Geometric inequality of warped product semi-slant submanifolds of locally product Riemannian manifolds

Rifaqat Ali\textsuperscript{1*} and Wan Ainun Mior Othman\textsuperscript{2}

Abstract: In the present article, we derive an inequality in terms of slant immersions and well define warping function for the squared norm of second fundamental form for warped product semi-slant submanifold in a locally product Riemannian manifold. Moreover, the equality cases are verified and generalized the inequality for semi-invariant warped products in locally Riemannain product manifold.

Subjects: Mathematical Analysis; Pure Mathematics; Engineering Mathematics; Mathematics

Keywords: Mean curvature; warped products; Riemannian manifolds; semi-slant immersions

1. Introduction

The notion of warped product manifolds plays very important roles not only in differential geometry but also in general relativity theory in physics. For example, Robertson-Walker space-times, asymptotically flat spacetime, Schwarzschild spacetime, and Reissner-Nordstrom spacetime are warped product manifolds (Hiepko, 1979). The geometry of warped products has a crucial role in differential geometry, as well as physical sciences. Bishop and O'Neill (1969) discovered the concept of warped product manifolds to derive an example of Riemannian manifolds of negative curvature, such manifolds are natural generalizations of Riemannian products manifolds. Therefore, many geometers are studied in Ali and Luarian (2017), Ali, Othman, and Ozel (2015), Ali and Ozel (2017), Ali, Uddin, and Othman (2017), Al-Solamy and Khan (2012), Al-Solamy, Khan,
and Uddin (2017), Atceken (2008, 2013), Chen (2001), Sahin (2006a, 2006b, 2006c). It is interesting to see that there exist no warped product semi-slant submanifolds of the forms $M = M_0 \times_{f} M_1$ and $M = M_1 \times_{f} M_2$ in a Kaehler manifold $M$ such that $M_1$ and $M_2$ are holomorphic and slant submanifolds, respectively (see Sahin, 2006b). While, Atceken (see examples 3.1 (Atceken, 2008)) has given an example on the existence of warped product semi-slant submanifold of the form $M = M_0 \times_{f} M_1$ in a locally product Riemannian manifold such that $M_1$ and $M_2$ are invariant and slant submanifolds, respectively. Hence, the geometry of warped product submanifolds in a locally product Riemannian manifold is different from the geometry of warped product submanifolds in Kaehler manifold. Therefore, we consider such a warped product semi-slant submanifold as mixed totally geodesic of locally product Riemannian manifold and obtain a geometric inequality for the length of the second fundamental form in terms of slant immersion and warping functions.

2. Preliminaries

Assume that $M$ be a manifold of dimension $m$ with a tensor field of such that

$$F^2 = I(F \neq \pm I),$$

where $F$ is a one-one tensor field and $I$ represent the identity transformation. Thus, $M$ is an almost product manifold with almost product structure $F$. If an almost product manifold $M$ admits a Riemannian metric $g$ satisfying

$$g(FU, FU) = g(U, V), \quad g(FU, V) = g(U, FV),$$

2010 Mathematics Subject Classification. 53C40 Primary 53C20 53C42 secondary.

Key words and phrases. Mean curvature, warped products, Riemannian manifolds, semi-slant immersions. For any $U, V \in \Gamma(TM)$, where $\Gamma(TM)$ denotes the set of all vector fields of $M$ then $M$ is said to be an almost product Riemannian metric manifold. Denote $\nabla$ the Levi-Civita connection on $M$ with respect to $g$. If $(\nabla_U F)V = 0$, for all $U, V \in \Gamma(TM)$, then $(M, g)$ is a locally product Riemannian manifold with Riemannian metric $g$ (see Sahin, 2006a).

Let $M$ be a submanifold of locally product Riemannian manifold $\bar{M}$ with an induced metric $g$. If $\nabla^\perp$ and $\nabla$ are induced Riemannian connections on normal bundle $T^\perp M$ and tangent bundle $TM$ and of $M$, respectively, then Gauss and Weingarten formulas are given by

\begin{align}
(i) \quad & \nabla_U V = \nabla_U V + h(U, V), \\
(ii) \quad & \nabla_U N = -A_N U + \nabla_U N,
\end{align}

for each $U, V \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, where $h$ and $A_N$ are the second fundamental form and shape operator for an immersion $M$ into $\bar{M}$. They are correlated as

