Almost sure asymptotic stability for some stochastic partial functional integrodifferential equations on Hilbert spaces

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Abstract: In this work, we study the asymptotic behavior of the mild solutions of a class of stochastic partial functional integrodifferential equation on Hilbert spaces. Using the stochastic convolution developed, we establish the exponential stability in $p$–mean square with $p \geq 2$. Also, pathwise exponential stability is proved for $p > 2$. We extend the result of an example is provided for illustration.

Subjects: Science; Mathematics & Statistics; Advanced Mathematics; Applied Mathematics; Statistics & Probability

Keywords: exponential stability in $p$-mean; almost sure asymptotic stability; resolvent operator; stochastic convolution; mild solution; predictable process

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PUBLIC INTEREST STATEMENT

One of the important problems in many branches of science and industry, e.g. engineering, management, finance, social science, is the specification of the stochastic process governing the behavior of an underlying quantity. We here use the term underlying quantity to describe any interested object whose value is known at present but is liable to change in the future. Typical examples are the number of cancer cells, number of HIV infected individuals, share price in a company, price of gold, oil or electricity. All these stochastic processes are studied by stochastic differential equations. Therefore, in this paper, we discuss the stability of stochastic partial integrodifferential equations using stochastic convolution.
1. Introduction
Stochastic delay differential equations (SDDEs) play an important role in many branches of science and industry. Such models have been used with great success in a variety of application areas, including biology, epidemiology, mechanics, economics, and finance. In the past few decades, qualitative theory of SDDEs has been studied intensively by many scholars. Here, we refer to Da Prato and Zabczyk (1992a) and references therein. In recent years, existence, uniqueness, stability, and other quantitative and qualitative properties of solutions to stochastic partial differential equations have been extensively investigated by several authors.


The stochastic integrodifferential equations are more general and still in a state of flux, with new basic results continuously emerging. Integrodifferential equations are important for investigating some problems raised from natural phenomena. They have applications in many areas such as physics, chemistry, economics, social sciences, finance, population dynamics, electrical engineering, medicine biology, ecology and other areas of science and engineering. Qualitative properties such as existence, uniqueness, optimality conditions, controllability and stability for various linear and non-linear stochastic partial integrodifferential equations have been extensively studied by many researchers, see for instance (Balachandran & Sakthivel, 2001; Diagana, Hernández, & Dos Santos, 2009; Dieye, Diop, & Ezzinbi, 2016a, 2016b, 2017; Diop, Ezzinbi, & Lo, 2012; Dos Santos, Guzzo, & Rabelo, 2010; Ezzinbi & Ghnimi, 2010; Sathya & Balachandran, 2012) and the references therein.

As the motivation of above-discussed works, we consider the following stochastic partial functional integrodifferential equation:

\[
\begin{align*}
\frac{dx(t)}{dt} &= A x(t) + \int_0^t \gamma(t-s)x(s)ds + f(t, x(t)) \, dt + g(t, x(t)) \, dw(t) \\
x(0) &= x_0,
\end{align*}
\]

(1)

where \( A \) generates a \( C_0 \)-semigroup on a separable Hilbert \( H \), \( \gamma(t) \) a closed linear operator on \( H \) with time-independent domain \( \mathcal{D}(A) \subset \mathcal{D}(\gamma) \). \( f : \mathbb{R}^+ \times H \to H \), \( g : \mathbb{R}^+ \times H \to \mathcal{L}(U, H) \) are Lipschitzian functions in \( x \in H \) and continuous in \( (t, x) \in \mathbb{R}^+ \times H \). \( w(t) \) is a Wiener process on the separable Hilbert space \( U \) with covariance operator \( Q \in \mathcal{L}(U) \). \( x_0 \) is a \( \mathcal{F}_0 \)-measurable \( H \)-valued square-integrable random variable.

In this work, our main aim is to study the exponential stability in \( p \)-th mean and also almost sure stability property of mild solutions for the system (1) by using the theory of resolvent operator as developed by Grimmer (1982) and the properties of stochastic convolution developed in Dieye, Diop, and Ezzinbi (2016c). The analysis of (1) when \( B=0 \) was initiated in Taniguchi (1995), where the authors proved the existence and stability of solutions by using a strict contraction principle. The main contribution of this paper is on finding conditions to assure the existence, uniqueness, and stability of impulsive neutral stochastic integrodifferential equations. Our paper expands the usefulness of stochastic integrodifferential equations since the literature shows results for existence and stability for such equations under semigroup theory.
The remaining of the paper is organized as follows. Section 2, presents notations and preliminary results. We study also the existence of the mild solutions of Equation (1). Section 3, shows the stability of the mild solutions. Finally, Section 4, presents an example that illustrates our results.

2. Stochastic processes and integrodifferential equations

Let X and Y be Banach spaces. \( L(X, Y) \) denotes the space of bounded linear operator from X to Y, simply \( L(X) \) when \( X = Y \). We will assume that \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P) \) is a complete filtered probability space. We are given a Q-Wiener process the probability space and having value in U a separable Hilbert space, one can construct \( w(t) \) as follows, \( w(t) := \sum_{n=1}^{\infty} \sqrt{\lambda_n} B_n(t)e_n \quad t \geq 0 \). where \( B_n(t)(n = 1, 2, 3, \ldots) \) is a sequence of real-valued standard Brownian motions mutually independent of \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P) \), \( \lambda_n \geq 0(n = 1, 2, 3, \ldots) \) are positive real numbers such that \( \sum_{n=1}^{\infty} \lambda_n < +\infty \). \( (e_n)_{n \geq 1} \) is a complete orthonormal basis in U, and \( Q \in L(U) \) is the incremental covariance operator of the \( w \) which is a symmetric nonnegative trace class operator defined by \( Q e_n = \lambda_n e_n \quad n = 1, 2, 3, \ldots \)

