Sampling theorem and efficiency comparison of three local minimum variance unbiased estimators of the mean and variance of the exponential distribution

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Abstract: This article continues the works of references to improve and perfect the sampling theorem of exponential distribution. First, the distribution of the sample range of exponential distribution is derived, and that the sample range is mutually independent of the sample minimum is proven. Then, this article derives the distribution of the difference between sample maximum and mean and demonstrates that the difference of these two statistics is mutually independent of the sample minimum. Thus, three local minimum variance unbiased estimators of the mean could be constructed. The estimator built by sample minimum and the difference between sample mean and minimum is precisely the uniformly minimum-variance unbiased estimator (UMVUE) of the mean. Similarly, three local minimum variance unbiased estimators of the variance are derived. At last, the efficiency comparison is made among the above three local minimum variance unbiased estimators of mean and variance of the exponential distribution.

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PUBLIC INTEREST STATEMENT

What is the sampling theorem of the exponential distribution? It includes the content about the distributions of the sample mean, sample maximum, sample minimum and their differences. It also includes the content of whether their differences are mutually independent of sample minimum. What is the local minimum variance unbiased estimation? Based on two mutually independent unbiased estimators, a kind of weighted linear unbiased estimators could be constructed, among which the one with the minimum variance is the local minimum variance unbiased estimation. One should remember that three local minimum variance unbiased estimators of mean and variance are not substituted for uniformly minimum variance unbiased estimators of mean and variance, respectively, but only rich in natural estimators.
1. Introduction
Sample minimum, sample maximum and sample mean are important statistics in exponential distribution. Sample minimum has an exponential distribution, and sample mean has a gamma distribution or Chi-square distribution with degree freedom of \( n \). The difference of sample mean and minimum has a gamma distribution or Chi-square distribution with degree freedom of \( n-1 \). The difference between sample mean and minimum is mutually independent of the sample minimum (Arnold, 1968; Gupta & Kundu, 2000; Marshall & Olkin, 1967).

This article derives the distribution of the difference of sample mean and minimum and demonstrates that the difference of these two statistics is mutually independent of the sample minimum is demonstrated (Cohen and Helm, 1973; Kundu & Gupta, 2009; Lawrance & Lewis, 1983; Nie, Sinha, & Hedayat, 2017).

Thus, the sampling theorem is improved. As natural corollary of the sampling theorem of the exponential distribution, a first local minimum variance unbiased estimators of expectation could be constructed by sample minimum and the difference between sample mean and minimum, which is precisely the UMVUE of the expectation. A second local minimum variance unbiased estimators of expectation could be constructed by sample minimum and sample range. A third local minimum variance unbiased estimators of expectation could be constructed by sample minimum and the difference between sample maximum and mean; similarly, three local minimum variance unbiased estimators of the variance are derived. At last, the efficiency comparison is made among the above three local minimum variance unbiased estimators of mean and variance of the exponential distribution (Al-Saleh & Al-Hadhrami, 2003; Baklizi & Dayyeh, 2003; Dixit & Nasiri, 2008; Guoan, Jianfeng, & Lihong, 2017; Li, 2016).

2. Sampling theorem of exponential distribution
The joint distribution of order statistics \((X_{(1)}, ..., X_{(n)})\) of exponential distribution is shown as follows:

\[
\begin{align*}
\text{Definition 2.1.} & \quad \text{If} \quad X \sim E(\alpha), X_1, ..., X_n \quad \text{is a sample with sample size} \quad n \quad \text{from} \quad X \sim E(\alpha), \quad (X_{(1)}, ..., X_{(n)}) \quad \text{has a joint density function:} \\
n(X_1, X_2, ..., X_n) = \frac{n!}{\alpha^n} \exp\left[-\frac{nX_0}{\alpha}\right], \quad X_{(1)} < X_{(2)} < ... < X_{(n)}, \quad \alpha > 0.
\end{align*}
\]

(1)

Then, we could say \((X_{(1)}, ..., X_{(n)})\) is from a multivariate order statistics exponential distribution.

