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Universally polar cohomogeneity two Riemannian manifolds of constant negative curvature

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Abstract: In this paper, we suppose that M is a Riemannian manifold of constant negative curvature under the action of a Lie subgroup G of $Iso(M)$ such that the maximum of the dimension of the orbits is $\dim M - 2$. Then, we study topological properties of M under some conditions.

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Keywords: Riemannian manifold; polar action; cohomogeneity; lie group; curvature
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1. Introduction

An isometric action of a Lie group on a Riemannian manifold M is called polar if there exists a connected closed submanifold Σ of M which intersects the orbits orthogonally and meets every orbit. Such a submanifold Σ is called a section of the group action. In the special case where the section is flat in the induced metric, the action is called hyperpolar. The classification problem of polar actions was initiated by Dadok (1985), who classified polar actions by compact Lie groups on Euclidean spaces. In Heintze, Palais, Terng, and Thorbergsson (1995), the authors mentioned the interest of classifying (hyper-)polar actions on symmetric spaces of compact type. Kollross (2007) classified polar actions on compact symmetric spaces with simple isometry group and rank greater than one. Berndt (2011) presented a survey about polar actions on Riemannian symmetric spaces,

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PUBLIC INTEREST STATEMENT

In differential geometry, a Riemannian manifold is said to be homogeneous if the geometric structure is similar for all points. For example, sphere is homogeneous, because there is no difference between geometric properties of the sphere in any two different points. But, ellipsoid with non-equal semi-axes is not homogeneous. In homogeneous Riemannian manifolds, a connected lie group acts transitively such that each action is isometric (preserves distance). For an example consider the group of all clockwise rotations around the origin of the sphere. A generalization of homogeneous Riemannian manifolds is cohomogeneity K Riemannian manifolds, where transitivity of the action is reduced to another weaker condition called cohomogeneity of the action. In special case, when $K = 0$, the manifold will be homogeneous.

The present paper is devoted to study topological properties of some cohomogeneity two Riemannian manifolds.

with emphasis on the noncompact case. This classification showed that these actions are in fact all hyperpolar. J. C. Diaz-Ramos and A. Kollross obtain a classification of polar actions with a fixed point on symmetric spaces (Diaz-Ramos & Kollross, 2011).

Let M^n be a connected and complete Riemannian manifold of dimension n , and let G be a closed and connected subgroup of the Lie group of all isometries of M . If $x \in M$ then $G(x) = \{gx : g \in G\}$ is the orbit containing x . The cohomogeneity of the action of G on M is defined by $Coh(G, M) = n - \max\{\dim G(x) : x \in M\}$. If $Coh(G, M) = m$ then M is called cohomogeneity m Riemannian manifold. If $Coh(G, M) = 0$ then M is called homogeneous Riemannian manifold. Kobayashi (1962) proved that a homogeneous Riemannian manifold M of negative curvature is simply connected. Recently, Riemannian manifolds of cohomogeneity one have been studied from different points of view. Alekseevsky and Alekseevsky (1993) gave a description of such manifolds in terms of Lie subgroup G . Podesta and Spiro (1996) got interesting results about Riemannian manifold of negative curvature and of cohomogeneity one. Among other results, they proved that, if M , $\dim(M) \geq 3$, is a Riemannian G -manifold of negative curvature and $Coh(G, M) = 1$, then either M is diffeomorphic to $R^k \times T^r$, $r + k = \dim(M)$, or $\pi_1(M) = Z$ and the principal orbits are covered by $S^{n-2} \times R$, $n = \dim(M)$.

In this paper, Mirzaie (2011a) studied topological properties and G -orbits of a flat Riemannian G -manifold M of cohomogeneity two, and in Mirzaie (2009) he characterized a Riemannian G -manifold of negative curvature and of cohomogeneity two from topological view point, under the condition that $M^G \neq \emptyset$.

In this paper, in combination of the concept of polarity and cohomogeneity, we study topological properties of a Riemannian G -manifold of constant negative curvature and of cohomogeneity two.

2. Preliminaries

In the following, we mention some facts needed for the proof of our theorems.