For this analysis, we recall the definition of H-valued stochastic integral with respect to the U-valued Q-Wiener process \( w \). Let \( L^2_0 = L_2(U_0, H) \) denote the space of all Hilbert–Schmidt operator from \( U_0 = Q^{1/2}(U) \) to H which is a separable Hilbert space, equipped with the following norm:

\[
\| \varphi \|_2^2 = \text{tr}(\varphi Q \varphi^*).
\]

Clearly, for any bounded operators \( \varphi \in L(U, H) \), this norm is given by

\[
\| \varphi \|_2^2 = \text{tr}(\varphi Q \varphi^*) = \sum_{n=1}^{\infty} \| \sqrt{\lambda_n} \varphi e_n \|_H^2.
\]

Let \( \varphi : (0, +\infty) \to L^2_0 \) be a predictable \( \mathcal{F}_t \)-adapted process such

\[
\int_0^t \mathbb{E} \| \varphi(s) \|_2^2 ds < \infty \quad \text{for } t > 0.
\]

Then, define the H-valued stochastic integral

\[
\int_0^t \varphi(s)dw(s)
\]

which is a continuous square integrable martingale. For more details on stochastic integrals, we refer to Da Prato and Zabczyk (1992a).

Next, we recall conditions that guarantee the existence of solution for the deterministic, integrodifferential equation

\[
\begin{cases}
v'(t) = Av(t) + \int_0^t \gamma(t-s)v(s)ds \quad \text{for } t \geq 0 \\
v(0) = v_0 \in H.
\end{cases}
\]

Def. 2.1. (Grimmer, 1982) A resolvent operator for Equation (2) is a bounded linear operator-valued function \( R(t) \in L(H) \) for \( t \geq 0 \), having the following properties:

(i) \( R(0) = I \) (the identity map of H) and \( \| R(t) \|_{L(H)} \leq N e^{\nu t} \) for some constants \( N > 0 \) and \( \nu \in \mathbb{R} \).
(ii) For each \( x \in H \), \( R(t)x \) is strongly continuous for \( t \geq 0 \).
(iii) \( R(t) \in L(Y) \) for \( t \geq 0 \). For \( x \in Y \), \( R(\cdot)x \in C^1(\mathbb{R}^+; H) \cap C(\mathbb{R}^+; Y) \) and
\[ R'(t)x = AR(t)x + \int_0^t Y(t - s)R(s)xds \\
= R(t)Ax + \int_0^t R(t - s)Y(s)xds \quad \text{for} \quad t \geq 0. \]

In the whole of this work, we assume that

(A1) The operator A is the infinitesimal generator of a \(C_0\)-semigroup \((S(t))_{t \geq 0}\) on \(H\).

(A2) For all \(t \geq 0\), \(Y(t)\) is closed linear operator from \(D(A)\) to \(H\) and \(Y(t) \in \mathcal{L}(Y, H)\). For any \(y \in Y\), the map \(t \to Y(t)y\) is bounded, differentiable and the derivative \(t \to Y'(t)y\) is bounded uniformly continuous on \(\mathbb{R}^+\).

The following theorem gives a satisfactory answer to the problem of existence of solutions.

**Theorem 2.1.** (Grimmer, 1982) Assume that (A1) – (A2) hold. Then there exists a unique resolvent operator for the Cauchy problem (2).

In the following, we give some results for the existence of solutions for the following integro-differential equation:

\[
\begin{aligned}
&v'(t) = Av(t) + \int_0^t Y(t - s)v(s)ds + q(t), \quad \text{for} \quad t \geq 0 \\
v(0) = v_0 \in H.
\end{aligned}
\]  

(3)

where \(q : \mathbb{R}^+ \to H\) is a continuous function.

**Definition 2.2.** (Grimmer, 1982) A continuous function \(v : \mathbb{R}^+ \to H\) is said to be a strict solution of Equation (3) if \(v \in C^1(\mathbb{R}^+; H) \cap \mathcal{C}(\mathbb{R}^+; Y)\) and \(v\) satisfies Equation (3).

**Theorem 2.2.** (Grimmer, 1982) Assume that (A1) – (A2) hold. If \(v\) is a strict solution of Equation (3), then

\[ v(t) = R(t)v_0 + \int_0^t R(t - s)q(s)ds \quad \text{for} \quad t \geq 0. \]

In order to set our problem, we make the following assumptions

(A3) There exist \(\gamma > 0\) and \(M \geq 1\) such that the resolvent operator \((R(t))_{t \geq 0}\) of Equation (2) satisfies

\[ \| R(t) \|_{\mathcal{L}(H)} \leq Me^{-\gamma t} \quad \text{for} \quad t \geq 0. \]  

(4)

The exponential stability of the resolvent operator will play a crucial role to prove the main results of this work. It has been studied in Grimmer (1982).