Notate \(n\bar{X} = \sum_{i=1}^{n} X_i = \sum_{j=1}^{n} X_{(j)}\), the sampling theorem is:

**Theorem 2.1.** If \(X \sim E(\alpha), X_1, ..., X_n\) is a sample from \(X \sim E(\alpha)\) with sample size \(n\), \((X_{(1)}, ..., X_{(n)})\) are the order statistics, then

\[
\begin{align*}
(1) & \quad \frac{2nX_1}{\alpha} - x^2(2), \quad \frac{2n(X-x_1)}{\alpha} - x^2(2(n-1)), \quad \bar{X} - X_{(1)} \quad \text{is mutually independent of} \quad X_{(1)}, \\
(2) & \quad 2(n-1) \ln \left(1 - \exp\left[-\frac{(X-nX_0)}{\alpha}\right]\right) - x^2(2), \quad X_{(1)} \quad \text{is mutually independent of} \quad (X_{(n)} - X_{(1)}), \\
(3) & \quad X_{(1)} \quad \text{is mutually independent of} \quad (X_{(n)} - \bar{X}). \quad \text{The density function of} \quad (X_{(n)} - \bar{X}): \\
& \quad f_{X_{(n)} - \bar{X}}(x) = \left[\sum_{k=0}^{n-2} (-1)^k \binom{n}{k+1} \frac{(n-1-k)^{n-2}}{\alpha^{n-1}} \exp\left(-\frac{(n-1)(k+1)x}{(n-1-k)\alpha}\right)\right] \cdot x > 0 \quad (2)
\end{align*}
\]

**Proof.** \(P(X_{(1)} > X_{(1)}) = P(X_1 > X_{(1)}, ..., X_n > X_{(1)}) = \exp\left[-\frac{nX_1}{\alpha}\right], \quad g(X_{(1)}) = \frac{n}{\alpha} e^{-\frac{nX_1}{\alpha}}, \quad x > 0, \quad \frac{2nX_{(1)}}{\alpha} - x^2(2).\)
Notate \( U_{i(1)} = X_{(i)} - X_{(1)} \), \( i = 2, \ldots, n \), \( V = X_{(1)} \),
\( u_{(i-1)} = x_{(i)} - x_{(1)} \), \( v = x_{(1)} \Rightarrow u_{(i-1)} + v, i = 2, \ldots, n, x_{(1)} = v \),

the joint distribution density of \( (U_{1}, \ldots, U_{n-1}, V) \) is:

\[
f(u_{1}, u_{2}, \ldots, u_{n-1}, v) = \frac{(n!)!}{a^{n}} \exp \left[ -\left( \sum_{i=1}^{n-1} u_{(i)} + nv \right) / \alpha \right], u_{(1)} < u_{(2)} < \ldots < u_{(n-1)}, v > 0 \tag{2.3}
\]

Therefore, \( (U_{1}, U_{2}, \ldots, U_{n-1}) \) is the order statistics of sample \( U_{1}, U_{2}, \ldots, U_{n-1} \), which is the sample from \( U \sim E(\alpha) \) with sample size \( -1, (U_{1}, U_{2}, \ldots, U_{n-1}) \) is mutually independent of \( X_{(1)} \),

\[
\frac{2(n-1)X_{(n)}}{a} = \frac{2 \sum_{i=1}^{n-1} u_{(i)}}{a} = 2 \frac{\sum_{i=1}^{n-1} u_{(i)}}{a} - \chi^{2}(2(n-1)), X_{(1)} \] is mutually independent of \( X_{(1)} \).

