Ali & Mior Othman, *Cogent Mathematics & Statistics* (2019), 6: 1602017
https://doi.org/10.1080/25742558.2019.1602017

cogent mathematics & statistics

Received: 09 December 2018
Accepted: 28 February 2019
First Published: 12 April 2019

*Corresponding author: Rifaqat Ali, Department of Mathematics, College of Science, King Khalid University, 9004 Abha, Saudi Arabia
E-mail: rrafat@kku.edu.sa; rifaqat.ali1@gmail.com

Reviewing editor:
Lishan Liu, Qufu Normal University, Qufu, China

Additional information is available at the end of the article

PURE MATHEMATICS | RESEARCH ARTICLE

Geometric inequality of warped product semi-slant submanifolds of locally product Riemannian manifolds

Rifaqat Ali[1]* and Wan Ainun Mior Othman[2]

Abstract: In the present article, we derive an inequality in terms of slant immersions and well define warping function for the squared norm of second fundamental form for warped product semi-slant submanifold in a locally product Riemannian manifold. Moreover, the equality cases are verified and generalized the inequality for semi-invariant warped products in locally Riemannain product manifold.

Subjects: Mathematical Analysis; Pure Mathematics; Engineering Mathematics; Mathematics; Mathematics

Keywords: Mean curvature; warped products; Riemannian manifolds; semi-slant immersions

1. Introduction

The notion of warped product manifolds plays very important roles not only in differential geometry but also in general relativity theory in physics. For example, Robertson-Walker space-times, asymptotically flat spacetime, Schwarzschild spacetime, and Reissner-Nordstrom spacetime are warped product manifolds (Hiepko, 1979). The geometry of warped products has a crucial role in differential geometry, as well as physical sciences. Bishop and O'Neill (1969) discovered the concept of warped product manifolds to derive an example of Riemannian manifolds of negative curvature, such manifolds are natural generalizations of Riemannian products manifolds. Therefore, many geometers are studied in Ali and Luarian (2017), Ali, Othman, and Ozel (2015), Ali and Ozel (2017), Ali, Uddin, and Othman (2017), Al-Solamy and Khan (2012), Al-Solamy, Khan,

ABOUT THE AUTHOR

Rifaqat Ali

Rifaqat Ali is an Assistant Professor, Department of Mathematics, College of Science, King Khalid University, Abha, Saudi Arabia. He completed his PhD from Aligarh Muslim University, Aligarh India in 2012. He has qualified graduate aptitude test in engineering (GATE) All India rank-270. His research interests are Complex Analysis, Approximation Theory and Differential Geometry. He is also working as principal and co-principal investigator in several ongoing projects of King Khalid University.

PUBLIC INTEREST STATEMENT

The extrinsic geometry of such warped product submanifolds actually is the mathematization that explicates our awareness of different concrete shapes in given ambient spaces and the intrinsic geometry of such warped product submanifolds is proper and Riemannian geometry. The study of warped products from this extrinsic point of view was initiated around the beginning of this century. Since then the study of warped product submanifolds from the extrinsic point of view has become a very active research subject in differential geometry and many nice results on this subject have been obtained by many geometers. In similar, we obtained the relation between the second fundamental form, the main extrinsic invariant, the main intrinsic invariants are the warping function of a warped product semi-slant submanifolds and slant angle.

and Uddin (2017), Atceken (2008, 2013), Chen (2001), Sahin (2006a, 2006b, 2006c). It is interesting to see that there exist no warped product semi-slant submanifolds of the forms $M = M_\theta \times_f M_T$ and $M = M_T \times_f M_\theta$ in a Kaehler manifold $\bar{M}$ such that M_T and M_θ are holomorphic and slant submanifolds, respectively (see Sahin, 2006b). While, Atceken (see examples 3.1 (Atceken, 2008)) has given an example on the existence of warped product semi-slant submanifold of the form $M = M_\theta \times_f M_T$ in a locally product Riemannian manifold such that M_T and M_θ are invariant and slant submanifolds, respectively. Hence, the geometry of warped product submanifolds in a locally product Riemannian manifold is different from the geometry of warped product submanifolds in Kaehler manifold. Therefore, we consider such a warped product semi-slant submanifold as mixed totally geodesic of locally product Riemannian manifold and obtain a geometric inequality for the length of the second fundamental form in terms of slant immersion and warping functions.

