The 2 x n seating derangements

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ABSTRACT. In this paper, we study the derangement of $2n$ persons sitting 2 rows and n columns. In how many ways can the $2n$ persons rearrange their seating in accordance with the following condition. Each seat is located by one person and reoccupied by another person and each person moves to a horizontal or a vertical or a diagonal neighboring seat. We establish the system of recurrence relations for the solution of this problem and provide the solution of the system of recurrence relations.

1. Introduction

Kennedy and Cooper [2] are interested in the following problem.

"A $m \times n$ classroom has m rows of n desks per row. The teacher requests each pupil to change his seat by going either to the seat in front, the one behind, the one to his left, on to the one on his right (of course not all these options are possible for all students). How many ways can this direcitve be carried out?"

The answer to the $2 \times n$ classroom is F_{n+1}^2, where F_n is the nth Fibonacci number[1, 2, 3]. The following problem, which were raised in Kenndy and Cooper [2] and still remain open, as follows.

"A $2 \times n$ classroom has 2 rows of n desks per row. The teacher requests each pupil to change his seat by going either to the seat in front or behind, the one to the left or the right, or to one of the diagonal seats (of course, not all these options are possible for all students). How may ways can this directive be carried out?"

The purpose of this paper is to study derangement for the $2 \times n$ classroom problem. We can restate the $2 \times n$ classroom problem as follows. Consider $2n$ person seated at the 2 rows of n seats per row; in how many ways the $2n$ person can be derangement such that each seat can located by one person and reoccuppied by another person and each person moves to a horizontal or a vetical or a diagonal neighboring seat.

2. Main Results.

For any positive integer n, suppose that $2n$ persons, are seated at the 2 rows of n seats per row.

Definition 2.1. Let n be a positive integer.

A $2 \times n$ seating derangement is any movement of person among the $2n$ seats, arranged in 2 rows and n columns, so that each seat is located by one person and reocuppied by another person and each person moves to a horizontal or a vertical or a diagonal neighboring seat.

For example, the figure of 2×4 seating derangement and 2×5 seating derangement are shown in Figure 1.(a) and (b) respectively.

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Key words and phrases. derangement, recurrence relation.

FIGURE 1. 2×4 and 2×5 seating derangements.

In Figure 1, □ denotes a seat and ⇀ denotes the movement of a person from one seat to another.

Next, we are interrested in the problem of a $2\times n$ seating derangement, in how many ways the seats can be derangement? What is the number of ways that this can be done?

We may find that it is not easy to get a direct answer to the problem. Let us consider some special cases. When $n=1$, it is clear that there is one and only one way to interchange the two seats in a 2×1 seating derangement as shown in Figure 2.

FIGURE 2. 2×1 seating derangement.

When $n=2$, there are exactly 9 ways to rearrange the 4 seats in a 2×2 seating derangement as shown in Figure 3.

FIGURE 3. 2×2 seating derangement.

When $n=3$, there are 33 different ways to rearrange the 6 seats on a 2×3 seating derangement as shown in Figure 4.

FIGURE 4. 2×3 seating derangement.

We observed that each figure in the 2×3 seating derangement(Figure 4) are ended by 2×1 and the 2×2 seating derangements. These exhibit endings that may be classified into 13 types, call type A, type B, type C, type D, type E, type F, type G, type H, type I, type J, type K, type L, type M as shown in Figure 5.

FIGURE 5. type A, type B, , type M.

Type B may be further subdivided into five kinds shown in Figure 6; similarly for type C, type D and type E.

FIGURE 6. type B.

There are 4 kinds of type F shown in the Figure 7, similarly for type G, type H and type I. It is easy to see that there is only one kind of type A, J, K, and L.

FIGURE 7. type F.

For convenience, let T_n be the set of all $2\times n$ seating derangements and t_n denote the number of the set T_n. As shown before, we have

$$t_1=1, t_2=9, t_3=33.$$

Next, we will find $t_n, n\geq 3$.

The set T_n may be partitioned into 13 subsets $A_n,B_n,C_n,D_n,E_n,F_n,G_n,H_n,I_n,J_n,K_n,L_n$ and M_n according to the 41 endings.(Figure 8, 9, 10,11)

FIGURE 8. A_n

FIGURE 9. B_n,C_n,D_n,E_n

FIGURE 10. F_n,G_n,H_n,I_n

FIGURE 11. J_n,K_n,L_n,M_n

Let $a_n,b_n,c_n,d_n,e_n,f_n,g_n,h_n,i_n,j_n,k_n,l_n$ and m_n be the cardinalities of $A_n,B_n,C_n,D_n,E_n,F_n,$ G_n,H_n,I_n,J_n,K_n,L_n and M_n, respectively.