Then, prove part (2).

\[
F_{(X_{1}, X_{2})}(x, y) = \Pr(X_{1} > x, X_{2} \leq y) = \Pr(x < X_{1} \leq y, \ldots, x < X_{n} \leq y) = \left[ \exp\left(-\frac{x}{\alpha}\right) - \exp\left(-\frac{y}{\alpha}\right) \right]^{n} \tag{2.4}
\]

Transform as \( U_{1} = X_{(1)}, U_{2} = X_{(n)} - X_{(1)} \),

\[
f_{(U_{1}, U_{2})}(u_{1}, u_{2}) = \frac{n(n-1)}{a^{2}} \left[ \exp\left(-\frac{u_{1}}{\alpha}\right) - \exp\left(-\frac{u_{2} + u_{1}}{\alpha}\right) \right]^{n-2} \exp\left[ -\frac{2u_{1} + u_{2}}{\alpha} \right] \tag{2.5}
\]

Therefore, \( U_{1} \) is mutually independent of \( U_{2} \).

\[
\frac{2nX_{(1)}}{a} \sim \chi^{2}(2), -2(n-1) \ln \left[ 1 - \exp\left( -\frac{X_{(1)}}{\alpha} \right) \right] \sim \chi^{2}(2).
\]

Notate \( U_{i(1)} = X_{(i)} - X_{(1)} \), \( i = 2, \ldots, n \), \( W_{(1)} = U_{(1)} \),

\[
W_{(k)} = kU_{(k)} - \sum_{i=1}^{k-1} U_{(i)}, k = 2, \ldots, n-1 \), then \( W_{(2)} - W_{(1)} = 2U_{(2)} - 2U_{(1)} > 0 \),

\[
W_{(k)} - W_{(k-1)} = kU_{(k)} - (k-1)U_{(k-1)} - U_{(k-1)} = k(U_{(k)} - U_{(k-1)}) > 0, k = 2, \ldots, n-1 \), the determinant is:

\[
|J| = \begin{vmatrix}
1 & 0 & 0 & \cdots & 0 \\
-1 & 2 & 0 & \cdots & 0 \\
-1 & -1 & 3 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
-1 & -1 & -1 & \cdots & n-1 \\
\end{vmatrix} = (n-1)! \tag{2.6}
\]

In \( \sum_{i=1}^{n-1} U_{i(1)} \), the coefficient of \( W_{(k)} \) is \( \frac{n}{k! \alpha^{k-1}} \), \( k = 1, \ldots, n-2 \), the coefficient of \( W_{(n-1)} \) is \( \frac{1}{(n-1)! \alpha^{n-2}} \). Derive the density function \( f_{W_{1}(1)}(w_{(n-1)}) \) of \( W_{(n-1)} \):

\[
f_{W_{1}(1)}(w_{(n-1)}) = \int_{0}^{w_{(n-1)}} dw_{1} \int_{w_{2}}^{w_{(n-1)}} dw_{2} \cdots \int_{w_{n-2}}^{w_{n-1}} dw_{n-2} \int_{w_{n-1}}^{w_{(n-1)}} \exp\left[ -\frac{w_{(n-1)}}{\alpha} \right] \exp\left[ \sum_{i=1}^{n-1} \frac{nw}{(n-1)! \alpha^{n-i}} \right] dw_{n-2} \tag{2.7}
\]
From $W_{(n-1)} = (n-1)(X_{(n)} - \bar{X})$, we could obtain $f_{(X_{(n)} - \bar{X})}(x) = (n-1)\frac{f_{W_{(n-1)}}((n-1)x)}{\sigma^2}$, then obtain:

$$
\begin{align*}
& f_{(X_{(n)} - \bar{X})}(x) = \left\{ \frac{(n-1)x^{n-2} - \left(1\right)^2(n-1)^2x^{n-2}}{n^{n-2}} \right\} \exp \left[ -\frac{nx}{(n-2)\alpha} \right] + \left(1\right)^2(n-1)^2x^{n-2} \\
& \exp \left[ -\frac{2nx}{(n-3)\alpha} \right] + \left(1\right)^2(n-1)^2x^{n-2} \exp \left[ -\frac{3nx}{(n-4)\alpha} \right] + \ldots + \left(1\right)^2(n-1)^2x^{n-2} \exp \left[ -\frac{n(n-2)x}{\alpha} \right] \\
& \left(\frac{n-1}{\alpha}\right) \exp \left[ -\frac{x}{\alpha} \right] = \left\{ \sum_{k=1}^{n-2} \left(-1\right)^k C_{n-1}^k \left(\frac{n-1-k}{\alpha}\right)^{n-2} \exp \left[ -\frac{knx}{(n-1-k)\alpha} \right] \right\} \left(\frac{n-1}{\alpha}\right) \exp \left[ -\frac{x}{\alpha} \right] \\
& = \left[ \sum_{k=1}^{n-2} \left(-1\right)^k C_{n-1}^k \left(\frac{n-1-k}{\alpha}\right)^{n-2} \left(\frac{n-1}{\alpha}\right) \exp \left[ -\frac{(n-1)(k+1)x}{(n-1-k)\alpha} \right] \right] x > 0. \quad (2.8)
\end{align*}
$$