2. Preliminaries

Assume that $\bar{M}$ be a manifold of dimension m with a tensor field of such that

$$F^2 = I(F \neq \pm I), \tag{2.1}$$

where F is a one-one tensor field and I represent the identity transformation. Thus, $\bar{M}$ is an almost product manifold with almost product structure F. If an almost product manifold $\bar{M}$ admits a Riemannian metric g satisfying

$$g(FU, FV) = g(U, V), \; g(FU, V) = g(U, FV), \tag{2.2}$$

2010 *Mathematics Subject Classification*. 53C40 Primary 53C20 53C42 secondary.

Key words and phrases. Mean curvature, warped products, Riemannian manifolds, semi-slant immersions. For any $U, V \in \Gamma(T\bar{M})$, where $\Gamma(T\bar{M})$ denotes the set of all vector fields of $\bar{M}$ then $\bar{M}$ is said to be an almost product Riemannian metric manifold. Denote $\bar{\nabla}$ the Levi-Civita connection on $\bar{M}$ with respect to g. If $(\bar{\nabla}_U F)V = 0$, for all $U, V \in \Gamma(T\bar{M})$, then $(\bar{M}, g)$ is a locally product Riemannian manifold with Riemannian metric g (see Sahin, 2006a).

Let M be a submanifold of locally product Riemannian manifold $\bar{M}$ with an induced metric g. If $\nabla^\perp$ and ∇ are induced Riemannian connections on normal bundle $T^\perp M$ and tangent bundle TM and of M, respectively, then Gauss and Weingarten formulas are given by

$$(i)\; \bar{\nabla}_U V = \nabla_U V + h(U, V),$$
$$(ii)\; \bar{\nabla}_U N = -A_N U + \nabla^\perp_U N, \tag{2.3}$$

for each $U, V \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, where h and A_N are the second fundamental form and shape operator for an immersion M into $\bar{M}$. They are correlated as

$$g(h(U, V), N) = g(A_N U, V). \tag{2.4}$$

For any $X \in \Gamma(TM)$, we can write

$$(i)\; FU = PU + \omega U,$$
$$(ii)\; FN = tN + fN,$$

where $PU(tN)$ and $\omega U(fN)$ are tangential and normal components of $FU(FN)$, respectively. The covariant derivatives of the endomorphism F as

$$(\bar{\nabla}_U F)V = \bar{\nabla}_U FV - F\bar{\nabla}_U V, \forall U, V \in \Gamma(T\bar{M}). \tag{2.6}$$

A submanifold M of a locally product Riemannian manifold $\bar{M}$ is said to be *totally umbilical* (and *totally geodesic* respectively) if

$$h(U,V) = g(U,V)H, \ \ \& \ h(U,V) = 0, \tag{2.7}$$

for all $U, V \in \Gamma(TM)$. Then H is a mean curvature vector of M given by $H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i)$, where n is the dimension of M and $\{e_1, e_2, \cdots, e_n\}$ is a local orthonormal frame of the tangent vector space TM. Furthermore, if $H = 0$, then M is *minimal* in $\bar{M}$.

Definition 2.1. *A submanifold M of a locally product Riemannian manifold $\bar{M}$, then for each non zero vector U tangent to M at a point p, the angle $\theta(U)$ between FU and T_pM is called a Wirtinger angle of U. Hence, M is said to be a slant submanifold if the Wirtinger angle is constant and it is independent from the choice of $U \in T_pM$ and $p \in M$. The holomorphic and totally real submanifolds are slant submanifolds with slant angle $\theta = 0$ and $\theta = \pi/2$, respectively. A slant submanifold is said to be proper if it is neither holomorphic nor totally real. More generally, a distribution $\mathcal{D}$ on M is called a slant distribution if the angle $\theta(X)$ between FX and $\mathcal{D}_x$ has same value of θ for each $x \in \bar{M}$ and a non zero vector $X \in \mathcal{D}_x$.*

Thus for a slant submanifold M, a normal bundle $T^{\perp}M$ can be expressed as

$$T^{\perp}M = \omega(TM) \oplus \nu, \tag{2.8}$$

where ν is an invariant normal bundle with respect to F orthogonal to $\omega(TM)$. We recall following result for a slant submanifold of a locally product Riemannian manifold given by H. Li (cf. Li & Li, 2005).