Fact 2.1 (Bredon, 1972). If M is a complete and connected Riemannian manifold and G is a connected subgroup of $Iso(M)$, and if \tilde{M} is the universal Riemannian covering manifold of M with the covering map $\kappa : \tilde{M} \rightarrow M$, then there is a connected covering \tilde{G} of G with the covering map $\pi : \tilde{G} \rightarrow G$, such that \tilde{G} acts isometrically and effectively on \tilde{M} , $\dim_{\tilde{G}} \tilde{M} = \dim_{\tilde{G}} \tilde{M}$ and

- (1) Each deck transformation δ of the covering $\kappa : \tilde{M} \rightarrow M$ maps \tilde{G} -orbits on to \tilde{G} -orbits.
- (2) If $x \in M$ and $\tilde{x} \in \tilde{M}$ such that $\kappa(\tilde{x}) = x$ then $\kappa(\tilde{G}(\tilde{x})) = G(x)$.
- (3) If G has a fixed point in M , then $\tilde{G} = G$ and \tilde{M}^G is the full inverse image of M^G (where $M^G = \{x \in M : G(x) = x\}$).

Theorem 2.2 (Kobayashi, 1962). A homogeneous Riemannian manifold of negative curvature is simply connected.

Theorem 2.3 (Podesta & Spiro, 1996). If M is a complete and connected cohomogeneity one Riemannian manifold of negative curvature, then either M is simply connected or $\pi_1(M) = Z^p$, $p \geq 1$.

Theorem 2.4 (Mirzaie & Kashani, 2002). Let M be a flat non-simply connected cohomogeneity one Riemannian manifold under the action of Lie group $G \subset Iso(M)$.

- (a) If there is a singular orbit, then $\pi_1(M) = Z^p$.

(b) If there is no singular orbit and $M/G = R$ then M is diffeomorphic to $R^r \times T^t$ for some non-negative integers $r, t, r + t = \dim M$.

Corollary 2.5 (Diaz-Ramos & Kollross, 2011). If G is a closed, connected and nontrivial subgroup of the isometry group of a Riemannian manifold M of nonpositive sectional curvature, then M^G is a totally geodesic submanifold of M and $\dim M^G < \text{Coh}(G, M)$.

Definition 2.6. If $G_1, G_2 \subset \text{Iso}(M)$ then we say that G_1 and G_2 are orbit equivalent if for each $x \in M, G_1(x) = G_2(x)$.

If M is a complete and simply connected Riemannian manifold of nonpositive curvature then the geodesics γ_1 and γ_2 in M are called asymptotic provided there exists a number $c > 0$ such that $d(\gamma_1(t), \gamma_2(t)) \leq c$ for all $t \geq 0$. The asymptotic relation is an equivalence relation on the set of all geodesic in M , the equivalence classes are called asymptotic classes. If γ is a geodesic in M , then we denote by $[\gamma]$ the asymptotic class of geodesics containing γ . The following set is by definition the infinity of M :

$$M(\infty) = \{[\gamma] : \gamma \text{ is a geodesic in } M\}.$$

For any $x \in M$ and $[\gamma] \in M(\infty)$, there exists a unique geodesic $\gamma_x \in [\gamma]$ such that $x \in \gamma_x$ and there is a unique hypersurface S_x , which contains x and is perpendicular to all elements of $[\gamma]$. The hypersurface S_x is called the horosphere determined by x and $[\gamma]$.

Consider the Lorentzian space $R^{n,1} (= R^{n+1})$ with a non-degenerate scalar product \langle, \rangle given by

$$\langle x, y \rangle = -x_1 y_1 + \sum_{i=2}^{n+1} x_i y_i$$

It is well known that any simply connected Riemannian manifold of constant negative curvature $c < 0$, is isometric to the hyperbolic space of curvature c defined by

$$H^n(c) = \{x \in R^{n,1} : \langle x, x \rangle = -r^2\}, \quad c = \frac{1}{r^2}$$

It is well known that each horosphere in $H^n(c)$ is isometric to R^{n-1} .