List some specific situations:

When $n = 3$:

$$
\begin{align*}
& f_{(X_{(3)} - \bar{X})}(x) = \left[ \frac{6}{32} \exp \left( -\frac{x}{\alpha} \right) - \frac{3}{16} \exp \left( -\frac{x}{2\alpha} \right) \right] x > 0, \text{ it is a mixed exponential distribution.}
\end{align*}
$$

When $n = 4$:

$$
\begin{align*}
& f_{(X_{(4)} - \bar{X})}(x) = \left[ \frac{27}{16\alpha} \exp \left( -\frac{x}{\alpha} \right) - \frac{9}{4\alpha} \exp \left( -\frac{x}{3\alpha} \right) + \frac{9}{16\alpha} \exp \left( -\frac{3x}{4\alpha} \right) \right] x > 0.
\end{align*}
$$

When $n = 5$:

$$
\begin{align*}
& f_{(X_{(5)} - \bar{X})}(x) = \left[ \frac{256}{125\alpha} \exp \left( -\frac{x}{\alpha} \right) - \frac{432}{125\alpha} \exp \left( -\frac{8x}{3\alpha} \right) + \frac{192}{125\alpha} \exp \left( -\frac{6x}{5\alpha} \right) - \frac{16}{125\alpha} \exp \left( -\frac{16x}{5\alpha} \right) \right] x > 0.
\end{align*}
$$

When $n = 6$:

$$
\begin{align*}
& f_{(X_{(6)} - \bar{X})}(w) = \left[ \frac{3125}{125\alpha} \exp \left( -\frac{x}{\alpha} \right) + \frac{600}{625\alpha} \exp \left( -\frac{5x}{3\alpha} \right) + \frac{25}{625\alpha} \exp \left( -\frac{5x}{2\alpha} \right) + \frac{50}{625\alpha} \exp \left( -\frac{10x}{5\alpha} \right) + \frac{25}{625\alpha} \exp \left( -\frac{25x}{5\alpha} \right) \right] x > 0.
\end{align*}
$$

3. Three local minimum variance unbiased estimators of expectation

**Theorem 3.1.** If $X - E(\alpha)$, $X_1$, ..., $X_n$ is the sample from $X - E(\alpha)$ with sample size $n$, $X_{(1)}$, ..., $X_{(n)}$ are the order statistics, then the local minimum variance unbiased estimator, which is based on $X_{(1)}$ and $\bar{X} - X_{(1)}$, is the UMVUE of expectation.

**Proof.** From Theorem 2.1: $\frac{2nX_{(n)} - x^2}{a^2} - \chi^2(2)$, $\frac{2nX_{(n)} - x^2}{a^2} - \chi^2(2(n-1))$, $\bar{X} - X_{(1)}$ is mutually independent of $X_{(1)}$, we obtain $nX_{(1)}$ and $n(X_{(n)} - x^2)$ are both unbiased estimator of $\alpha$, and the effective unbiased estimator is $\hat{\alpha}_0 = c(nX_{(1)}) + (1 - c)\frac{n(X_{(n)} - x^2)}{n-1}$,

$$
\begin{align*}
& c = \frac{D \frac{n(X_{(n)} - x^2)}{n-1}}{D(nX_{(1)}) + D \frac{n(X_{(n)} - x^2)}{n-1}} = \frac{\alpha^2}{\frac{\alpha^2}{n-1}} = \frac{\alpha^2}{\frac{\alpha^2}{n-1}} = \frac{\alpha^2}{\frac{\alpha^2}{n-1}} = \frac{\alpha^2}{\frac{\alpha^2}{n-1}} = \frac{1}{\frac{1}{n}}.
\end{align*}
$$

Plug in and get $\hat{\alpha}_0 = \bar{X}$, which is the UMVUE of the expectation.