Theorem 2.1. *If M is a submanifold of a locally product Riemannian manifold $\bar{M}$, then M is a slant submanifold if and only if there exists a constant $\lambda \in [0, 1]$ such that $P^2 = \lambda I$. In this case, θ is a slant angle of M, and then it satisfies $\lambda = \cos^2\theta$.*

Therefore, the following identities which are consequences from the Theorem 2.1

$$g(PU, PV) = \cos^2\theta g(U, V), \tag{2.9}$$

$$g(\omega U, \omega V) = \sin^2\theta g(U, V), \tag{2.10}$$

for any $U, V \in \Gamma(TM)$. Now let $\{e_1, e_2, \ldots e_n\}$ be an orthonormal basis of the tangent space TM and e_r belonging to the orthonormal basis $\{e_{n+1}, e_{n+2}, \ldots e_m\}$ of the normal bundle $T^{\perp}M$. Then we define

$$h_{ij}^r = g(h(e_i, e_j), e_r) \text{ and } ||h||^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)). \tag{2.11}$$

As a consequence for a differentiable function $\varphi : M \to \mathbb{R}$, we have

$$\| \nabla\varphi\|^2 = \sum_{i=1}^{n} (e_i(\varphi))^2, \tag{2.12}$$

where gradient $\nabla\varphi$ is defined by $g(\nabla\varphi, X) = X\varphi$, for any $X \in \Gamma(TM)$.

3. Semi-slant submanifolds

Semi-slant submanifolds were described by Papaghiuc (1994). These submanifolds are generalizations of CR-submanifolds with slant angle $\theta = \pi/2$.

Definition 3.1. *A submanifold M of an almost complex manifold $\bar{M}$ is called a semi-slant submanifold if there exist two orthogonal distributions $\mathcal{D}$ and $\mathcal{D}^{\theta}$ such that*

(i) $TM = \mathcal{D} \oplus \mathcal{D}^{\theta}$,
(ii) $\mathcal{D}$ *is holomorphic*, i.e., $F(\mathcal{D}) \subseteq \mathcal{D}$,
(iii) $\mathcal{D}^{\theta}$ *is slant distribution with slant angle* $\theta \neq 0, \pi/2$.

The dimensions of *holomorphic* distribution $\mathcal{D}$ and *slant* distribution $\mathcal{D}^\theta$ of semi-slant submanifold of a locally product Riemannian manifold $\bar{M}$ are denoted by m_1 and m_2 respectively. Then M is *holomorphic* if $m_2 = 0$ and *slant* if $m_1 = 0$. It is called proper semi-slant if the slant angle different from 0 and $\pi/2$. Moreover, if ν is an *invariant* subspace under the endomorphism F of normal bundle $T^\perp M$, then, in case of semi-slant submanifold, the normal bundle $T^\perp M$ can be decomposed as $T^\perp M = \omega\mathcal{D}^\theta \oplus \nu$. A semi-slant submanifold is said to be a *mixed totally geodesic*, if $h(X, Z) = 0$, for any $X \in \Gamma(\mathcal{D}^\theta)$ and $Z \in \Gamma(\mathcal{D})$.

4. Warped product submanifolds with the form $M_\theta \times_f M_T$

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds with a $f : M_1 \to (0, \infty)$, a positive differentiable function on M_1, we define on the product manifold $M_1 \times M_2$ with metric $g = \pi^* g_1 + (fo\pi)\gamma^* g_2$, where π and γ are natural projections on M_1 and M_2. Under these condition the product manifold is called warped product of M_1 and M_2, it is denoted by $M_1 \times_f M_2$ and f is called warping function. So we have the following lemma

Lemma 4.1 *Let $M = M_1 \times_f M_2$ be a warped product manifold. Then for any $X, Y \in \Gamma(TM_1)$ and $Z, W \in \Gamma(TM_2)$, we have*

(*i*) $\nabla_X Y \in \Gamma(TM_1)$.