Then, the number of the $2\times n$ seating derangement is

$$t_n=a_n+b_n+c_n+d_n+e_n+f_n+g_n+h_n+i_n+j_n+k_n+l_n+m_n.$$

Theorem 2.2. The number of the $2\times n$ seating derangement is

$$t_n=w_n+4x_n+4y_n+4z_n,$$

where w_n,x_n,y_n,z_n satisfy the recurrence matrix

$$\begin{pmatrix}1&4&4&4\\1&2&2&0\\0&2&2&0\\1&0&0&0\end{pmatrix}\begin{pmatrix}w_n\\x_n\\y_n\\z_n\end{pmatrix}=\begin{pmatrix}w_{n+1}\\x_{n+1}\\y_{n+1}\\z_{n+1}\end{pmatrix},$$

and $w_1=1, x_1=y_1=z_1=0.$

Proof. It is clear that any member of T_n becomes a member of A_{n+1} when an extra column (Figure 2) is attached to the end, and conversely. Then

$$a_{n+1}=a_n+b_n+c_n+d_n+e_n+f_n+g_n+h_n+i_n+j_n+k_n+l_n+m_n.$$

Next, there is a 1-1 correspondences between $A_n\cup B_n\cup D_n\cup F_n\cup H_n$ and B_{n+1}, shown in

Figure 12. Then

$$b_{n+1} = a_n + b_n + d_n + f_n + h_n.$$

Similar correspondences, we have

$$c_{n+1} = a_n + c_n + e_n + g_n + i_n,$$
$$d_{n+1} = a_n + c_n + e_n + g_n + i_n,$$
$$e_{n+1} = a_n + b_n + d_n + f_n + h_n,$$
$$f_{n+1} = b_n + c_n + f_n + h_n,$$
$$g_{n+1} = b_n + c_n + g_n + i_n,$$
$$h_{n+1} = b_n + c_n + f_n + h_n, \text{and}$$
$$i_{n+1} = b_n + c_n + g_n + i_n.$$

Since A_n is a one-to-one correspondence to J_{n+1}, we have

$$j_{n+1} = a_n.$$

Similarly, we have

$$k_{n+1} = l_{n+1} = m_{n+1} = a_n.$$

From the 2×1 seating derangement, we obtain that $a_1 = 1$, and $b_1 = c_1 = d_1 = e_1 = f_1 = g_1 = h_1 = i_1 = j_1 = k_1 = l_1 = m_1 = 0.$

For $n \geq 1$, we derive $a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1}, e_{n+1}, f_{n+1}, g_{n+1}, h_{n+1}, i_{n+1}, j_{n+1}, k_{n+1}$, l_{n+1}, and m_{n+1} form the following recurrence relation.

$$a_{n+1} = a_n + b_n + c_n + d_n + e_n + f_n + g_n + h_n + i_n + j_n + k_n + l_n + m_n,$$
$$b_{n+1} = a_n + b_n + d_n + f_n + h_n,$$
$$c_{n+1} = a_n + c_n + e_n + g_n + i_n,$$
$$d_{n+1} = a_n + c_n + e_n + g_n + i_n,$$
$$e_{n+1} = a_n + b_n + d_n + f_n + h_n,$$
$$f_{n+1} = b_n + c_n + f_n + h_n,$$
$$g_{n+1} = b_n + c_n + g_n + i_n,$$
$$h_{n+1} = b_n + c_n + f_n + h_n,$$
$$i_{n+1} = b_n + c_n + g_n + i_n,$$
$$j_{n+1} = a_n,$$
$$k_{n+1} = a_n,$$

$$l_{n+1} = a_n, \text{ and}$$
$$m_{n+1} = a_n.$$

FIGURE 12. 1-1 correspondence between $A_n \cup B_n \cup D_n \cup F_n \cup H_n$ and B_{n+1}.

We notice that

$$b_{n+1} = e_{n+1},$$
$$c_{n+1} = d_{n+1},$$
$$f_{n+1} = h_{n+1},$$
$$g_{n+1} = i_{n+1}, \text{ and}$$
$$j_{n+1} = k_{n+1} = l_{n+1} = m_{n+1}.$$

$b_{n+1} = e_{n+1}, c_{n+1} = d_{n+1}, f_{n+1} = h_{n+1}, g_{n+1} = i_{n+1}$ and $j_{n+1} = k_{n+1} = l_{n+1} = m_{n+1}$. It follows that

$$a_{n+1} = a_n + 2b_n + 2c_n + 2f_n + 2g_n + 4j_n.$$

Since $f_{n+1} - g_{n+1} = 2(f_n - g_n)$ and $f_1 = 0 = g_1$, we have $f_{n+1} = g_{n+1}$ and $b_{n+1} = c_{n+1}$.