$$g(h(U, V), N) = g(A_N U, V).$$

For any $X \in \Gamma(TM)$, we can write

\begin{align}
(i) \quad & FU = PU + \omega U, \\
(ii) \quad & FN = tN + fN,
\end{align}

where $PU(tN)$ and $\omega U(fN)$ are tangential and normal components of $FU(FN)$, respectively. The covariant derivatives of the endomorphism $F$ as

$$\nabla_U FV = \nabla_U FV - F \nabla_U V, \forall U, V \in \Gamma(TM).$$

A submanifold $M$ of a locally product Riemannian manifold $\bar{M}$ is said to be totally umbilical (and totally geodesic respectively) if
\[ h(U, V) = g(U, V)H, \quad h(U, V) = 0. \tag{2.7} \]

for all \( U, V \in \Gamma(TM) \). Then \( H \) is a mean curvature vector of \( M \) given by \( H = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i) \), where \( n \) is the dimension of \( M \) and \( \{e_1, e_2, \ldots, e_n\} \) is a local orthonormal frame of the tangent vector space \( TM \). Furthermore, if \( H = 0 \), then \( M \) is minimal in \( M \).

**Definition 2.1.** A submanifold \( M \) of a locally product Riemannian manifold \( \tilde{M} \), then for each non zero vector \( U \) tangent to \( M \) at a point \( p \), the angle \( \vartheta(U) \) between \( FU \) and \( T_pM \) is called a Wirtinger angle of \( U \). Hence, \( M \) is said to be a slant submanifold if the Wirtinger angle is constant and it is independent from the choice of \( U \in T_pM \) and \( p \in M \). The holomorphic and totally real submanifolds are slant submanifolds with slant angle \( \vartheta = 0 \) and \( \vartheta = \pi/2 \), respectively. A slant submanifold is said to be proper if it is neither holomorphic nor totally real. More generally, a distribution \( D \) on \( M \) is called a slant distribution if the angle \( \vartheta(X) \) between \( FX \) and \( D_x \) has same value of \( \vartheta \) for each \( x \in M \) and a non zero vector \( X \in D_x \).

Thus for a slant submanifold \( M \), a normal bundle \( T^\perp M \) can be expressed as
\[ T^\perp M = \omega(TM) \oplus \nu, \tag{2.8} \]

where \( \nu \) is an invariant normal bundle with respect to \( F \) orthogonal to \( \omega(TM) \). We recall following result for a slant submanifold of a locally product Riemannian manifold given by H. Li (cf. Li & Li, 2005).

**Theorem 2.1.** If \( M \) is a submanifold of a locally product Riemannian manifold \( \tilde{M} \), then \( M \) is a slant submanifold if and only if there exists a constant \( \lambda \in [0, 1] \) such that \( P^2 = \lambda I \). In this case, \( \theta \) is a slant angle of \( M \), and then it satisfies \( \lambda = \cos^2 \theta \).

Therefore, the following identities which are consequences from the Theorem 2.1
\[ g(PU, PV) = \cos^2 \vartheta g(U, V), \tag{2.9} \]
\[ g(\omega U, \omega V) = \sin^2 \vartheta g(U, V), \tag{2.10} \]

for any \( U, V \in \Gamma(TM) \). Now let \( \{e_1, e_2, \ldots, e_n\} \) be an orthonormal basis of the tangent space \( TM \) and \( e_i \) belonging to the orthonormal basis \( \{e_{n+1}, e_{n+2}, \ldots, e_m\} \) of the normal bundle \( T^\perp M \). Then we define
\[ h_{ij} = g(h(e_i, e_j), e_i) \text{ and } ||h||^2 = \sum_{i,j=1}^{m} g(h(e_i, e_j), h(e_i, e_j)). \tag{2.11} \]

As a consequence for a differentiable function \( \varphi : M \to \mathbb{R} \), we have
\[ ||\nabla \varphi||^2 = \sum_{i=1}^{m} (e_i(\varphi))^2, \tag{2.12} \]

where gradient \( \nabla \varphi \) is defined by \( g(\nabla \varphi, X) = X\varphi \), for any \( X \in \Gamma(TM) \).

### 3. Semi-slant submanifolds

Semi-slant submanifolds were described by Papaghiuc (1994). These submanifolds are generalizations of CR-submanifolds with slant angle \( \theta = \pi/2 \).