(*ii*) $\nabla_Z X = \nabla_X Z = (X \ln f)Z$.

(*iii*) $\nabla_Z W = \nabla'_Z W - g(Z, W)\nabla \ln f$,

where ∇ and ∇' are the Levi-Civita connections on M_1 and M_2 respectively. Thus $\nabla \ln f$ is the gradient of $\ln f$ is defined as $g(\nabla \ln f, U) = U \ln f$. If the warping function f is constant, then the warped product manifold $M = M_1 \times_f M_2$ is called trivial, otherwise non-trivial. Furthermore, in a warped product manifold $M = M_1 \times_f M_2$, M_1 is totally geodesic and M_2 is totally umbilical submanifold in M, respectively (cf. Bishop & O'Neill, 1969). There are two types of warped product semi-slant submanifolds $M = M_\theta \times_f M_T$ and $M = M_T \times_f M_\theta$. For the second case, we have following non-existence theorem from Atceken (2008).

Theorem 4.1. *Assume that $\bar{M}$ is a locally Riemannian product manifold and M is a submanifold of $\bar{M}$. Then there exists no a warped product semi-slant submanifold $M = M_T \times_f M_\theta$ in $\bar{M}$ such that M_T is an invariant submanifold and M_θ is a proper slant submanifold of $\bar{M}$.*

Now, we develop some important lemmas for first type warped product for later use in the inequality and we refer for example to see their existence, Example 4.1 in Atceken (2008).

Lemma 4.2. *Let $M = M_\theta \times_f M_T$ be a warped product semi-slant submanifold of a locally product Riemannian manifold $\bar{M}$. Then*

$$g(h(X, FY), \omega Z) = -(Z \ln f)g(X, Y) \tag{4.1}$$

$$g(h(X, FY), \omega PZ) = -(PZ \ln f)g(X, Y), \tag{4.2}$$

for any $Z \in \Gamma(TM_\theta)$ and $X, Y \in \Gamma(TM_T)$.

PROOF. If $Z \in \Gamma(TM_\theta)$ and $X, Y \in \Gamma(TM_\perp)$, we have

$$g(h(X, FX), \omega Z) = g(\bar{\nabla}_X FX, \omega Z).$$

From (2.2) and (2.5) (i), we get

$$g(h(X, FX), \omega Z) = g(F\bar{\nabla}_X X, FZ) + g(\bar{\nabla}_X FX, PZ).$$

From the fact that X and Z are orthogonal, we obtain

$$g(h(X, FX), \omega Z) = -g(\bar{\nabla}_X Z, X) + g(\bar{\nabla}_X PZ, FX).$$

Then from (2.3) (i), we derive

$$g(h(X, FX), \omega Z) = -g(\nabla_X Z, X) + g(\bar{\nabla}_X PZ, FX).$$

Using Lemma 4.1 (ii), we arrive at

$$g(h(X, FX), \omega Z) = -(Z \ln f) g(X, X) + g(X, FX)(PZ \ln f).$$

As X and FX are orthogonal to each other by the definition of $(1, 1)$ tensor field F, the second term of last equation should be zero. Then we get

$$g(h(X, FX), \omega Z) = -(Z \ln f)||X||^2.$$

Replacing X by $X + Y$ in the above equation and from the property of linearity, we get the first result of lemma. Now interchanging Z by PZ, we obtain

$$g(h(X, FY), \omega PZ) = -(PZ \ln f) g(X, Y).$$

It completes the proof of the lemma. □

Lemma 4.3. *Let $M = M_\theta \times_f M_T$ be a warped product semi-slant submanifold of a locally product Riemannian manifold $\bar{M}$. Then*

$$g(h(X, X), \omega PZ) = g(h(FX, FX), \omega PZ) = (Z \ln f)\cos^2\theta ||X||^2, \tag{4.3}$$

$$g(h(X, X), \omega Z) = g(h(FX, FX), \omega Z) = (PZ \ln f)||X||^2, \tag{4.4}$$

for any $Z \in \Gamma(TM_\theta$ and $X \in \Gamma(TM_T)$.