**Theorem 3.2.** If $X - E(\alpha)$, $X_1$, ..., $X_n$ is the sample from $X - E(\alpha)$ with sample size $n$, $X_{(1)}$, ..., $X_{(n)}$ are the order statistics, then the local minimum variance unbiased estimator, which is based on $X_{(1)}$ and $(X_{(n)} - X_{(1)})$, is $\hat{\alpha}_1 = c_1 \hat{\alpha}_1 + (1 - c_1)\hat{\alpha}_2$, here
\[
\hat{a}_{11} \text{ is the unbiased estimator based on } X_{(1)}, \quad \hat{a}_{12} = nX_{(1)}, \quad \hat{a}_{12} \text{ is the unbiased estimator based on } (X(n) - X_{(1)}); \quad \hat{a}_{12} = \frac{X_{(n) - X_{(1)}}}{n^2}. 
\]

**Proof.**

\[
E(X(n) - X_{(1)}) = \int_0^\infty x^{\frac{n-1}{\alpha}} |1 - \exp(-\frac{x}{\alpha})|^{n-2} \exp(-\frac{x}{\alpha}) dx = -x[1 - (1 - \exp(-\frac{x}{\alpha})^{n-1}] \bigg|_0^\infty + \int_0^\infty [1 - (1 - \exp(-\frac{x}{\alpha})]^{n-1} dx = \int_0^\infty [\sum_{k=1}^{n-1} (-1)^{k+1} \frac{C_k}{n-1} x \exp(-\frac{kx}{\alpha})] dx = \sum_{k=1}^{n-1} (-1)^{k+1} \frac{C_k}{n-1} \alpha^2, 
\]

\[
E(\hat{a}_{12}) = E\left(\frac{X_{(n)} - X_{(1)}}{n^2}\right) = \alpha, \quad \text{from Theorem 2.1:} \quad \hat{a}_{11} \text{ is independent of } \hat{a}_{12}, \quad \hat{a}_{11} \text{ and } \hat{a}_{12} \text{ are both unbiased estimator of } \alpha, \quad \text{when } 0 < c_1 \leq 1, \quad \hat{a}_1 = c_1\hat{a}_{11} + (1 - c_1)\hat{a}_{12} \text{ is the unbiased estimator of } \alpha, \quad D(\hat{a}_1) = c_1^2 D(\hat{a}_{11}) + (1 - c_1)^2 D(\hat{a}_{12}), \quad \text{take derivative of } c_1, \text{ make it equal to 0 and get:} 
\]

\[
2c_1 D(\hat{a}_{11}) - 2(1 - c_1)D(\hat{a}_{12}) = 0 \quad \Rightarrow \quad c_1 = \frac{D(\hat{a}_{12})}{D(\hat{a}_{11})}, 
\]

here, \(D\hat{a}_{11} = DnX_{(1)} = \frac{d^2}{n^2} \frac{2nX_{(1)}}{\alpha} = \frac{d^2}{\alpha} \times 2 = \alpha^2, \)

\[
E(X(n) - X_{(1)})^2 = \int_0^\infty x^\frac{n-1}{\alpha} \left[1 - \exp\left(-\frac{x}{\alpha}\right)\right]^{n-2} \exp\left(-\frac{x}{\alpha}\right) dx = -x^2 \left[1 - \exp\left(-\frac{x}{\alpha}\right)\right]^{n-1} \bigg|_0^\infty + \int_0^\infty 2x[1 - \exp\left(-\frac{x}{\alpha}\right)]^{n-1} dx = 2 \int_0^\infty \left[\sum_{k=1}^{n-1} (-1)^{k+1} \frac{C_k}{n-1} x \exp\left(-\frac{kx}{\alpha}\right)\right] dx = 2 \sum_{k=1}^{n-1} (-1)^{k+1} \frac{C_k}{n-1} \alpha^2, 
\]