**Definition 3.1.** A submanifold \( M \) of an almost complex manifold \( M \) is called a semi-slant submanifold if there exist two orthogonal distributions \( D \) and \( D^0 \) such that
- \( TM = D \oplus D^0 \),
- \( D \) is holomorphic, i.e., \( F(D) \subseteq D \),
- \( D^0 \) is slant distribution with slant angle \( \theta \neq 0, \pi/2 \).
The dimensions of holomorphic distribution $\mathcal{D}$ and slant distribution $\mathcal{D}^\theta$ of a locally product Riemannian manifold $M$ are denoted by $m_1$ and $m_2$, respectively. Then $M$ is holomorphic if $m_2 = 0$ and slant if $m_1 = 0$. It is called proper semi-slant if the slant angle different from $0$ and $\pi/2$. Moreover, if $\nu$ is an invariant subspace under the endomorphism $F$ of normal bundle $T^\bot M$, then, in case of semi-slant submanifold, the normal bundle $T^\bot M$ can be decomposed as $T^\bot M = \omega\mathcal{D}^\theta \oplus \nu$. A semi-slant submanifold is said to be a mixed totally geodesic, if $h(X, Z) = 0$, for any $X \in \Gamma(\mathcal{D})$ and $Z \in \Gamma(\mathcal{D})$.

4. Warped product submanifolds with the form $M_M \times f_M T$

Let $(M_1, g_1)$ and $(M_2, g_2)$ be two Riemannian manifolds with a $f : M_1 \rightarrow (0, \infty)$, a positive differentiable function on $M_1$, we define on the product manifold $M_1 \times M_2$ with metric $g = \pi^* g_1 + (f \pi)^* g_2$, where $\pi$ and $\gamma$ are natural projections on $M_1$ and $M_2$. Under these condition the product manifold is called warped product manifold of $M_1$ and $M_2$, it is denoted by $M_1 \times f_2 M_2$ and $f$ is called warping function. So we have the following lemma

**Lemma 4.1** Let $M = M_1 \times f_1 M_2$ be a warped product manifold. Then for any $X, Y \in \Gamma(T M_1)$ and $Z, W \in \Gamma(T M_2)$, we have

(i) $\nabla_2 Y \in \Gamma(T M_1)$.

(ii) $\nabla_2 X = \nabla X Z = (X \ln f) Z$.

(iii) $\nabla_2 W = \nabla W - g(Z, W) \nabla \ln f$.

where $\nabla$ and $\nabla'$ are the Levi-Civita connections on $M_1$ and $M_2$ respectively. Thus $\nabla \ln f$ is the gradient of $\ln f$ is defined as $g(\nabla \ln f, U) = U \ln f$. If the warping function $f$ is constant, then the warped product manifold $M = M_1 \times f_2 M_2$ is called trivial, otherwise non-trivial. Furthermore, in a warped product manifold $M = M_1 \times f_1 M_2$, $M_1$ is totally geodesic and $M_2$ is totally umbilical submanifold in $M$, respectively (cf. Bishop & O'Neill, 1969). There are two types of warped product semi-slant submanifolds $M = M_0 \times f_2 M_1$ and $M = M_1 \times f_1 M_0$. For the second case, we have following non-existence theorem from Atceken (2008).

**Theorem 4.1.** Assume that $M$ is a locally Riemannian product manifold and $M$ is a submanifold of $M$. Then there exists no a warped product semi-slant submanifold $M = M_1 \times f_1 M_2$ in $M$ such that $M_1$ is an invariant submanifold and $M_2$ is a proper slant submanifold of $M$.

Now, we develop some important lemmas for first type warped product for later use in the inequality and we refer for example to see their existence, Example 4.1 in Atceken (2008).

**Lemma 4.2.** Let $M = M_0 \times f_2 M_1$ be a warped product semi-slant submanifold of a locally product Riemannian manifold $M$. Then

\[ g(h(X, FY), \omega Z) = -(Z \ln f) g(X, Y) \] \hspace{1cm} (4.1)

\[ g(h(X, FY), PZ) = -(PZ \ln f) g(X, Y), \] \hspace{1cm} (4.2)

for any $Z \in \Gamma(T M_0)$ and $X, Y \in \Gamma(T M_1)$.