PROOF. Suppose that $X \in \Gamma(TM_\theta)$ and (2.5) (i), we have

$$g(h(X, X), \omega PZ) = g(\bar{\nabla}_X X, FPZ) - g(\bar{\nabla}_X X, P^2 Z),$$

for $Z \in \Gamma(TM_T)$. Then from Theorem 2.1, implies that

$$g(h(X, X), \omega PZ) = g(\bar{\nabla}_X FX, PZ) - \cos^2\theta g(\bar{\nabla}_X X, Z).$$

Since FX and PZ are orthogonal then, we obtain

$$g(h(X, X), \omega PZ) = -g(\bar{\nabla}_X PZ, FX) + \cos^2\theta g(\bar{\nabla}_X Z, X).$$

From Lemma 4.1 (ii), we arrive at

$$g(h(X, X), \omega PZ) = -(PZ \ln f) g(X, FX) + (Z \ln f)\cos^2\theta g(X, X).$$

Finally, we obtain

$$g(h(X, X), \omega PZ) = (Z \ln f)\cos^2\theta g(X, X). \tag{4.5}$$

If interchanging X by FX and using Riemannian metric property in the above equation we get the second assertion of the first part of the lemma. Now replacing Z by PZ in (4.3), then we get

$$g(h(X, X), \omega P^2 Z) = (PZ \ln f)\cos^2\theta g(X, X).$$

Thus using Theorem 2.1, in left hand side of the above equation fora slant submanifold, we reach the second part of lemma. Again replacing X by FX then we get final result of lemma. It completes the proof of the lemma. □

5. An inequality for semi-slant warped product submanifolds

In this section, we obtain a geometric inequality for a warped product semi-slant submanifold in terms of the second fundamental form and the warping function with mixed totally geodesic submanifold. Now, we describe an orthonormal frame for a semi-slant submanifold, which we shall use in the proof of inequality theorem.

Let $M = M_\theta \times_f M_T$ be an $m = 2\alpha + 2\beta$-dimensional warped product semi-slant submanifold of $2n$-dimensional locally product Riemannian manifold $\bar{M}$ such that the dimension of M_θ is $d_1 = 2\alpha$ and the dimension of M_T is $d_2 = 2\beta$, where M_θ and M_T are the integral manifolds of $\mathcal{D}^\theta$ and $\mathcal{D}$, respectively. We consider $\{e_1, e_2, \ldots e_\beta, e_{\beta+1} = Fe_1, \ldots e_{2\beta} = Fe_\beta\}$ and $\left\{e_{2\beta+1} = e_1^*, \ldots e_{2\beta+\alpha} = e_q^*, e_{2\beta+\alpha+1} = e_{\alpha+1}^* = \sec\theta Pe_1^*, \ldots e_{2\beta+2\alpha} = e_{2q}^* = \sec\theta Pe_\alpha^*\right\}$ which are orthonormal frames of $\mathcal{D}$ and $\mathcal{D}^\theta$ respectively. Thus the orthonormal frames of the normal sub bundles, $\omega\mathcal{D}^\theta$ and invariant sub bundle ν, respectively are $\{e_{m+1} = \bar{e}_1 = \csc\theta\omega e_1^*, \ldots e_{m+\alpha} = \bar{e}_\alpha = \csc\theta\omega e_\alpha^*, \ldots e_{m+\alpha+1} = \bar{e}_{\alpha+1} = \csc\theta\sec\theta\omega Pe_1^*, \ldots e_{m+2\alpha} = \bar{e}_{2\alpha} = \csc\theta\sec\theta\omega Pe_\alpha^*\}$ and $\{e_{m+2\alpha+1}, \ldots e_{2n}\}$.