\[
D(X(n) - X_{(1)}) = 2 \left[\sum_{k=1}^{n-1} (-1)^{k+1} \frac{C_k}{n-1} \alpha^2\right] - \left[\sum_{k=1}^{n-1} (C_k \frac{(-1)^{k+1} \alpha^2}{n-1})\right]^2, 
\]

\[
D\hat{a}_{12} = D\left(\frac{X(n) - X_{(1)}}{n^2}\right) = \left[\frac{\sum_{k=1}^{n-1} (-1)^{k+1} \frac{C_k}{n-1} \alpha^2}{n^2}\right] - \left[\frac{\sum_{k=1}^{n-1} (C_k \frac{(-1)^{k+1} \alpha^2}{n-1})}{n^2}\right]^2, 
\]

when \(c_1 = \frac{2 \left[\sum_{k=1}^{n-1} (-1)^{k+1} \frac{C_k}{n-1} \alpha^2\right] - \left[\sum_{k=1}^{n-1} (C_k \frac{(-1)^{k+1} \alpha^2}{n-1})\right]^2}{2 \left[\sum_{k=1}^{n-1} (-1)^{k+1} \frac{C_k}{n-1} \alpha^2\right] + 2 \left[\sum_{k=1}^{n-1} (-1)^{k+1} \frac{C_k}{n-1} \alpha^2\right]} = \hat{\alpha}_1 \text{ is the local minimum variance unbiased estimator of expectation based on } X_{(1)} \text{ and } (X(n) - X_{(1)}). \)

**Theorem 3.3:** If \(X - E(\alpha), X_1, ..., X_n \text{ is a sample from } X - E(\alpha) \text{ with sample size } n, \quad X_{(1)}, ..., X_{(n)} \text{ are the order statistics, then the local minimum variance unbiased estimator of expectation based on } X_{(1)} \text{ and } (X(n) - X_{(1)}) \text{ is} \)

\[
\hat{a}_2 = c_2 \hat{a}_{21} + (1 - c_2)\hat{a}_{22} 
\]

where \(c_2 = 1 - \left(\frac{\mu_1(n)}{\mu_2(n)}\right)^2, \quad \hat{a}_{21} \text{ is the unbiased estimator of expectation based on } X_{(1)}, \quad \hat{a}_{22} = nX_{(1)}, \quad \hat{a}_{22} \text{ is the unbiased estimator of expectation based on } (X(n) - X), \quad \hat{a}_{22} = \left(\frac{X_{(n)} - X_{(1)}}{n}\right), \quad \mu_1(n), \quad \mu_2(n) \text{ are the coefficients of } \alpha \text{ and } \alpha^2 \text{ from } E(X(n) - X) \text{ and } E((X(n) - X))^2, \text{ respectively.} \quad \mu_1(n) = \left[\sum_{k=1}^{n-1} \frac{(-1)^{k+1} \frac{C_k}{n-1} \alpha^2}{n^2(n-1)^2(k+1)^3}\right]. \
Proof. $\bar{X}_{21} = nX_{11}$, $E\bar{X}_{21} = \alpha$, $D\bar{X}_{21} = \alpha^2$, similar to Theorem 3.2, we only need to compute the expectation, second moment and variance of $(X_{(n)} - \bar{X})$.

\[
\mathbb{E}(X_{(n)} - \bar{X}) = \mu_1(n) = \left[ \sum_{k=0}^{n-2} (-1)^k C_{n-1}^k \frac{(n-1-k)(n-1)}{\alpha^2} \exp\left(-\frac{(n-1)(k+1)x}{\alpha}\right) \right], \quad x > 0.
\]

Similarly, $E(X_{(n)} - \bar{X})^2 = 2 \left[ \sum_{k=0}^{n-2} (-1)^k C_{n-1}^k (n-1-k)^2 \right] \alpha^2$, let $\mu_2(n)$ denote the coefficient of $\alpha^2$, then $E\bar{X}_{22} = E(\frac{\bar{X} - X_{(n)}}{\mu_2(n)}) = \alpha$.

4. Three local minimum variance unbiased estimators of variance

Let $DX = \alpha^2 = \lambda$.