(A4) The functions \(f\) and \(g\) are Lipschitz continuous. Let \(L_1, L_2, K > 0\) be such that for every \(x, y \in H\) and \(t \geq 0\) the following conditions are satisfied:

\[ \| f(t, x) - f(t, y) \|_H \leq L_1 \| x - y \|_H, \]  

(5)

\[ \| g(t, x) - g(t, y) \|_2 \leq L_2 \| x - y \|_H, \]  

(6)

\[ \| f(t, x) \|_H^2 + \| g(t, x) \|_2^2 \leq K(1 + \| x \|_H^2). \]  

(7)

**2.1. Existence of the mild solution of Equation (1)**

The next definition introduces the concept of solution for the stochastic system (1).
Definition 2.3. A mild solution of the integrodifferential Equation (1) on \([0, T]\) is a stochastic process \(x : [0, T] \to H\) defined on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\) such that \(x\) is \(\mathcal{F}_t\)-adapted predictable on \([0, T]\) satisfies with probability one, \(\int_0^T \| x(t) \|_H^2 \, dt < \infty\) and

\[
x(t) = R(t)x_0 + \int_0^t R(t-s)f(s, x(s)) \, ds + \int_0^t R(t-s)g(s, x(s)) \, dw(s) \quad \text{for } t \in [0, T],
\]

(8)

Let \(I = [0, T]\) and \(\Psi(I, H) = \Psi\) denote the space of \(\mathcal{F}_t\)-adapted predictable random process with values in the Hilbert space \(H\) satisfying \(\sup_{t \in I} E \| x(t) \|_H^2 < \infty\).

We define the following norm on \(\Psi\) by \(\| x \| = \sup_{t \in I} \sqrt{E \| x(t) \|_H^2}\) is a norm. Then, we have

Lemma 2.3 (Dieye et al., 2016c) \((\Psi, \| \cdot \|)\) is a Banach space.

Theorem 2.4. If hypotheses (A1), (A2) and (A4) hold, then for each initial datum \(x_0\) \(\mathcal{F}_0\)-measurable \(X\)-valued square-integrable random variable, the integrodifferential Equation (1) has a unique mild solution on \([0, T]\).

Proof. Let \(T > 0\). We define for \(x, y \in \Psi\) the following applications on \(\Psi \times \Psi\) by

\[
\delta_t(x, y) := \sup_{0 \leq s < t} E \| x(s) - y(s) \|_H^2,
\]

(9)

\[
d(x, y) := \sqrt{\delta_t(x, y)}.
\]

(10)

Since \(\sup_{t \in I} \sqrt{E \| x(t) \|_H^2} = \sqrt{\sup_{t \in I} E \| x(t) \|_H}\), then \(d(x, y) = |x - y|\), therefore \((\Psi, d)\) becomes a complete metric space. Now, we define the following map:

\[
(\mathcal{G}x)(t) = R(t)x_0 + \int_0^t R(t-s)f(s, x(s)) \, ds + \int_0^t R(t-s)g(s, x(s)) \, dw(s) \quad \text{for } t \in [0, T].
\]

(11)

Note that a fixed point of \(\mathcal{G}\) is a mild solution of Equation (1). Using the same arguments developed in Dieye et al. (2016c), we obtain that \(\mathcal{G}\) applied \(\Psi\) to itself. Moreover, we have the following estimation:

\[
E \| (\mathcal{G}x)(t) - (\mathcal{G}y)(t) \|^2 \leq C(t) \int_0^t \delta_s(x, y) \, ds.
\]

(12)

where \(C(t) = 2(M_T L_1)^2 t + 2(M L_2)^2\) with \(M_T = \sup_{t \in [0, T]} \| R(t) \|_{L(H)}\). Taking the supremum over \([0, t]\), we get that

\[
\delta_t(\mathcal{G}x, \mathcal{G}y) \leq C(T) \int_0^t \delta_s(x, y) \, ds.
\]

(13)

By iterative process involving repeated substitution of the expression (13) into itself, we obtain after \(n\) iterations the following inequality:

\[
\delta_t(\mathcal{G}^n x, \mathcal{G}^n y) \leq (C^t)^n/n! \delta_t(x, y)
\]

(14)

where \(C = C(T)\) and \(\mathcal{G}^n\) denotes the \(n\)-fold composition of the operator \(\mathcal{G}\). Hence, by taking \(t = T\) the above inequality gives that

\[
d(\mathcal{G}^n x, \mathcal{G}^n y) \leq \sqrt{(CT)^n/n!} d(x, y).
\]

(15)

The constants \(C\) and \(T\) being finite, then for \(n\) large enough: \(0 < (CT)^n/n! < 1\) and hence the \(n\)-th iterate \(\mathcal{G}^n\) of the operator \(\mathcal{G}\) is a contraction on the metric space \((\Psi(I, H), d)\). Since this is
a complete metric space it follows from Banach fixed point Theorem that for $n$ sufficiently large, $\mathcal{G}^n$ has a unique fixed point, that is a unique fixed point also of $\mathcal{G}$. We deduce that the mild solution exists on $[0, T]$, this is true for any $T > 0$ which means, we have a global existence.

3. Exponential stability of the mild solutions Equation (1)
We are mainly interested in the stability properties of the mild solutions, we consider the following equation:

$$x(t) = R(t)x_0 + \int_0^t R(t-s)f(s, x(s))ds + \int_0^t R(t-s)g(s, x(s))dw(s) \quad \text{for } t \geq 0$$

(16)

instead of Equation (1).

3.1. Exponential stability in the $p$-th mean
In the next, we discuss the exponential asymptotic stability in the $p$-th mean of mild solutions Equation (1). From now on, let $x(t) = x(t, x_0)$ denote the solution of Equation (16) with an initial value $x_0$ random variable and we always assume that $x_0$ is $\mathcal{F}_0$ measurable with $E \| x_0 \|_{\mathcal{H}}^p < \infty$ ($p \geq 2$) and $x_0$ is independent of $w(t)$.