Theorem 2.7 (Di Scala & Olmos, 2001). Let G be a connected Lie subgroup of the isometries of hyperbolic space H^n . Then, one of the following assertions is true:

- (i) G has a fixed point.
- (ii) G has a unique nontrivial totally geodesic orbit.
- (iii) All orbits are included in horospheres centered at the same point at infinity.

Theorem 2.8 (Diaz-Ramos & Sanchez et al., 2013). Let G act polarly on H^n . Then the action of G is orbit equivalent to:

- (a) The action of $SO(1, m) \times K$, where $m \in \{0, \dots, n-1\}$, and K is a compact group acting polarly on R^{n-m} .
- (b) The action of $N \times K$, where N is the nilpotent part of the Iwasawa decomposition of $SO(1, m)$, $m \in \{2, \dots, n\}$, and K is a compact group acting polarly on R^{n-m} .

Remark 2.9 (see (Eberlein & O'Neil, 1973), pp. 57, 58). Let \tilde{M} be a complete and simply connected Riemannian manifold of strictly negative curvature, and let S be a horosphere in \tilde{M} determined by

asymptotic class of geodesics $[\gamma]$. The function $f: \tilde{M} \rightarrow \mathbb{R}, f(p) = \lim_{t \rightarrow \infty} d(p, \gamma(t)) - t$, is called a Bussmann function.

For each point $p \in \tilde{M}$ there is a point $\eta_S(p)$ in S , which is the unique point in S of minimal distance from p , and the following map is a homeomorphism:

$$\phi: \tilde{M} \rightarrow S \times \mathbb{R}, \quad \phi(p) = (\eta_S(p), f(p)).$$

Fact 2.10 (Do Carmo, 1992). If M is nonsimply connected Riemannian manifold of negative curvature and there is a geodesic in \tilde{M} such that $\Delta(\gamma) = \gamma$ then Δ is isomorphic to $(\mathbb{Z}, +)$.

Definition 2.11. We say that a non-simply connected Riemannian G -manifold M is universally polar, when the covering group \tilde{G} of G acts polarly on its universal Riemannian covering manifold \tilde{M} .

Theorem 2.12 (Heidari & Mirzaie, 2018). Let $M^n, n \geq 3$, be a nonsimply connected Riemannian G -manifold of constant negative curvature and of cohomogeneity two. Then either M is universally polar or it is diffeomorphic to $S^1 \times \mathbb{R}^{n-1}$ or $B^2 \times \mathbb{R}^{n-2}$ (B^2 is the Moebius band).

3. Results

Fact 3.1. Let M^n be a closed, connected and nonsimply connected Riemannian manifold of constant negative curvature and of cohomogeneity two under the action of connected Lie group $G \subset \text{Iso}(M)$ such that $\tilde{M}^{\tilde{G}} = \phi$. Then one of the following is true:

- (a) There is a positive integer number p such that $\pi_1(M) = \mathbb{Z}^p$.
- (b) M is homeomorphic to $M_1 \times \mathbb{R}$, where M_1 is a flat cohomogeneity one G -submanifold of M .

Proof: Since M^n is a Riemannian manifold of constant negative curvature, then H^n is its universal covering manifold. We denote the covering maps by $k: H^n \rightarrow M^n$. Since $\tilde{M}^{\tilde{G}} = \phi$, by Theorem 2.8, one of the following is true:

- (i) \tilde{G} has a unique nontrivial totally geodesic orbit.
- (ii) All \tilde{G} -orbits of H^n are included in horospheres centered at the same point at infinity.

We consider each case separately.

(i) We denote the unique totally geodesic orbit of the action of \tilde{G} on H^n by Q . By the fact that Δ maps \tilde{G} -orbits of H^n onto \tilde{G} -orbits and since Q is the unique totally geodesic orbit, then $\Delta(Q) = Q$. If $\dim Q = 1$, then Q is a geodesic in H^n . Thus by Fact 2.12, $\Delta = \mathbb{Z}$ and we get part (a). If $\dim Q > 1$, put $W = k(Q)$. Since Q is the unique totally geodesic \tilde{G} -orbit in H^n , then W is unique totally geodesic G -orbit in M . Thus W is homogeneous and of constant negative curvature and by Theorem 2.2, W is simply connected. Therefore, Δ is trivial and M is simply connected which is contradiction.