Theorem 5.1. *Let $M = M_\theta \times_f M_T$ be a m-dimensional mixed totally geodesic warped product semi-slant submanifold of $2n$-dimensional locally product Riemannian manifold $\bar{M}$ such that M_T is holomorphic submanifold of dimension d_2 and M_θ is a proper slant submanifold of dimension d_1 of $\bar{M}$. Then*

(i) *The squared norm of the second fundamental form of M is given by*

$$\|h\|^2 \geq 4\beta\csc^2\theta\left\|\nabla^\theta \ln f\right\|^2. \tag{5.1}$$

(ii) *The equality holds in (5.1), if $h(\mathcal{D}, \mathcal{D}) \subseteq \nu$ and M_θ is totally geodesic in $\bar{M}$. Moreover, M_T can not be minimal.*

PROOF. By the definition of second fundamental form, we have

$$\|h\|^2 = \left\|h(\mathcal{D}^\theta, \mathcal{D}^\theta)\right\|^2 + \|h(\mathcal{D}, \mathcal{D})\|^2 + 2\left\|h(\mathcal{D}^\theta, \mathcal{D})\right\|^2.$$

Since, M is mixed totally geodesic, then we get

$$\|h\|^2 = \|h(\mathcal{D}, \mathcal{D})\|^2 + \left\|h(\mathcal{D}^\theta, \mathcal{D}^\theta)\right\|^2 \tag{5.2}$$

Leaving second term and using (2.11) in first term, we obtain

$$||h||^2 \geq \sum_{l=m+1}^{2n} \sum_{r,k=1}^{2\beta} g(h(e_r, e_k), e_l)^2.$$

The above expression can be written as in the components of $\omega\mathcal{D}^\theta$ and ν, then we derive

$$\begin{aligned} ||h||^2 \geq & \sum_{l=1}^{\alpha} \sum_{r,k=1}^{2\beta} g(h(e_r, e_k), \bar{e}_l)^2 + \sum_{l=\alpha+1}^{2\alpha} \sum_{r,k=1}^{2\beta} g(h(e_r, e_k), \bar{e}_l)^2 \\ & + \sum_{l=m+2\alpha+1}^{2n} \sum_{r,k=1}^{2\beta} g(h(e_r, e_k), e_l)^2. \end{aligned} \tag{5.3}$$

We will remove the last term and using the adapted frame for $\omega\mathcal{D}^\theta$, we derive

$$||h||^2 \geq \csc^2\theta \sum_{j=1}^{\alpha} \sum_{r,k=1}^{2\beta} g(h(e_r, e_k), \omega e_j^*)^2 + \csc^2\theta\sec^2\theta \sum_{j=1}^{\alpha} \sum_{r,k=1}^{2\beta} g(h(e_r, e_k), \omega Pe_j^*)^2.$$

Again using the adapted frame for $\mathcal{D}$ and the fact that second fundamental form is symmetric, then we get

$$\begin{aligned}||h||^2 \geq \csc^2\theta \sum_{j=1}^{\alpha}\sum_{r,k=1}^{\beta} g(h(e_r,e_k),\omega e_j^*)^2 \\ + 2\csc^2\theta \sum_{j=1}^{\alpha}\sum_{r,k=1}^{\beta} g(h(e_r,Fe_k),\omega e_j^*)^2 \\ + \csc^2\theta \sum_{j=1}^{\alpha}\sum_{r,k=1}^{\beta} g(h(Fe_r,Fe_k),\omega e_j^*)^2 \\ + \csc^2\theta\sec^2\theta \sum_{j=1}^{\alpha}\sum_{r,k=1}^{\beta} g(h(e_r,e_k),\omega Pe_j^*)^2 \\ + 2\csc^2\theta\sec^2\theta \sum_{j=1}^{\alpha}\sum_{r,k=1}^{\beta} g(h(e_r,Fe_k),\omega Pe_j^*)^2 \\ + \csc^2\theta\sec^2\theta \sum_{j=1}^{\alpha}\sum_{r,k=1}^{\beta} g(h(Fe_r,Fe_k),\omega Pe_j^*)^2.\end{aligned}$$

