algebra. *Asian-European Journal of Mathematics*, *11*(2), 1850050–9. doi:10.1142/S179355711850050X

Sheikh-Mohseni, S., Saeedi, F., & Badrkhani Asl, M. (2015). On special subalgebras of derivations of Lie algebras. *Asian-European Journal of Mathematics*, *8*(2), 1550032, 12. doi:10.1142/S1793557115500321

Tôgô, S. (1955). On the derivations of Lie algebras. *Journal of Science Hiroshima University Series A*, *19*, 71–77.

Tôgô, S. (1961). On the derivation algebras of Lie algebras. *Canadian Journal of Mathematics*, *13*, 201–216. doi:10.4153/CJM-1961-017-8

Tôgô, S. (1964). Derivations of Lie algebras. *Journal of Science Hiroshima University Series A-I*, *28*, 133–158.

Tôgô, S. (1967). Outer derivations of Lie algebras. *Transactions of the American Mathematical Society*, *128*, 264–276. doi:10.2307/1994323

Zadeh, L. A. (1965). Fuzzy sets. *Information Control*, *8*, 338–353. doi:10.1016/S0019-9958(65)90241-X