Theorem 4.1. If $X - E(\sqrt{\lambda})$, $X_1, ..., X_n$ is a sample from $X - E(\sqrt{\lambda})$ with sample size $n$, $X_{(1)}, ..., X_{(n)}$ are the order statistics, then the local minimum variance unbiased estimator, which is based on $(X_{(1)})^2$ and $(X - X_{(1)})^2$, is

\[
\hat{\lambda}_0 = d_0 \hat{\lambda}_{01} + (1 - d_0) \hat{\lambda}_{02}
\]

where $d_0 = \frac{4n+2}{4n+1}$, $\hat{\lambda}_{01}$ is the unbiased estimator based on $(X_{(1)})^2$, $\hat{\lambda}_{01} = \frac{n^2(X_{(1)})^2}{2n}$, $\hat{\lambda}_{02}$ is the unbiased estimator based on $(X - X_{(1)})^2$, $\hat{\lambda}_{02} = \frac{n^2(X - X_{(1)})^2}{2n}$.

Proof. From Theorem 2.1, $\frac{2nX_{(1)}}{(\alpha)} - \chi^2(2)$, $\frac{2n(X - X_{(1)}) - \chi^2(2(n-1))}{(\alpha)}$ is mutually independent of $X_{(1)}$. Obtain: $\frac{n^2(X_{(1)})^2}{2n}$ and $\frac{n^2(X - X_{(1)})^2}{2n}$ are both unbiased estimator of $\lambda$, and the effective unbiased estimator is $\hat{\lambda}_0 = d_0 \frac{n^2(X_{(1)})^2}{2n} + (1 - d_0) \frac{n^2(X - X_{(1)})^2}{2n}$, here

\[
\mathbb{E}(\frac{2nX_{(1)}}{(\sqrt{\lambda})})^4 = \int_0^\infty x^4 \frac{1}{2} \exp\left(-\frac{x}{2}\right) dx = 16 \times 24 = 3 \times 2^7,
\]

\[
D\left(\frac{n^2(X_{(1)})^2}{2}\right) = 6\lambda^2 - \lambda^2 = 5\lambda^2;
\]

\[
\mathbb{E}(\frac{2n(X - X_{(1)})}{(\sqrt{\lambda})})^4 = \int_0^\infty x^4 \frac{1}{2n-1}(n-1)^{n-2} \exp\left(-\frac{x}{2}\right) dx = 16 \times (n+2)(n+1)n(n-1),
\]

\[
D\left(\frac{n^2(X - X_{(1)})^2}{n-1}\right) = \mathbb{E} \left(\frac{n^2(X - X_{(1)})^2}{n-1}\right)^4 = \left(\frac{(n+2)(n+1)}{n(n-1)} - 1\right) \lambda^2 = \frac{(4n+2)\lambda^2}{n(n-1)};
\]

\[
d_0 = D\left(\frac{n^2(X - X_{(1)})^2}{n-1}\right) + D\left(\frac{n^2(X_{(1)})^2}{n}\right) = \frac{(4n+2)\lambda^2}{5n^2} + \frac{(4n+2)\lambda^2}{5n^2} = \frac{4n+2}{5n^2 - n + 2},
\]
Plug in and get \( \hat{\lambda}_0 = \frac{(2n^2-n^2)(X_1)^2}{5n^2-n^2} + \frac{5n^2(X-X_1)^2}{5n^2-n^2}. \)

**Theorem 4.2.** If \( X - E(\sqrt{\lambda}) \), \( X_1, \ldots, X_n \) is a sample from \( X - E(\sqrt{\lambda}) \) with sample size \( n \), \( X_1, \ldots, X_n \) are the order statistics, then the local minimum variance unbiased estimator, which is based on \( (X_1)^2 \) and \( (X(n) - X_1)^2 \), is \( \hat{\lambda}_1 = d_1 \hat{\lambda}_{11} + (1 - d_1) \hat{\lambda}_{12}, \)

where \( d_1 = \frac{6 \sum_{i=1}^{n-1} (-1)^{k+1} C_n^k}{5\sum_{i=1}^{n-1} (-1)^{k+1} C_n^k} - \frac{\sum_{i=1}^{n-1} (-1)^{k+1} C_n^k}{\sum_{i=1}^{n-1} (-1)^{k+1} C_n^k} \)

\( \hat{\lambda}_{11} \) is the unbiased estimator based on \( (X_1)^2 \), \( \hat{\lambda}_{12} = \frac{n^2(X_0)^2}{2 \sum_{i=1}^{n-1} (-1)^{k+1} C_n^k} \).