(ii) In this case without loss of generality, we suppose that all \tilde{G} -orbits of H^n are included in horospheres determined by an asymptotic geodesics of class $[\gamma]$. If S is a horosphere determined by $[\gamma]$ then $\tilde{G}(S) = S$. The homeomorphism ϕ mentioned in Remark 2.11, induces a homeomorphism

$$\phi_1: M = \frac{\tilde{M}}{\tilde{G}} \rightarrow \frac{S}{\tilde{G}} \times \mathbb{R}. \text{ It is well known that each horosphere in } H^n \text{ is isometric to } \mathbb{R}^{n-1}. \text{ Thus } \frac{S}{\tilde{G}} \text{ is a flat Riemannian } G\text{-manifold of cohomogeneity one, and we get part (b).}$$

Lemma 3.2. Let M^n be a closed, connected and non-simply connected Riemannian manifold of constant negative curvature and of cohomogeneity two under the action of a connected Lie group

$G \subset \text{Iso}(M)$ such that \tilde{G} is orbit equivalent to $SO(1, m) \times K$, where $K \subset \text{Iso}(R^{n-m})$ is compact and $m \in \{0, \dots, n-1\}$. Then one of the following is true:

- (a) M is homeomorphic to $R^r \times T^l$ for some non-negative integers $r, t, r+t = \dim M$ (T^l is a l -torus).
- (b) There is a positive integer p such that $\pi_1(M) = Z^p$.
- (c) M is homeomorphic to $M_1 \times R$, where M_1 is a flat cohomogeneity one G -submanifold of M without singular orbit and $\frac{M_1}{G} = S^1$.

Proof: Since $SO(1, m)$ acts transitively on H^m , then $\tilde{M}^{SO(1,m) \times K} = \phi$. Thus, by Fact 3.1, we get that $\pi_1(M) = Z^p$ or M is homeomorphic to $M_1 \times R$, where M_1 is a flat cohomogeneity one G -submanifold of M . In the second case, we can consider the following cases.

Case 1. There is a singular orbit for the G -action on M_1 .

Case 2. There is no singular orbit for the G -action on M_1 .

Case 1. In this case, by Theorem 2.4 (a), $\pi_1(M_1) = Z^p, p \geq 1$, thus $\pi_1(M) = Z^p$.

Case 2. In this case, if $\frac{M_1}{G} = R$ then by Theorem 2.4 (b), we get part (a) of the lemma, otherwise M is homeomorphic to $M_1 \times R$, where M_1 is a flat cohomogeneity one G -submanifold of M without singular orbit and $\frac{M_1}{G} = S^1$ then we get part (c) of the lemma.

Lemma 3.3. Let M^n be a complete, connected and non-simply connected Riemannian manifold of constant negative curvature and of cohomogeneity two under the action of a connected Lie group $G \subset \text{Iso}(M)$ such that \tilde{G} is orbit equivalent to $\tilde{G}_1 \times \tilde{G}_2$, where \tilde{G}_1 is a nilpotent subgroup of $SO(1, m)$, $m \in \{2, \dots, n\}$, and $\tilde{G}_2 \subset \text{Iso}(R^{n-m})$ is compact. Then one of the following is true:

- (a) M is homeomorphic to $R^r \times T^l$ for some non-negative integers $r, t, r+t = \dim M$ (T^l is a l -torus).
- (b) There is a positive integer number p such that $\pi_1(M) = Z^p$.
- (c) M is homeomorphic to $M_1 \times R$, where M_1 is a flat cohomogeneity one G -submanifold of M without singular orbit and $\frac{M_1}{G} = S^1$.

Proof: Since M is a Riemannian G -manifold of constant negative curvature and of cohomogeneity two and \tilde{G} is orbit equivalent to $\tilde{G}_1 \times \tilde{G}_2$, then one of the following is true:

- (i) $\text{Coh}(\tilde{G}_1, H^m) = 0, \text{Coh}(\tilde{G}_1, H^m) = 2$;
- (ii) $\text{Coh}(\tilde{G}_1, H^m) = 1, \text{Coh}(\tilde{G}_1, H^m) = 1$;
- (iii) $\text{Coh}(\tilde{G}_1, H^m) = 2, \text{Coh}(\tilde{G}_2, R^{n-m}) = 0$.