**PROOF.** If $Z \in \Gamma(T M_0)$ and $X, Y \in \Gamma(T M_1)$, we have

\[ g(h(X, FX), \omega Z) = g(\nabla_X FX, \omega Z). \]

From (2.2) and (2.5) (i), we get

\[ g(h(X, FX), \omega Z) = g(F \nabla_X FX, Z) + g(\nabla_X FX, PZ). \]

From the fact that $X$ and $Z$ are orthogonal, we obtain
\[ g(h(X, FX), \omega Z) = -g(\nabla_X Z, X) + g(\nabla_X PZ, FX). \]

Then from (2.3) (i), we derive
\[ g(h(X, FX), \omega Z) = -g(\nabla_X Z, X) + g(\nabla_X PZ, FX). \]

Using Lemma 4.1 (ii), we arrive at
\[ g(h(X, FX), \omega Z) = -(Z \ln f)g(X, X) + g(X, FX)(PZ \ln f). \]

As \( X \) and \( FX \) are orthogonal to each other by the definition of \((1, 1)\) tensor field \( F \), the second term of last equation should be zero. Then we get
\[ g(h(X, FX), \omega Z) = -Z \ln f||X||^2. \]

Replacing \( X \) by \( X + Y \) in the above equation and from the property of linearity, we get the first result of lemma. Now interchanging \( Z \) by \( PZ \), we obtain
\[ g(h(X, FY), \omega PZ) = -(PZ \ln f)g(X, Y). \]

It completes the proof of the lemma. \( \square \)

**Lemma 4.3.** Let \( M = M_\theta \times M_\tau \) be a warped product semi-slant submanifold of a locally product Riemannian manifold \( M \). Then
\[ \begin{align*}
g(h(X, X), \omega PZ) &= g(h(FX, FX), \omega PZ) = (Z \ln f)\cos^2 \theta ||X||^2, \\
g(h(X, X), \omega Z) &= g(h(FX, FX), \omega Z) = (PZ \ln f)||X||^2,
\end{align*} \tag{4.3} \tag{4.4} \]
for any \( Z \in \Gamma(TM_\theta) \) and \( X \in \Gamma(TM_\tau) \).

**PROOF.** Suppose that \( X \in \Gamma(TM_\theta) \) and (2.5) (i), we have
\[ g(h(X, X), \omega PZ) = g(\nabla_X X, FPZ) - g(\nabla_X X, P^2Z). \]

for \( Z \in \Gamma(TM_\tau) \). Then from Theorem 2.1, implies that
\[ g(h(X, X), \omega PZ) = g(\nabla_X FX, PZ) - \cos^2 \theta g(\nabla_X X, Z). \]

Since \( FX \) and \( PZ \) are orthogonal then, we obtain
\[ g(h(X, X), \omega PZ) = -g(\nabla_X PZ, FX) + \cos^2 \theta g(\nabla_X X, Z). \]

From Lemma 4.1 (ii), we arrive at
\[ g(h(X, X), \omega PZ) = -(PZ \ln f)g(X, FX) + (Z \ln f) \cos^2 \theta g(X, X). \]

Finally, we obtain
\[ g(h(X, X), \omega PZ) = (Z \ln f)\cos^2 \theta g(X, X). \tag{4.5} \]

If interchanging \( X \) by \( FX \) and using Riemannian metric property in the above equation we get the second assertion of the first part of the lemma. Now replacing \( Z \) by \( PZ \) in (4.3), then we get
\[ g(h(X, X), \omega P^2Z) = (PZ \ln f)\cos^2 \theta g(X, X). \]

Thus using Theorem 2.1, in left hand side of the above equation fora slant submanifold, we reach the second part of lemma. Again replacing \( X \) by \( FX \) then we get final result of lemma. It completes the proof of the lemma. \( \square \)
5. An inequality for semi-slant warped product submanifolds

In this section, we obtain a geometric inequality for a warped product semi-slant submanifold in terms of the second fundamental form and the warping function with mixed totally geodesic submanifold. Now, we describe an orthonormal frame for a semi-slant submanifold, which we shall use in the proof of inequality theorem.