Set $w_n = a_n, x_n = b_n, y_n = f_n$, and $z_n = j_n$. Therefore, the number of the $2 \times n$ seating derangement is

$$t_n = w_n + 4x_n + 4y_n + 4z_n$$

where

$$w_{n+1} = w_n + 4x_n + 4y_n + 4z_n,$$
$$x_{n+1} = w_n + 2x_n + 2y_n,$$
$$y_{n+1} = 2x_n + 2y_n, \text{ and}$$
$$z_{n+1} = w_n,$$

and $w_1 = 1, x_1 = y_1 = z_1 = 0.$

These results are summarized in the matrix equation:

$$\begin{pmatrix} 1 & 4 & 4 & 4 \\ 1 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} w_n \\ x_n \\ y_n \\ z_n \end{pmatrix} = \begin{pmatrix} w_{n+1} \\ x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix},$$

where

$$\begin{pmatrix} w_1 \\ x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

□

The following table shows the values of w_n, x_n, y_n, z_n, and t_n in Theorem 2.2 when $n = 1, 2, \ldots, 10$.

n	w(n)	x(n)	y(n)	z(n)	t(n)
1	1	0	0	0	1
2	1	1	0	1	9
3	9	3	2	1	33
4	33	19	10	9	185
5	185	91	58	33	913
6	913	483	298	185	4777
7	4777	2475	1562	913	24577
8	24577	12851	8074	4777	127385
9	127385	66427	41850	24577	658801
10	658801	343939	216554	127385	3410313

Theorem 2.3. Let n be a positive integer.

The number of the $2\times n$ seating derangement is

$$t_n = \left(\frac{1}{2}\alpha - 2\right) A\alpha^{n+2} + \left(\frac{1}{2}\beta - 2\right) B\beta^{n+2} + \left(\frac{1}{2}\gamma - 2\right) C\gamma^{n+2},$$

where

$$A = \frac{2}{\alpha(\alpha-\beta)(\alpha-\gamma)}, B = \frac{2}{\beta(\beta-\alpha)(\beta-\gamma)}, C = \frac{2}{\gamma(\gamma-\alpha)(\gamma-\beta)},$$

and

$$\alpha = \frac{5}{3} + \frac{2\sqrt{37}}{3}\cos\left(\frac{1}{3}\arccos\left(\frac{-1}{37\sqrt{37}}\right)\right),$$

$$\beta = \frac{5}{3} + \frac{\sqrt{37}}{3}\left(-\cos\left(\frac{1}{3}\arccos\left(\frac{-1}{37\sqrt{37}}\right)\right) + \sqrt{3}\sin\left(\arccos\left(\frac{-1}{37\sqrt{37}}\right)\right)\right),$$

$$\gamma = \frac{5}{3} - \frac{\sqrt{37}}{3}\left(\cos\left(\frac{1}{3}\arccos\left(\frac{-1}{37\sqrt{37}}\right)\right) + \sqrt{3}\sin\left(\arccos\left(\frac{-1}{37\sqrt{37}}\right)\right)\right).$$

Proof. By Theorem 2.2, the number of the $2\times n$ seating derangement is

(1) $$t_n = w_n + 4x_n + 4y_n + 4z_n$$

where

(*) $$\begin{cases} w_{n+1} &= w_n + 4x_n + 4y_n + 4z_n, \\ x_{n+1} &= w_n + 2x_n + 2y_n, \\ y_{n+1} &= 2x_n + 2y_n, \text{ and} \\ z_{n+1} &= w_n. \end{cases}$$

for positive integer n and $w_1 = 1, x_1 = y_1 = z_1 = 0.$

The system of equations in (*) are reduced to a single equation.