We study (i), (ii), and (iii) separately.

(i) Since the action of \tilde{G}_1 on H^m is transitive and $\tilde{G}_1 \subset SO(1, m)$, then by Lemma 3.2, we get parts (a), (b), or (c) of the lemma.

(ii) Since $\text{Coh}(\tilde{G}_2, R^{n-m}) = 1$ then $(R^{n-m})^{\tilde{G}_2}$ has only one point that we denote it by y . For each $x = (h, y) \in H^m \times \{y\}$ we have $\tilde{G}(x) = (\tilde{G}_1 \times \tilde{G}_2)(h, y)$. Then

$$\widetilde{G}(H^m \times \{y\}) = (\widetilde{G}_1 \times \widetilde{G}_2)(H^m \times \{y\}) = H^m \times \{y\} \quad (*)$$

Consider the set $W = H^m \times \{y\}$. By (*), $\widetilde{G}(W) = W$. So W must be a Riemannian \widetilde{G} -manifold of constant negative curvature and of cohomogeneity one. If $x = (h, a) \in H^m \times R^{n-m}$ and $a \neq y$, since y is the only fixed point of the action \widetilde{G}_2 on R^{n-m} , then $\dim \widetilde{G}_2(a) \geq 1$ and $\dim \widetilde{G}(x) = \dim (\widetilde{G}_1 \times \widetilde{G}_2)(h, a) \geq m$. So, by dimensional reasons and the fact that each $\delta \in \Delta$ maps orbits to orbits, for each $h \in H^m$, there is $h' \in H^m$ such that

$$\delta(\widetilde{G}_1(h) \times \{y\}) = (\widetilde{G}_1(h') \times \{y\}).$$

Thus $\Delta(W) = W$. Now, put $T = k(W)$, Since $\widetilde{G}(W) = W$ and $\text{Coh}(\widetilde{G}, W) = 1$, then $G(T) = T$ and $\text{Coh}(G, T) = 1$. Thus, T is a Riemannian manifold of constant negative curvature and of cohomogeneity one. So by Theorem 2.3, $\pi_1(T) = Z^p$, $p \geq 1$ and

$$\pi_1(M) = \Delta = \pi_1(T) = Z^p.$$

(iii) This case cannot occur, because \widetilde{G}_2 has a fixed point.

The following theorem is similar to a theorem in Mirzaie (2011b), but with assumption of universal polarity we get more precise conclusions about the topology of M .

Theorem 3.4. *Let M^n be a complete, connected and non-simply connected Riemannian manifold of constant negative curvature and of cohomogeneity two, under the action of a Lie subgroup G of isometries such that it is universally polar. Then one of the following is true.*

- (a) M is homeomorphic to $R^r \times T^l$ for some non-negative integers $r, t, r + t = \dim M$ (T^l is a l -torus).
- (b) There is a positive integer number p such that $\pi_1(M) = Z^p$.
- (c) M is homeomorphic to $M_1 \times R$, where M_1 is a flat cohomogeneity one G -submanifold of M without singular orbit and $\frac{M_1}{G} = S^1$.

Proof: $\widetilde{M} = H^n$ is the universal Riemannian covering manifold of M . Consider \widetilde{G} as Fact 2.1, let $k : H^n \rightarrow M$ be the covering map and let Δ be the deck transformation group of the covering $k : H^n \rightarrow M$. Since the \widetilde{G} -action is polar on H^n , we can consider cases (1) and (2) of Theorem 2.8 for the \widetilde{G} -action on H^n .

Case 1. In this case, by Lemma 3.2 we get parts (a), (b), or (c) of the lemma.

Case 2. In this case, by Lemma 3.3 we get parts (a), (b), or (c) of the lemma.

Now by Theorems 2.12 and 3.4, we get the following theorem.