Let \( M = M_\theta \times M_f \) be an \( m = 2\alpha + 2\beta \)-dimensional warped product semi-slant submanifold of \( 2n \)-dimensional locally product Riemannian manifold \( M \) such that the dimension of \( M_\theta \) is \( d_1 = 2\alpha \) and the dimension of \( M_f \) is \( d_2 = 2\beta \), where \( M_\theta \) and \( M_f \) are the integral manifolds of \( D^\theta \) and \( D^f \), respectively. We consider \( \{ e_1, e_2, \ldots, e_n, e_{\beta+1} = Fe_1, \ldots, e_\beta = Fe_\beta \} \) and \( \{ e_{\beta+1} = e_1^*, e_{\beta+1} = e_2^*, \ldots, e_{2\beta+1} = e\thetaPe_1^*, \ldots, e_{2\beta+2\alpha} = e\theta Pe_\beta^* \} \) which are orthonormal frames of \( D \) and \( D^\theta \) respectively. Thus the orthonormal frames of the normal sub bundles, \( \omega D^\theta \) and invariant sub bundle \( \nu \), respectively are \( \{ e_{m+1} = e_1 = \csc \theta \omega e_1^*, \ldots, e_{m+\alpha} = e_\alpha = \csc \theta \omega e_\alpha^*, \ldots, e_{m+\alpha+1} = e_{\alpha+1} = \csc \theta \sec \theta \omega Pe_1^*, \ldots, e_{m+2\alpha} = e_{2\alpha} = \csc \theta \sec \theta \omega Pe_\beta^* \} \) and \( \{ e_{m+2\alpha+1}, \ldots, e_{2\beta n} \} \).

**Theorem 5.1.** Let \( M = M_\theta \times M_f \) be a \( m \)-dimensional mixed totally geodesic warped product semi-slant submanifold of \( 2n \)-dimensional locally product Riemannian manifold \( M \) such that \( M_f \) is holomorphic submanifold of dimension \( d_2 \) and \( M_\theta \) is a proper slant submanifold of dimension \( d_1 \) of \( M \). Then

(i) The squared norm of the second fundamental form of \( M \) is given by

\[
\|h\|^2 \geq 4\alpha \csc^2 \theta \|\nabla^\theta \ln f \|^2 \quad (5.1)
\]

(ii) The equality holds in \( (5.1) \), if \( h(D, D) \subseteq \nu \) and \( M_\theta \) is totally geodesic in \( M \). Moreover, \( M_f \) can not be minimal.

**Proof.** By the definition of second fundamental form, we have

\[
\|h\|^2 = \|h(D^f, D^\theta)\|^2 + \|h(D, D^\theta)\|^2 + 2\|h(D^\theta, D^f)\|^2.
\]

Since, \( M \) is mixed totally geodesic, then we get

\[
\|h\|^2 = \|h(D, D)\|^2 + \|h(D^\theta, D^f)\|^2 \quad (5.2)
\]

Leaving second term and using \( (2.11) \) in first term, we obtain

\[
\|h\|^2 \geq 2n \sum_{i=m+1} \sum_{k=1} g(h(e_i, e_i))^2.
\]

The above expression can be written as in the components of \( \omega D^\theta \) and \( \nu \), then we derive

\[
\|h\|^2 \geq \sum_{l=1} \sum_{r=1} g(h(e_l, e_l))^2 \sum_{l=1} \sum_{r=1} g(h(e_r, e_r))^2
\]

\[
+ 2n \sum_{l=m+1} \sum_{k=1} g(h(e_l, e_l))^2.
\]

(5.3)

We will remove the last term and using the adapted frame for \( \omega D^\theta \), we derive

\[
\|h\|^2 \geq \csc^2 \theta \sum_{j=1} \sum_{k=1} g(h(e_j, e_j))^2 + \csc^2 \theta \sec^2 \theta \sum_{j=1} \sum_{k=1} g(h(e_j, e_j))^2.
\]

\[
= \csc^2 \theta \sum_{j=1} \sum_{k=1} g(h(e_j, e_j))^2 + \csc^2 \theta \sec^2 \theta \sum_{j=1} \sum_{k=1} g(h(e_j, e_j))^2.
\]

\[
= \csc^2 \theta \sum_{j=1} \sum_{k=1} g(h(e_j, e_j))^2 + \csc^2 \theta \sec^2 \theta \sum_{j=1} \sum_{k=1} g(h(e_j, e_j))^2.
\]

\[
= \csc^2 \theta \sum_{j=1} \sum_{k=1} g(h(e_j, e_j))^2 + \csc^2 \theta \sec^2 \theta \sum_{j=1} \sum_{k=1} g(h(e_j, e_j))^2.
\]