(2) $$y_{n+3} - 5y_{n+2} - 4y_{n+1} + 16y_n = 0.$$

The characteristic equation is $t^3 - 5t^2 - 4t + 16 = 0.$

The zeros of this polynomial are

$$\alpha = \frac{5}{3} + \frac{2\sqrt{37}}{3}\cos\left(\frac{1}{3}\arccos\left(\frac{-1}{37\sqrt{37}}\right)\right),$$

$$\beta = \frac{5}{3} + \frac{\sqrt{37}}{3}\left(-\cos\left(\frac{1}{3}\arccos\left(\frac{-1}{37\sqrt{37}}\right)\right) + \sqrt{3}\sin\left(\arccos\left(\frac{-1}{37\sqrt{37}}\right)\right)\right),$$

$$\gamma = \frac{5}{3} - \frac{\sqrt{37}}{3}\left(\cos\left(\frac{1}{3}\arccos\left(\frac{-1}{37\sqrt{37}}\right)\right) + \sqrt{3}\sin\left(\arccos\left(\frac{-1}{37\sqrt{37}}\right)\right)\right).$$

The general solution of equation (2) is

(3) $$y_n = A\alpha^n + B\beta^n + C\gamma^n,$$

and using the intial conditions, $y_1 = y_2 = 0$ and $y_3 = 2,$ we obtain

$$A = \frac{2}{\alpha(\alpha-\beta)(\alpha-\gamma)},$$

$$B = \frac{2}{\beta(\beta-\alpha)(\beta-\gamma)},$$

$$C = \frac{2}{\gamma(\gamma-\alpha)(\gamma-\beta)}.$$

We substitute equation (3) into $w_n = \frac{1}{2}y_{n+2} - 2y_{n+1},$ which yields

$$w_n = \left(\frac{1}{2}\alpha - 2\right)A\alpha^{n+1} + \left(\frac{1}{2}\beta - 2\right)B\beta^{n+1} + \left(\frac{1}{2}\gamma - 2\right)C\gamma^{n+1}.$$

Since $t_n = w_{n+1}$ (by Theorem 2.2), we have

$$t_n = \left(\frac{1}{2}\alpha - 2\right)A\alpha^{n+2} + \left(\frac{1}{2}\beta - 2\right)B\beta^{n+2} + \left(\frac{1}{2}\gamma - 2\right)C\gamma^{n+2}.$$

This completes the proof. □

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PUBLIC INTEREST STATEMENT

A **2× *n* seating derangement** is any movement of persons among the 2*n* seats, arranged in 2 rows and *n* columns, so that each seat is located by one person and reoccupied by another person and each person moves to a horizontal or a vertical or a diagonal neighboring seat. How many ways can the directive in a 2× *n* seating derangement be carried out? What is the number of ways that this can be done? The objective of this paper is to solve a 2×*n* seating derangement. The system of recurrence relations for the solution of this problem is established and the solution of the system of recurrence relations is obtained.

ABOUT THE AUTHORS

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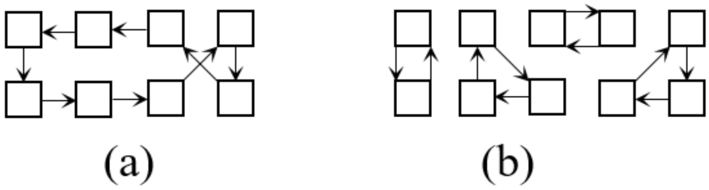
figure1



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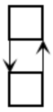

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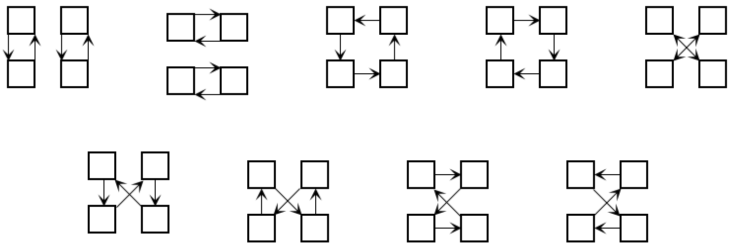