Then using Lemma 4.2 and Lemma 4.3, we arrive at

$$\begin{aligned}||h||^2 \geq 2\csc^2\theta \sum_{j=1}^{\alpha}\sum_{r,k=1}^{\beta} \left\{(Pe_j^* \ln f)g(e_r,e_k)\right\}^2 \\ + 2\csc^2\theta \sum_{j=1}^{\alpha}\sum_{r,k=1}^{\beta} \left\{(e_j^* \ln f)g(e_r,e_k)\right\}^2 \\ + 2\csc^2\theta\cos^2\theta \sum_{j=1}^{\alpha}\sum_{r,k=1}^{\beta} \left\{(e_j^* \ln f)g(e_r,e_k)\right\}^2 \\ + 2\csc^2\theta\sec^2\theta \sum_{j=1}^{\alpha}\sum_{r,k=1}^{\beta} \left\{(Pe_j^* \ln f)g(e_r,e_k)\right\}^2.\end{aligned}$$

Thus combining first and second terms and using the property of trigonometric identities in the third and fourth terms, we get

$$\begin{aligned}\|h\|^2 \geq 2\beta\csc^2\left\|\nabla^\theta \ln f\right\|^2 + 2\beta\csc^2\theta \sum_{j=1}^{\alpha}(e_j^* \ln f)^2 - 2\beta \sum_{j=1}^{\alpha}(e_j^* \ln f)^2 \\ + 2\beta\csc^2\theta \sum_{j=1}^{q}(Pe_j^* \ln f)^2 + 2\beta\sec^2\theta \sum_{j=1}^{q}(Pe_j^* \ln f)^2.\end{aligned}$$

Last the above equation can be modified as

$$||h||^2 \geq 4\beta\{\csc^2\theta||\nabla^\theta \ln f||^2\} + 2\beta \sum_{j=1}^{q}(\sec\theta Pe_j^* \ln f)^2 - 2\beta \sum_{j=1}^{q}(e_j^* \ln f)^2.$$

From definition of adapted frame for $\mathcal{D}^\theta$, finally, we obtain

$$\|h\|^2 \geq 4\beta\csc^2\theta\left\|\nabla^\theta \ln f\right\|^2.$$

If the equality holds, from the leaving terms in (5.2) and (5.3), we obtain the following conditions, i.e., M_θ is totally geodesic in $\bar{M}$ and $h(\mathcal{D},\mathcal{D}) \subset \nu$. So the equality case holds. It is completed proof of the theorem. □

6. Conclusion remark

If we assume that the slant angle $\theta = \frac{\pi}{2}$, then warped product semi-slant submanifold $M_\theta \times_f M_T$ becomes a warped product semi-invariant submanifold of type $M_\perp \times_f M_T$ of a locally product Riemannian manifold, in this case, Theorem 5.1 is generalized to the inequality theorem which was obtained by Sahin (2006a). Therefore, we say that Theorem 5.1 in Sahin (2006c) is trivial case of our derived Theorem 5.1, that is

Theorem 6.1. *Let $M = M_\perp \times_f M_T$ be a m-dimensional mixed totally geodesic warped product semi-invariant submanifold of 2n-dimensional locally product Riemannian manifold $\bar{M}$ such that M_T is holomorphic submanifold of dimension d_2 and $M_\perp$ is a anti-invariant submanifold of dimension d_1 of $\bar{M}$. Then*

(i) *The squared norm of the second fundamental form of M is given by*

$$\|h\|^2 \geq 4\beta \csc^2\theta \left\|\nabla^\theta \ln f\right\|^2. \tag{6.1}$$

(ii) *The equality holds in (5.1), if $h(\mathcal{D}, \mathcal{D}) \subseteq \nu$ and M_θ is totally geodesic in $\bar{M}$. Moreover, M_T can not be minimal.*

Author details
Rifaqat Ali[1]
E-mail: rrafat@kku.edu.sa
Wan Ainun Mior Othman[2]
E-mail: wanainun@um.edu.my
[1] Department of Mathematics, College of Science, King Khalid University, Abha 9004, Saudi Arabia.
[2] Institute of Mathematical Science, University of Malaya, Kaula Lumpur, 50603, Malaysia.