**Definition 3.1.** Let $p \geq 2$ be an integer. The mild solution $x(t, x_0)$ of Equation (1) is said to be globally exponentially asymptotically stable in the $p$-th mean if there exist $\rho > 0$ and $L \geq 1$ such that, for any mild solution of Equation (1), $y(t, y_0)$ corresponding to an initial value $y_0$ with $E \| y_0 \|_{\mathcal{H}}^p < \infty$, the following inequality holds:

$$E \| x(t, x_0) - y(t, y_0) \|_{\mathcal{H}}^p \leq Le^{-\rho t}E \| x_0 - y_0 \|_{\mathcal{H}}^p \quad \text{for } t \geq 0.$$ 

(17)

**Theorem 3.1.** Let $p \geq 2$ be an integer and let $x(t, x_0)$ and $y(t, y_0)$ be solutions of Equation (16) with initial values $x_0$ and $y_0$ respectively. Supposes that (A1) – (A4) hold true. Then, the following inequality holds:

$$E \| x(t, x_0) - y(t, y_0) \|_{\mathcal{H}}^p \leq \beta e^{-\alpha p t}E \| x_0 - y_0 \|_{\mathcal{H}}^p \quad \text{for } t \geq 0$$

(18)

where

$$\alpha = 3^{p-1}M^p(\sigma_1 + \sigma_2), \quad \beta = 3^{p-1}M^p, \quad \sigma_1 = (1/p)^{p-1}L_{1,2}^p.$$

$$\sigma_2 = c_p L_{1,2}^p \left( \frac{p - 2}{2(p - 1)} \right)^{(p-2)/2} \text{ and } c_p = \left( \frac{p(p - 1)}{2} \right)^{p/2}.$$

**Proof.** Let $x$ and $y$ be solutions of Equation (16) with initial values $x_0$ and $y_0$ respectively. Then, we have

$$x(t) = R(t)x_0 + \int_0^t R(t-s)f(s, x(s))ds + \int_0^t R(t-s)g(s, x(s))dw(s) \quad \text{for } t \geq 0,$$

and

$$y(t) = R(t)y_0 + \int_0^t R(t-s)f(s, y(s))ds + \int_0^t R(t-s)g(s, y(s))dw(s) \quad \text{for } t \geq 0.$$

Thus it follows that

$$E \| x(t) - y(t) \|_{\mathcal{H}}^p \leq 3^{p-1}E \| R(t)x_0 - y_0 \|_{\mathcal{H}}^p$$

$$+ 3^{p-1}E \left[ \int_0^t R(t-s)\left| f(s, x(s)) - f(s, y(s)) \right| dw(s) \right]_{\mathcal{H}}^p$$
\[ + 3^{P-1}E \left\| \int_0^t R(t-s)g(s,x(s)) - g(s,y(s))dw(s) \right\|^p \]
\[ \leq \sum_{j=1}^n \xi_j(t). \]  \hfill (19)

Now, we compute the terms on the right-hand side of the above inequality. From assumption (A3), we have
\[
\xi_1(t) \leq 3^{P-1}E \left\| R(t)|x_0 - y_0| \right\|^p \\
\leq 3^{P-1}(Me^{-\tau t})E \left\| x_0 - y_0 \right\|^p \\
\leq 3^{P-1}M^p e^{-\tau t}E \left\| x_0 - y_0 \right\|^p, \]  \hfill (20)

From the assumptions (A3) and (A3), we obtain
\[
\xi_2(t) \leq 3^{P-1}E \left\| \int_0^t R(t-s)|f(s,x(s)) - f(s,y(s))|ds \right\|^p \\
\leq 3^{P-1}E \left( \int_0^t \left\| R(t-s)|f(s,x(s)) - f(s,y(s))| \right\|_H ds \right)^p \\
\leq 3^{P-1}E \left( \int_0^t Me^{-\gamma(t-s)} \left\| f(s,x(s)) - f(s,y(s)) \right\|_H ds \right)^p \\
\leq 3^{P-1}M^p \left( \int_0^t e^{-\gamma(t-s)} \left\| f(s,x(s)) - f(s,y(s)) \right\|_H ds \right)^p \\
\leq 3^{P-1}M^p \left( \int_0^t e^{-\gamma(t-s)} ds \right)^{P-1} \left( E \int_0^t e^{-\gamma(t-s)} \left\| f(s,x(s)) - f(s,y(s)) \right\|^p_H ds \right)^\frac{1}{p} \\
\leq 3^{P-1}M^p \left( \int_0^t e^{-\gamma(t-s)} ds \right)^{P-1} E \left( \int_0^t e^{-\gamma(t-s)} \left\| x(s)) - y(s) \right\|^p_H ds \right)^\frac{1}{p} \\
\leq 3^{P-1}M^p \left( \frac{1}{\gamma} \right)^{P-1} L_1^p E \int_0^t e^{-\gamma(t-s)} \left\| x(s)) - y(s) \right\|^p_H ds \\
\leq 3^{P-1}M^p \sigma_1 \int_0^t e^{-\gamma(t-s)} E \left\| x(s)) - y(s) \right\|^p_H ds, \]  \hfill (21)

where \( \sigma_1 = (1/\gamma)^{P-1} L_1^p \). To estimate \( \xi_3(t) \), we recall the following result