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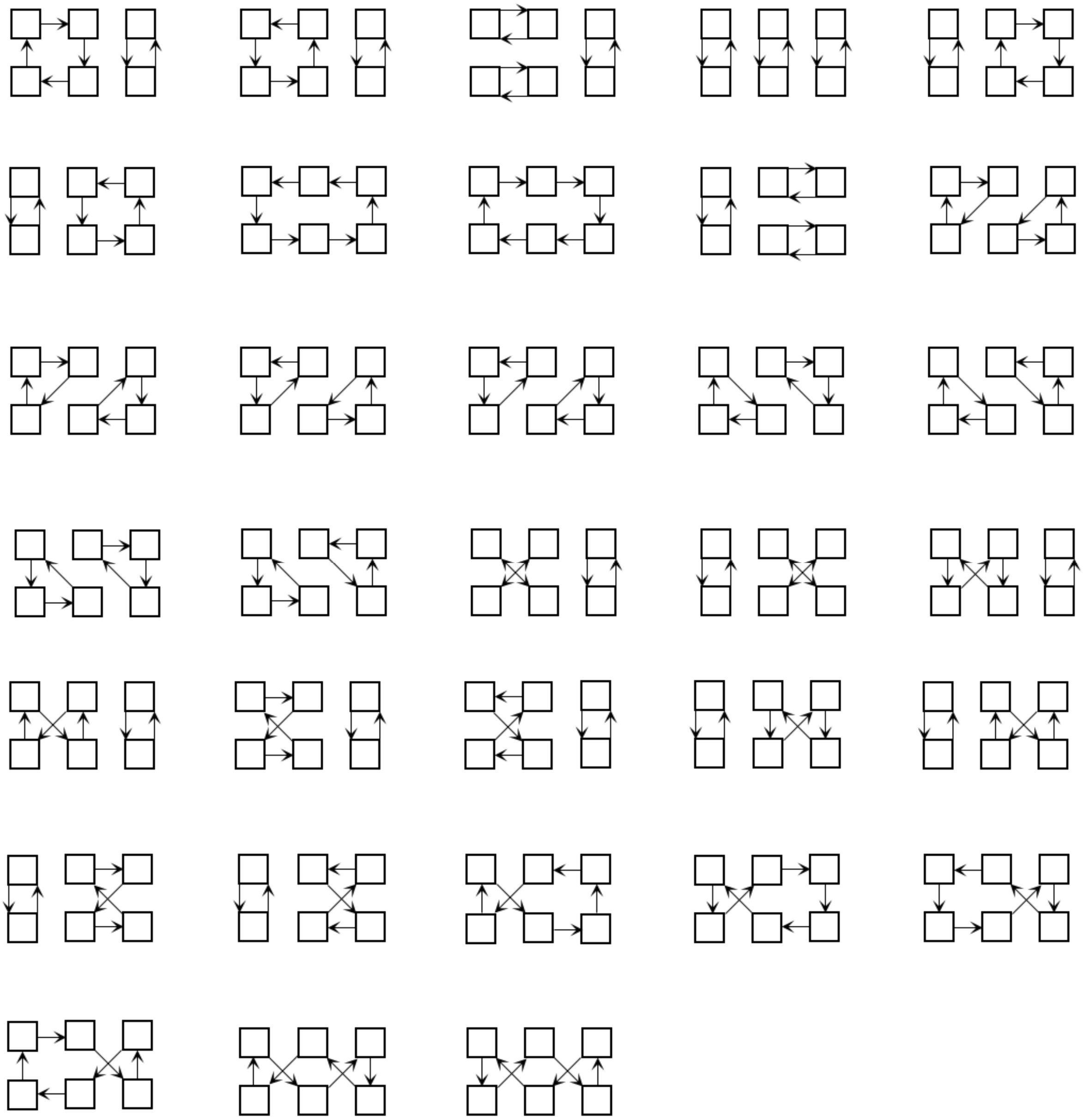

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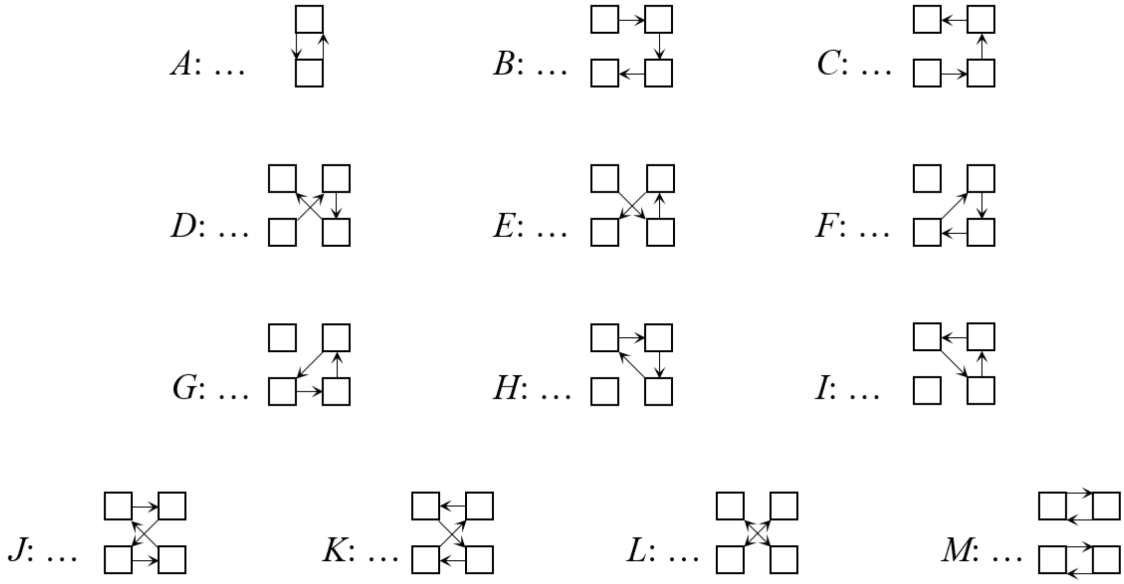