Citation information
Cite this article as: Geometric inequality of warped product semi-slant submanifolds of locally product Riemannian manifolds, Rifaqat Ali & Wan Ainun Mior Othman, *Cogent Mathematics & Statistics* (2019), 6: 1602017.

References

Ali, A., & Luarian, P. (2017). Geometry of warped product immersions of Kenmotsu space forms and its applications to slant immersions. Journal of Geometry and Physics, 114, 276–290. doi:10.1016/j.geomphys.2016.12.001

Ali, A., Othman, W. A. M., & Ozel, C. (2015). Some inequalities for warped product pseudo-slant submanifolds of nearly Kenmotsu manifolds. Journal of Inequalities and Applications, 2015, 291. doi:10.1186/s13660-015-0802-5

Ali, A., & Ozel, C. (2017). Geometry of warped product pointwise semi-slant submanifolds of cosymplectic manifolds and its applications. International Journal of Geometric Methods in Modern Physics, *14*(3), 175002.

Ali, A., Uddin, S., & Othman, W. A. M. (2017). Geometry of warped product pointwise semi-slant submanifolds of Kaehler manifolds. Filomat, 32, 3771–3788. doi:10.2298/FIL1712771A

Al-Solamy, F. R., & Khan, M. A. (2012). Warped product submanifold of Riemannian product manifolds. Abstract and Applied Analysis, Article ID 724898 doi:10.1155/2012/724898

Al-Solamy, F. R., Khan, V. A., & Uddin, S. (2017). Geometry of warped product semi-slant submanifolds of nearly Kaehler manifolds. Results in Mathematics, 71(3–4), 783–799. doi:10.1007/s00025-016-0581-4

Atceken, M. (2008). Warped product semi-slant submanifolds in locally Riemannian product manifolds. Bulletin of the Australian Mathematical Society, 77, 177–186. doi:10.1017/S0004972708000191

Atceken, M. (2013). Contact CR-warped product submanifolds of Kenmotsu space forms. The Bulletin of the Iranian Mathematical Society, 39, 415–429.

Bejancu, A. (1984). Semi-invariant submanifolds of locally product Riemannian manifolds. *Annals of West University of Timisoara*, *22*, 3-11.

Bishop, R. L., & O'Neill, B. (1969). Manifolds of negative curvature. The Transactions of the American Mathematical Society, 145, 1–9. doi:10.1090/S0002-9947-1969-0251664-4

Chen, B. Y. (1990). Slant immersions, Bull. Bulletin of the Australian Mathematical Society, 41, 135–147. doi:10.1017/S0004972700017925

Chen, B. Y. (2001). Geometry of warped product CR-submanifold in Kaehler manifolds II. Monatshefte für Mathematik, 134, 103–119. doi:10.1007/s006050170002

Hiepko, S. (1979). Eine inner kennzeichungder verzerrten produckt. Mathematics Annals, 244, 209–215. doi:10.1007/BF01421206

Li, H., & Li, X. (2005). Semi-slant submanifolds of locally product manifold. The Georgian Mathematical Journal, 12(2), 273–282.

Papaghiuc, N. (1994). Semi-slant submanifold of Kaehler manifold. The Alexandru Ioan Cuza University, 40, 55–61.

Sahin, B. (2006a). Warped product semi-invariant submanifolds of a locally product Riemannian manifold. Bulletin Mathematique de la Societe des Sciences Mathematiques de Roumanie, 49 (97)(4), 383–394.

Sahin, B. (2006b). Non-existence of warped product submanifolds of Kaehler manifolds. Geometriae Dedicata, 117, 195–202. doi:10.1007/s10711-005-9023-2

Sahin, B. (2006c). Slant submanifolds of an almost product Riemannian manifold. Journal of the Korean Mathematical Society, 43(4), 717–732. doi:10.4134/JKMS.2006.43.4.717

Uddin, S., & Alqahtani, L. S. (2016). Chen type inequality for warped product immersions in cosymplectic space forms. Journal of Nonlinear Sciences and Applications, 9, 2914–2921. doi:10.22436/jnsa