**Lemma 3.2. (Da Prato & Zabczyk, 1992a)** For any \( r \geq 1 \) and for an \( L_2^0 \)-predictable process \( \Phi(t) \) we have the following inequality
\[
\sup_{u \in [0,T]} E \left\| \int_0^t \Phi(s) dw(s) \right\|_{H}^{2r} \leq \left( r(2r - 1) \right)^r \left( \int_0^t E \left\| \Phi(s) \right\|_{L_2^0}^{2r} ds \right)^{1/r} \text{ for } t \in [0,T]. \]  \hfill (22)
By using Hölder’s inequality we obtain that

$$
\xi_4(t) \leq 3^{p-1} E \left\| \int_0^t R(t-s) [g(s,x(s)) - g(s,y(s))] dw(s) \right\|^p
$$

$$
\leq 3^{p-1} \sup_{u \in [0,t]} E \left\| \int_0^u R(t-s) [g(s,x(s)) - g(s,y(s))] dw(s) \right\|^{2(p/2)}
$$

$$
\leq 3^{p-1} c_p \left( \int_0^t \left( E \left\| R(t-s) [g(s,x(s)) - g(s,y(s))] \right\|_2^{p/2} ds \right)^{p/2}
$$

$$
\leq 3^{p-1} c_p M^p L^p_2 \left( \int_0^t \left( e^{-\gamma(t-s)} E \left\| x(s) - y(s) \right\|_H^{p/2} ds \right)^{p/2}
$$

$$
\leq 3^{p-1} c_p M^p L^p_2 \left( \int_0^t e^{-\gamma(t-s)} E \left\| x(s) - y(s) \right\|_H^{p/2} ds \right)^{p/2}
$$

$$
\leq 3^{p-1} c_p M^p L^p_2 \left[ \int_0^t e^{-\gamma(t-s)} ds \right]^{p/2} \times
$$

$$
\times \left[ \int_0^t \left\{ E \left\| x(s) - y(s) \right\|_H^{p/2} \right\}^{p/2} ds \right]^{p/2}
$$

$$
\leq 3^{p-1} c_p M^p \sigma_2 \int_0^t e^{-\gamma(t-s)} E \left\| x(s) - y(s) \right\|_H^{p/2} ds
$$

$$
\leq 3^{p-1} M^p \sigma_2 \int_0^t e^{-\gamma(t-s)} E \left\| x(s) - y(s) \right\|_H^{p/2} ds.
$$

where \( \sigma_2 = \frac{C_0 \mu P^p (2-\gamma)}{(2-\gamma)^2} \left( P^{p-1} \right)^{p/2} \) and \( c_p = \left( \frac{P^{p-1}}{2} \right)^{p/2} \). We remark if \( p = 2 \), the inequality (23) holds with convention \( 0^0 := 1 \). From inequalities (20), (21) and (23), one can see that the inequality (19) becomes

$$
E \left\| x(t) - y(t) \right\|_H^p \leq 3^{p-1} M^p e^{-\gamma t} E \left\| x_0 - y_0 \right\|_H^p
$$

$$
+ 3^{p-1} M^p (\sigma_1 + \sigma_2) \left( \int_0^t e^{-\gamma(t-s)} E \left\| x(u) - y(u) \right\|_H^p du \right) \text{ for } t \geq 0.
$$

that is,

$$
e^{-\gamma t} E \left\| x(t) - y(t) \right\|_H^p \leq 3^{p-1} M^p e^{-(\gamma-\gamma t)} E \left\| x_0 - y_0 \right\|_H^p
$$

$$
+ 3^{p-1} M^p (\sigma_1 + \sigma_2) \left( \int_0^t e^{-\gamma t} E \left\| x(u) - y(u) \right\|_H^p du \right) \text{ for } t \geq 0.
$$

Hence Gronwall’s inequality yields

$$
e^{-\gamma t} E \left\| x(t) - y(t) \right\|_H^p \leq 3^{p-1} M^p e^{-(\gamma-\gamma t)} E \left\| x_0 - y_0 \right\|_H^p \exp \left\{ 3^{p-1} M^p (\sigma_1 + \sigma_2) t \right\} \text{ for } t \geq 0,
$$

that is,
\[ E \| x(t) - y(t) \|_H^p \leq \beta E \| x_0 - y_0 \|_H^p e^{-(\gamma - \alpha)t} \quad \text{for } t \geq 0, \]  

(26)

which consequently completes the proof.

Consequently, we have the following result as corollary.

**Corollary 3.3.** Suppose that all hypotheses of Theorem 3.1 hold and let \( \gamma > \alpha/\beta \). Then mild solutions of Equation (1) are globally exponentially asymptotically stable in the p-th mean.

**Corollary 3.4.** Assume that all hypotheses of Theorem 3.1 hold, \( f(t,0) = 0 \) and \( g(t,0) = 0 \) and \( \gamma > \alpha/\beta \). Then

\[ E \| x(t) \|_H^p \leq E \| x_0 \|_H^p e^{-(\gamma - \alpha)t} \quad \text{for } t \geq 0. \]  

(27)

### 3.2. Almost sure asymptotic stability

In this subsection, we state the pathwise asymptotic stability for the mild solutions of equation (1). Due to the properties of the stochastic convolution, we study the case \( p > 2 \). At first, we need the following lemma

**Lemma 3.5.** (Dieye et al., 2016c) Suppose that the hypothesis (A1), (A2) and (A3) are satisfied. Let \( \psi : [0, +\infty) \to \mathcal{L}_2^0 \) be a predictable, \( \mathcal{F}_t \)-adapted process with \( \int_0^t E \| \psi(s) \|_2^p \, ds < +\infty \) for some integer \( p > 2 \) and any \( t \geq 0 \). Then there exists a constant \( \kappa_p > 0 \) such that for any \( n \in \mathbb{N} \), the following holds

\[ E \sup_{n \leq t \leq n+1} \left[ \int_n^t R(t-s)\psi(s)dw(s) \right]^p \leq \kappa_p t^{p \gamma} \int_n^{n+1} E \| \psi(s) \|_{L_2}^p \, ds. \]  

(28)

**Theorem 3.6.** Suppose that (A1) – (A4) hold. Let \( p > 2 \) be an integer, \( x(t,x_0) \) and \( y(t,y_0) \) be solutions of Equation (16) with initial values \( x_0 \) and \( y_0 \) respectively. If \( \gamma > \alpha/\beta \) then there exists \( T(\omega) > 0 \) such that for \( t \geq T(\omega) \), we have

\[ \| x(t,x_0) - y(t,y_0) \|_H^p \leq E \| x_0 - y_0 \|_H^p e^{-(\gamma - \alpha)t/2} \quad \text{P a.s.} \]  