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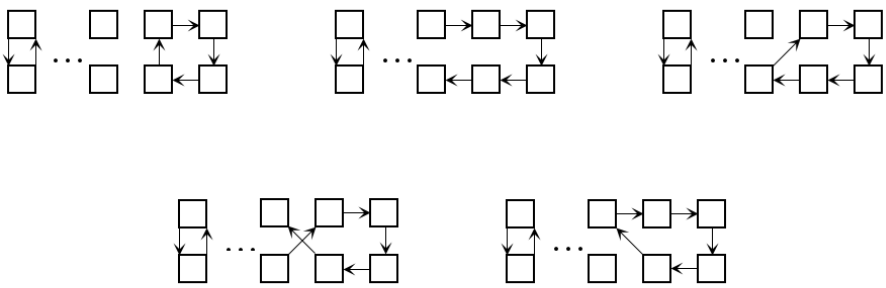

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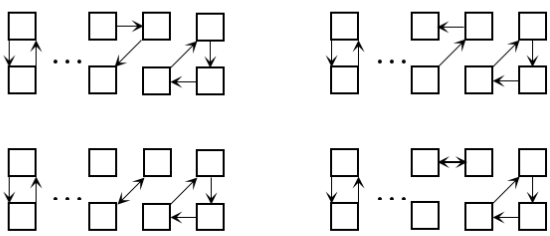

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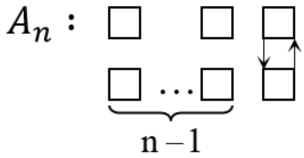
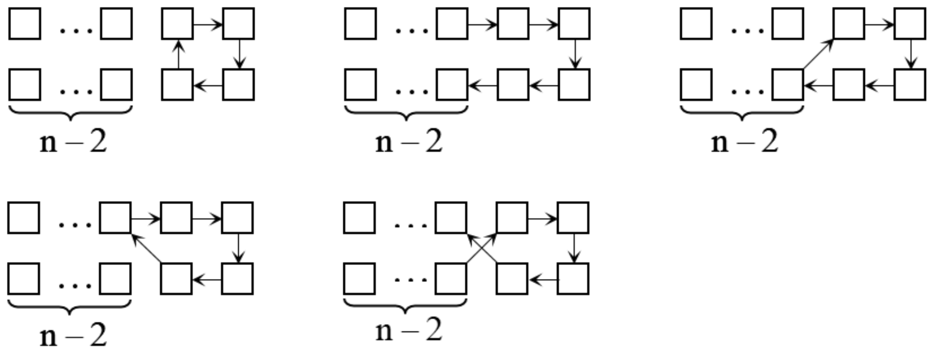
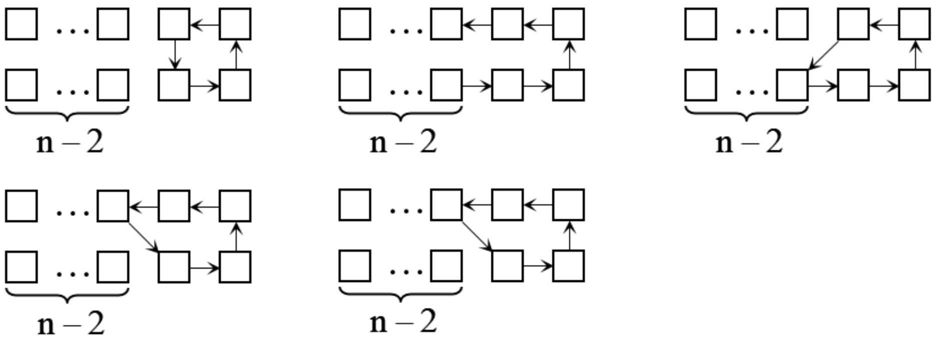