\[
= \csc^2 \theta \sum_{j=1} \sum_{k=1} g(h(e_j, e_j))^2 + \csc^2 \theta \sec^2 \theta \sum_{j=1} \sum_{k=1} g(h(e_j, e_j))^2.
\]
Again using the adapted frame for $D$ and the fact that second fundamental form is symmetric, then we get

$$
\|h\|^2 \geq \csc^2 \theta \sum_{j=1}^{\alpha} \sum_{k=1}^{\beta} g(h(e_j, e_k), \omega e_j^*)^2
+ 2\csc^2 \theta \sum_{j=1}^{\alpha} \sum_{k=1}^{\beta} g(h(e_j, F e_k), \omega e_j^*)^2
+ \csc^2 \theta \sum_{j=1}^{\alpha} \sum_{k=1}^{\beta} \{e_j^* \ln f \} g(e_j, e_k)^2
+ 2\csc^2 \theta \csc \theta \sum_{j=1}^{\alpha} \sum_{k=1}^{\beta} \{e_j^* \ln f \} g(e_j, e_k)^2
+ 2\csc^2 \theta \csc \theta \sum_{j=1}^{\alpha} \sum_{k=1}^{\beta} \{Pe_j^* \ln f \} g(e_j, e_k)^2.
$$

Then using Lemma 4.2 and Lemma 4.3, we arrive at

$$
\|h\|^2 \geq 2\csc^2 \theta \sum_{j=1}^{\alpha} \sum_{k=1}^{\beta} \{Pe_j^* \ln f \} g(e_j, e_k)^2
+ 2\csc^2 \theta \sum_{j=1}^{\alpha} \sum_{k=1}^{\beta} \{e_j^* \ln f \} g(e_j, e_k)^2
+ 2\csc^2 \theta \csc \theta \sum_{j=1}^{\alpha} \sum_{k=1}^{\beta} \{e_j^* \ln f \} g(e_j, e_k)^2
+ 2\csc^2 \theta \csc \theta \sum_{j=1}^{\alpha} \sum_{k=1}^{\beta} \{Pe_j^* \ln f \} g(e_j, e_k)^2.
$$

Thus combining first and second terms and using the property of trigonometric identities in the third and fourth terms, we get

$$
\|h\|^2 \geq 2\beta \csc^2 \theta \|\nabla^D \ln f\|^2 + 2\beta \csc^2 \theta \sum_{j=1}^{\alpha} \{e_j^* \ln f \}^2 - 2\beta \sum_{j=1}^{\alpha} \{e_j^* \ln f \}^2
+ 2\beta \csc^2 \theta \sum_{j=1}^{\alpha} \{Pe_j^* \ln f \}^2 + 2\beta \csc^2 \theta \sum_{j=1}^{\alpha} \{Pe_j^* \ln f \}^2.
$$

Last the above equation can be modified as

$$
\|h\|^2 \geq 4\beta \csc^2 \theta \|\nabla^D \ln f\|^2 + 2\beta \sum_{j=1}^{\alpha} \{\sec \theta Pe_j^* \ln f \}^2 - 2\beta \sum_{j=1}^{\alpha} \{e_j^* \ln f \}^2.
$$

From definition of adapted frame for $D^\theta$, finally, we obtain

$$
\|h\|^2 \geq 4\beta \csc^2 \theta \|\nabla^D \ln f\|^2.
$$

If the equality holds, from the leaving terms in (5.2) and (5.3), we obtain the following conditions, i.e., $M_\theta$ is totally geodesic in $M$ and $h(D, D) \subset \nu$. So the equality case holds. It is completed proof of the theorem.

6. Conclusion remark

If we assume that the slant angle $\theta = \frac{\pi}{2}$, then warped product semi-slant submanifold $M_\theta \times T M_T$ becomes a warped product semi-invariant submanifold of type $M_\theta \times T M_T$ of a locally product Riemannian manifold, in this case, Theorem 5.1 is generalized to the inequality theorem which
The squared norm of the second fundamental form of $M$ is given by
\[
\|A\|_2^2 \geq \frac{4}{\csc^2 \theta} \|\nabla^T \ln f\|_2^2.
\] (6.1)

(ii) The equality holds in (5.1), if $h(D, D) \leq \nu$ and $M_r$ is totally geodesic in $M$. Moreover, $M_r$ can not be minimal.

References