(29)

**Proof.** Let \( n \) be a sufficiently large integer and \( I_n \) denote the interval \([n, n+1]\). Then for \( t \in I_n \), we have

\[ x(t) = x(t,x_0) = R(t-n)x(n) + \int_n^t R(t-s)f(s,x(s))ds + \int_n^t R(t-s)g(s,x(s))dw(s). \]

\[ y(t) = y(t,y_0) = R(t-n)y(n) + \int_n^t R(t-s)f(s,y(s))ds + \int_n^t R(t-s)g(s,y(s))dw(s). \]

It follows that

\[ \| x(t) - y(t) \|_H \leq \| R(t-n)(x(n) - y(n)) \|_H + \left\| \int_n^t R(t-n)[f(s,x(s)) - f(s,y(s))]ds \right\|_H \]

\[ + \left\| \int_n^t R(t-n)[g(s,x(s)) - g(s,y(s))]dw(s) \right\|_H. \]

Thus, for any fixed \( \epsilon > 0 \), we obtain that
\[
P \left[ \sup_{t \in I_n} \| x(t) - y(t) \|_H > \epsilon \right] \leq P \left[ \sup_{t \in I_n} \| R(t - n) [x(n) - y(n)] \|_H > \epsilon / 3 \right] \\
+ P \left[ \sup_{t \in I_n} \left\| \int_n^t R(t - s) [f(s, x(s)) - f(s, y(s))] ds \right\|_H > \epsilon / 3 \right] \\
+ P \left[ \sup_{t \in I_n} \left\| \int_n^t R(t - s) [g(s, x(s)) - g(s, y(s))] dw(s) \right\|_H > \epsilon / 3 \right] \\
\leq (3/\epsilon) P \left[ \sup_{t \in I_n} \| R(t - n) [x(n) - y(n)] \|_H^p \right] \\
+ (3/\epsilon) P \left[ \sup_{t \in I_n} \left\| \int_n^t R(t - s) [f(s, x(s)) - f(s, y(s))] ds \right\|_H^p \right] \\
+ (3/\epsilon) P \left[ \sup_{t \in I_n} \left\| \int_n^t R(t - s) [g(s, x(s)) - g(s, y(s))] dw(s) \right\|_H^p \right] \\
:= \Gamma_1 + \Gamma_2 + \Gamma_3
\]

By Theorem 3.1 and Holder’s inequality, we have

\[
\Gamma_1 = (3/\epsilon) P \left[ \sup_{t \in I_n} \| R(t - n) [x(n) - y(n)] \|_H^p \right] \\
\leq (3/\epsilon) P \left[ \sup_{t \in I_n} M^p e^{-\beta (t-n)} \| x(n) - y(n) \|_H^p \right] \\
\leq (3M/\epsilon)^p P \| x_0 - y_0 \|_H^p e^{-\beta (t-n)}.
\]

Moreover,

\[
\Gamma_2 = (3/\epsilon) P \left[ \sup_{t \in I_n} \left\| \int_n^t R(t - s) [f(s, x(s)) - f(s, y(s))] ds \right\|_H^p \right] \\
\leq (3/\epsilon) P \left[ \sup_{t \in I_n} \left( \int_n^t \| R(t - s) [f(s, x(s)) - f(s, y(s))] \|_H ds \right)^p \right] \\
\leq (3ML_1/\epsilon)^p P \left[ \sup_{t \in I_n} \left( \int_n^t e^{-\gamma (t-s)} \| x(s) - y(s) \|_H ds \right)^p \right] \\
\leq (3ML_1/\epsilon)^p P \left[ \sup_{t \in I_n} \left( \int_n^t \| x(s) - y(s) \|_H ds \right)^p \right] \\
\leq (3ML_1/\epsilon)^p P \left[ \sup_{t \in I_n} \int_n^t \| x(s) - y(s) \|_H^p ds \right] \\
\leq (3ML_1/\epsilon)^p \sup_{t \in I_n} \int_n^t \| x(s) - y(s) \|_H^p ds
\]
\[ \leq (3ML_1/e)^{p} \int_{n}^{n+1} E \| x(s) - y(s) \|_{H}^p \, ds \]
\[ \leq (3ML_1/e)^{p} \int_{n}^{n+1} \beta E \| x_0 - y_0 \|_{H}^p e^{-(\gamma - a)s} \, ds \]
\[ \leq (3ML_1/e)^{p}(\beta/(\gamma - a)) E \| x_0 - y_0 \|_{H}^p \left( e^{-(\gamma - a)n} - e^{-(\gamma - a)(n+1)} \right) \]
\[ \leq (3ML_1/e)^{p}(\beta/(\gamma - a)) E \| x_0 - y_0 \|_{H}^p e^{-(\gamma - a)n} \]

Using Lemma 3.5, we get that
\[ \Gamma_1 = (\epsilon/3)^{p} E \left[ \sup_{t \in I_n} \left\| \int R(t - n)g(s, x(s)) - g(s, y(s)) \right\|_{H}^p \right] \]
\[ \leq (3ML_2/e)^{p} k_p \int_{n}^{n+1} E \| x(s) - y(s) \|_{H}^p \, ds \]
\[ \leq (3ML_2/e)^{p}(\beta \kappa_p/(\gamma - a)) E \| x_0 - y_0 \|_{H}^p e^{-(\gamma - a)n}. \]