figure9

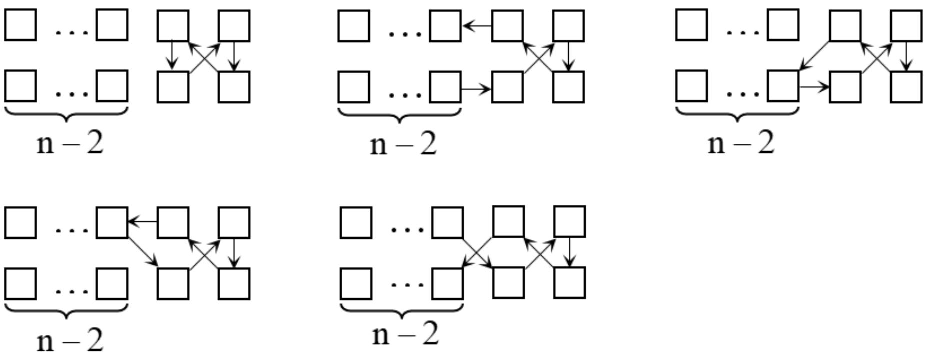
B_n :



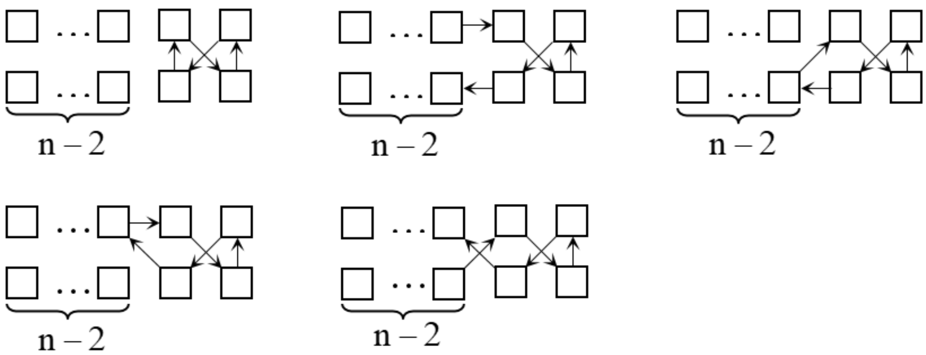
C_n :



D_n :



E_n :



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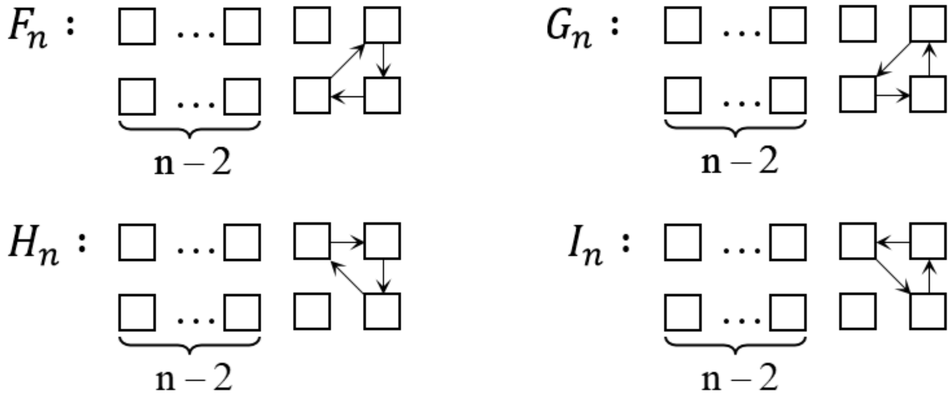

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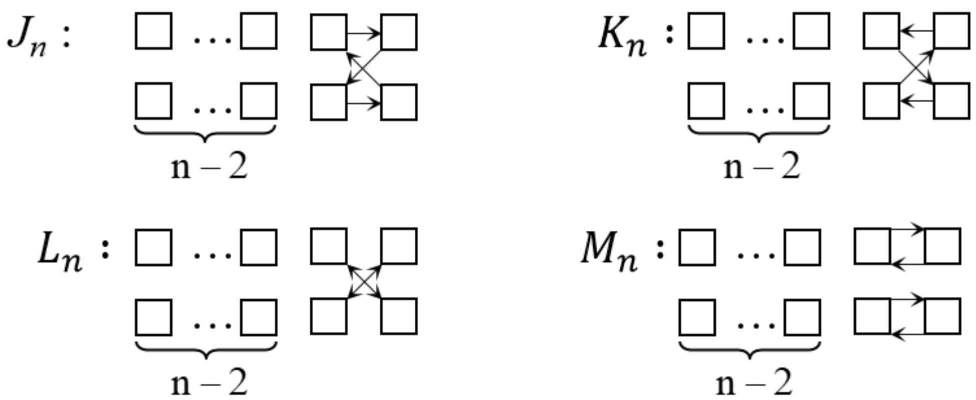