Corollary 3.5. *Let M^n , $n \geq 3$, be a nonsimply connected Riemannian G -manifold of constant negative curvature and of cohomogeneity two. Then one of the following is true.*

- (a) M is diffeomorphic to $S^1 \times R^{n-1}$ or $B^2 \times R^{n-2}$ (B^2 is the Moebius band).
- (b) M is homeomorphic to $R^r \times T^l$ for some non-negative integers $r, t, r + t = \dim M$ (T^l is a l -torus).
- (c) There is a positive integer number p such that $\pi_1(M) = Z^p$.
- (d) M is homeomorphic to $M_1 \times R$, where M_1 is a flat cohomogeneity one G -submanifold of M without singular orbit and $\frac{M_1}{G} = S^1$.

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References

- Alekseevsky, A., & Alekseevsky, D. (1993). Riemannian G-manifolds with one dimensional orbit space. *Annals of Global Analysis and Geometry*, 11, 197–211.
- Berndt, J. (2011). Polar actions on symmetric spaces. *Proceedings of The Fifteenth International Workshop on Diff. Geom* (Vol. 15, pp. 1–10).
- Bredon, G. E. (1972). *Introduction to compact transformation groups*. New York, London: Academic Press.
- Dadok, J. (1985). Polar coordinates induced by actions of compact Lie groups. *Transactions Amer Mathematical Society*, 288, 125–137. doi:10.1090/S0002-9947-1985-0773051-1
- Di Scala, A. J., & Olmos, C. (2001). The geometry of homogeneous submanifolds of hyperbolic space. *Mathematical Z*, 237, 199–209. doi:10.1007/PL00004860
- Diaz-Ramos, J. C., & Kollross, A. (2011). Polar actions with a fixed point. *Differential Geometry and Its Applications*, 29, 20–25. doi:10.1016/j.difgeo.2011.01.001
- Diaz-Ramos, J. C., Sanchez, M. Romero, A. (2013). *Recent trends in lorentzian geometry*. Springer Proceedings in Mathematics & Statistics. doi:10.1007/978-1-4614-4897-6-14.
- Do Carmo, M. P. (1992). *Riemannian geometry*. Boston, MA: Birkhauser Boston, Inc.
- Eberlein, P., & O'Neil, B. (1973). Visibility manifolds. *Pacific Journal Mathematical*, 46, 45–109. doi:10.2140/pjm.1973.46.45
- Heidari, M., & Mirzaie, R. (2018). Polarity of cohomogeneity two actions on negatively curved space forms. *To Appear in the Bulletin of the Korean Mathematical Society*, (vol. 55, pp.1433-1440).
- Heintze, E., Palais, R. S., Terng, C.-L., & Thorbergsson, G. (1995). *Hyperpolar actions on symmetric spaces. Geometry, Topology, and physics, Conf. proc. Lecture Note Geom. Topology, IV* (pp. 214–245). Cambridge, MA: Int. Press
- Kobayashi, S. (1962). Homogeneous Riemannian manifolds of negative curvature. *Tohoku Mathematical Journal*, 14, 413–415. doi:10.2748/tmj/1178244077
- Kollross, A. (2007). Polar actions on symmetric spaces. *Journal of Differential Geometry*, 77, 425–482. doi:10.4310/jdg/1193074901
- Mirzaie, R. (2009). On negatively curved G-manifolds of low cohomogeneity. *Hokkaido Mathematical Journal*, 38, 797–803. doi:10.14492/hokmj/1258554244
- Mirzaie, R. (2011a). On orbits of isometric actions on flat Riemannian manifolds. *Kyushu Journal Mathematical*, 65, 383–393. doi:10.2206/kyushujm.65.383
- Mirzaie, R. (2011b). On Riemannian manifolds of constant negative curvature. *Journal Korean Mathematical Society*, 48, 23–31. doi:10.4134/JKMS.2011.48.1.023
- Mirzaie, R., & Kashani, S. M. B. (2002). On cohomogeneity one flat Riemannian manifolds. *Glasgow Mathematical Journal*, 44, 185–190. doi:10.1017/S0017089502020189
- Podesta, F., & Spiro, A. (1996). Some topological properties of cohomogeneity one Riemannian manifolds with negative curvature. *Annals of Global Analysis and Geometry*, 14, 69–79. doi:10.1007/BF00128196



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