It follows that
\[ P \left[ \sup_{t \in I_n} \| x(t) - y(t) \|_{H} > \epsilon \right] \leq (m/e^p) E \| x_0 - y_0 \|_{H}^p e^{-(\gamma - a)n}, \tag{30} \]
where \( m = (3M)^p \beta + (3ML_1)^p(\beta/(\gamma - a)) + (3ML_2)^p(\beta \kappa_p/(\gamma - a)). \)

Now, for each integer \( n \), we set \( \epsilon_n = \left( E \| x_0 - y_0 \|_{H}^p e^{-(\gamma - a)(n+1)/2(2p)} \right)^{1/p} \). Then, it follows
\[ P \left[ \sup_{t \in I_n} \| x(t) - y(t) \|_{H} > \left( E \| x_0 - y_0 \|_{H}^p \right)^{1/p} e^{-(\gamma - a)(n+1)/2(2p)} \right] \leq m\epsilon^p e^{-(\gamma - a)n/2}, \tag{31} \]
where \( \delta = e^{-(\gamma - a)/2}. \)

We consider the subset of \( \Omega \) given by \( E_n = \left( \sup_{t \in I_n} \| x(t) - y(t) \| > \left( E \| x_0 - y_0 \|_{H}^p \right)^{1/p} e^{-(\gamma - a)(n+1)/2(2p)} \right), \quad n \geq 1. \)

By using inequality (31) and applying Borel-Cantelli’s Lemma it follows that
\[ P(\lim \sup_{n} E_n) = 0. \]

Then, the set of all \( \omega \) such that there exists an infinite number index \( n \) with \( \omega \in E_n \) is a negligible set. Hence, for almost sure, there exists infinite number index \( n(\omega) \) such that \( \omega \notin E_{n(\omega)} \), i.e
\[ \sup_{t \in I_{n(\omega)}} \| x(t) - y(t) \| \leq \left( E \| x_0 - y_0 \|_{H}^p \right)^{1/p} e^{-(\gamma - a)(n(\omega)+1)/2(2p)} \quad P \, a.s.. \]

Let \( T(\omega) \) be the lower bound of index \( n(\omega) \) almost surely.

For any \( t \geq T(\omega) \), there exists a positive integer \( n_0(\omega) \) such that \( t \in I_{n_0(\omega)} \). For \( t \in I_{n_0(\omega)} \), we have
\[ \sup_{t \in I_{n_0(\omega)}} \| x(t) - y(t) \| \leq \left( E \| x_0 - y_0 \|_{H}^p \right)^{1/p} e^{-(\gamma - a)(n_0(\omega)+1)/2(2p)} \quad P \, a.s.. \]

It follows that
\[ \| x(t) - y(t) \| \leq \left( E \| x_0 - y_0 \|_{H}^p \right)^{1/p} e^{-(\gamma - a)t/2(2p)} \quad P \, a.s.. \]
Hence, for $t \geq T(\omega)$, we have
\[
\|x(t) - y(t)\|^p \leq E \|x_0 - y_0\|^p e^{-\|p\| t/2} \quad \text{P.a.s.}
\] (32)

**Corollary 3.7.** Under the hypotheses of Theorem 3.6, if $f(t, 0) = 0, g(t, 0) = 0$, then the zero solution is almost sure asymptotically stable.

### 4. Application

Consider the following stochastic partial functional integrodifferential equation:

\[
\begin{align*}
\frac{\partial}{\partial t} u(t, \xi) & = \frac{\partial^2}{\partial \xi^2} u(t, \xi) + \int_0^t b(t - s) \frac{\partial^2}{\partial s^2} u(s, \xi) ds + k^2(u(t, \xi)) + \sqrt{2/\pi} \sin(\theta) k^2 (u(t, \xi) (\xi)) \frac{d\omega(t)}{dt} \quad \text{for } t \geq 0 \\
u(t, 0) & = u(t, \pi) = 0 \quad \text{for } t \geq 0, \xi \in [0, \pi] \\
u(0, \xi) & = u_0(\xi) \quad \text{for } \xi \in [0, \pi].
\end{align*}
\] (33)

where $\omega(t)$ denotes the standard $\mathbb{R}$-valued Brownian motion, $b : \mathbb{R} \to \mathbb{R}$ is continuous.

Let $H = L_2(0, \pi)$ with the norm $\| \cdot \|_H$ and $e_n := \sqrt{2/\pi} \sin(n \cdot), n = 1, 2, 3, \ldots$ denote the completed orthonormal basis in $H$. $k^1, k^2 : \mathbb{R} \to \mathbb{R}$ are bounded Lipschitz functions and $u_0 : (0, \pi) \to \mathbb{R}$ is a given continuous function.

Let $(\Omega, \mathcal{F}, \mathcal{F}_{t \geq 0}, \mathbb{P})$ be a complete filtered probability space where the Brownian motion $w$ is defined. Let $w(t) := \sum_{n=1}^{\infty} \sqrt{\lambda_n} \mathbb{B}_n(t) e_n$ ($\mathbb{B}_1 = \omega, \lambda_1 = 1, \lambda_n = 0, n > 1$), where $\mathbb{B}_n(t)$ are one-dimensional standard Brownian motion mutually independent of $(\Omega, \mathcal{F}, \mathcal{F}_1)_{t \geq 0}, \mathbb{P})$. Let $U = H, (\mathcal{Q} e_n = \lambda_n e_n)$ Then $w$ is a $u = X$ valued $\mathcal{Q}$-Brownian motion.

Define $A : D(A) \subset H \to H$ by $A = \partial^2 / \partial \xi^2$, with domain $D(A) = H^2(0, \pi) \cap H^1_0(0, \pi)$.