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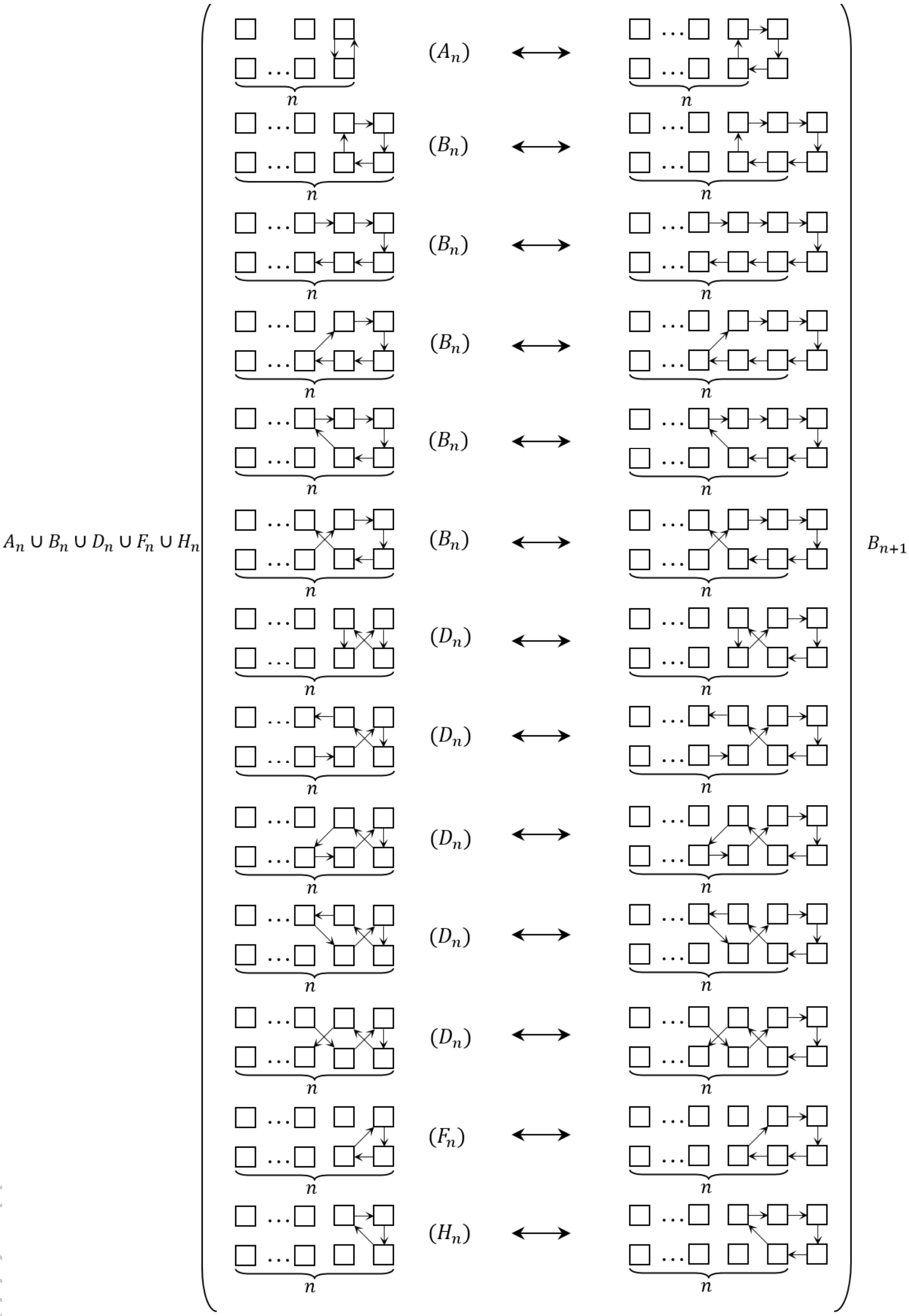
$A_n \cup B_n \cup D_n \cup F_n \cup H_n$
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B_{n+1}