Then $Av = -\sum_{n=1}^{\infty} n^2 \langle v, e_n \rangle e_n, v \in D(A)$, where $e_n, n = 1, 2, 3, \ldots$, is, also the orthonormal set of eigenvectors of $A$. It is well known that $A$ is the infinitesimal generator of a strongly continuous semigroup on $H (S(t))_{t \geq 0}$, thus, (A1) is true. Moreover, $\| S(t) \| \leq e^{-t} \leq 1 = M, t \geq 0$.

Let $Y : D(A) \subset H \to H$ be the operator defined by $Y(t)z = b(t)Az$ for $t \geq 0$ and $z \in D(A)$.

Let
\[
x(t)(\xi) = u(t, \xi), \quad 0 \leq t \leq T, \xi \in [0, \pi] \\
f(t, x)(\xi) = k^1(x(\xi)), \quad 0 \leq t \leq T, \xi \in [0, \pi] \\
(g(t, x))u(\xi) = k^2(u(\xi))u(\xi), \quad (t, \xi, y) \in [0, T] \times [0, \pi] \times U \\
x_0 = u_0.
\]

Then, Equation (33) takes the following form

\[
\begin{align*}
\frac{dx(t)}{dt} & = A x(t) + \int_0^t Y(t - s)x(s)ds + f(t, x(t)) + g(t, x(t)) \frac{d\omega(t)}{dt} \quad \text{for } t \geq 0, \\
x(0) & = x_0.
\end{align*}
\] (34)
Clearly $f$ and $g$ satisfy the assumption (A4) with $L_1 = L_{k_1}$ and $L_2 = L_{k_2}$ where $L_{k_1}$ and $L_{k_2}$ are the Lipschitz constants of the functions $k^1$ and $k^2$, respectively.

Moreover, if $b$ is bounded and $C^1$ function such that $b'$ is bounded and uniformly continuous, then (A1) and (A2) are satisfied, and hence, by Theorem 2.1, Equation (2) has a resolvent operator $(R(t))_{t \geq 0}$ on $H$.

**Proposition 4.1.** [Dieye et al., 2016c] Suppose that $b$ is bounded and $C^1$ function such that $b'$ is bounded and uniformly continuous and $b(t) \leq \frac{1}{2}e^{-\beta t}$ for all $t \geq 0$ where $\beta > a > 1$. Then the resolvent operator of the abstract form of Equation (34) decays exponentially to zero. Specifically $\| R(t) \| \leq e^{-\gamma t}$ where $\gamma = 1 - 1/a$.

In the next, we assume that $b$ is bounded and $C^1$ function such that $b'$ is bounded and uniformly continuous and $b(t) \leq \frac{1}{2}e^{-\beta t}$ for all $t \geq 0$ where $\beta > a > 1$.

Therefore, by Theorem 2.4 the existence and uniqueness of the mild solution of the stochastic partial functional integrodifferential Equation (33) is true.

Considering the case $p = 2$, we have the following constants:

\[
c_p = \left( \frac{p(p - 1)}{2} \right)^{p/2} = \left( \frac{2(2 - 1)}{2} \right)^{2/2} = 1, \tag{35}\]

\[
\sigma_1 := (1/\gamma)^{p-1}L_1^p = \gamma^{-1}L_1^2, \tag{36}\]

\[
\sigma_2 := c_p L_2^p \left( \frac{p - 2}{2(p - 1)\gamma} \right)^{(p-2)/2} = L_2^2. \tag{37}\]

Hence $\alpha := 3^{p-1}M^p(\sigma_1 + \sigma_2) = 9(\gamma^{-1}L_1^2 + L_2^2)$. Then, by Corollary 3.3, we have

**Proposition 4.2.** The system (33) is mean square globally exponentially asymptotically stable provided that:

\[
\left( \frac{L_1^2}{\gamma^2} + \frac{L_2^2}{\gamma} \right) < (2/9). \tag{38}\]

For $p = 4$, we have

\[
c_p = \left( \frac{p(p - 1)}{2} \right)^{p/2} = \left( \frac{4(4 - 1)}{2} \right)^{4/2} = 36, \tag{39}\]

\[
\sigma_1 := (1/\gamma)^{p-1}L_1^p = \gamma^{-1}L_1^4, \tag{40}\]

\[
\sigma_2 := c_p L_2^p \left( \frac{p - 2}{2(p - 1)\gamma} \right)^{(p-2)/2} = \frac{36L_2^4}{3\gamma} = \frac{12L_2^4}{\gamma}. \tag{41}\]

Therefore $\alpha = 3^{p-1}M^p(\sigma_1 + \sigma_2) = 3^4 - 1 \left( \frac{L_1^4}{\gamma^2} + \frac{12L_2^4}{\gamma} \right) = 27 \left( \frac{L_1^4}{\gamma^2} + \frac{12L_2^4}{\gamma} \right)$.

Moreover, by Theorem 3.6, we have also that

**Proposition 4.3** If

\[
\left( \frac{L_1^4}{\gamma^2} + \frac{12L_2^4}{\gamma^2} \right) \left( \frac{4}{27} \Leftrightarrow \gamma \right) \alpha/4. \tag{42}\]
Then the mild solution of the system (33) is exponentially asymptotically stable. If \( p = 4 \) the pathwise exponential stability is true.

**Theorem 4.1.** If \( k^1(0) = k^2(0) = 0 \) and condition (39) holds, then zero solutions of the stochastic partial functional integrodifferential Equation (33), is almost sure asymptotically stable.

**5. Conclusion**

In this paper, we have initiated a study on stochastic neutral partial functional differential equations in a real separable Hilbert space. By using stochastic convolution, existence and uniqueness have been discussed. Further, the exponential stability of the moments of the mild solution as well its sample paths have been studied.

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