**Proof:** \( E(X(n) - X_1)^2 \) is independent of \( \hat{\lambda}_{12}, \hat{\lambda}_{11} \) and \( \hat{\lambda}_{12} \) are both unbiased estimator of \( \lambda \), when \( 0 \leq d_1 \leq 1 \), \( \hat{\lambda}_1 = d_1 \hat{\lambda}_{11} + (1 - d_1) \hat{\lambda}_{12} \) is the unbiased estimator of \( \lambda \), \( D(d_1 \hat{\lambda}_{11} + (1 - d_1) \hat{\lambda}_{12}) = d_1^2 D(\hat{\lambda}_{11}) + (1 - d_1)^2 D(\hat{\lambda}_{12}) \), take derivative of \( d_1 \), make it equal to 0 and get: \( d_1 = \frac{D(\hat{\lambda}_{12})}{D(\hat{\lambda}_{11}) + D(\hat{\lambda}_{12})} \), here \( \hat{\lambda}_{11} = 5 \lambda^2 \).

\[ E(X(n) - X_1)^2 = \int_0^\infty x^4 \left[ 1 - \exp\left( -\frac{x}{\sqrt{\lambda}} \right) \right] \exp\left( -\frac{x}{\sqrt{\lambda}} \right) dx = \int_0^\infty x^4 \left[ 1 - \exp\left( -\frac{x}{\sqrt{\lambda}} \right) \right] \exp\left( -\frac{x}{\sqrt{\lambda}} \right) dx = 4 \sum_{i=1}^{n-1} (-1)^{k+1} C_n^k \int_0^\infty x^3 \exp\left( -\frac{kx}{\sqrt{\lambda}} \right) dx = 24 \sum_{i=1}^{n-1} (-1)^{k+1} C_n^k \frac{1}{k^2}. \]

\[ D\hat{\lambda}_{12} = \left( \frac{(X(n) - X_1)^2}{4(\sum_{i=1}^{n-1} (-1)^{k+1} C_n^k)} \right)^2 \lambda^2 - \frac{\left( \sum_{i=1}^{n-1} (-1)^{k+1} C_n^k \right)^2 (\sum_{i=1}^{n-1} (-1)^{k+1} C_n^k)}{\left( \sum_{i=1}^{n-1} (-1)^{k+1} C_n^k \right)^2}. \]

when

\[ d_1 = \frac{6 \sum_{i=1}^{n-1} (-1)^{k+1} C_n^k}{5\sum_{i=1}^{n-1} (-1)^{k+1} C_n^k} - \frac{\sum_{i=1}^{n-1} (-1)^{k+1} C_n^k}{\sum_{i=1}^{n-1} (-1)^{k+1} C_n^k} \]

\( \hat{\lambda}_1 \) is the local minimum variance unbiased estimator of variance based on \( (X_1)^2 \) and \( (X(n) - X_1)^2 \).

**Theorem 4.3.** If \( X - E(\sqrt{\lambda}) \), \( X_1, \ldots, X_n \) is the sample from \( X - E(\sqrt{\lambda}) \) with sample size \( n \), \( X_1, \ldots, X_n \) are the order statistics, then the local minimum variance unbiased estimator of variance based on \( (X_1)^2 \) and \( (X(n) - X_1)^2 \) is
\[
\hat{\lambda}_2 = d_2 \hat{\lambda}_{21} + (1 - d_2) \hat{\lambda}_{22}
\]  
(4.3)

Here \( d_2 = \frac{\mu(n) - \mu(n)}{\mu(n + k) \mu(n)} \), \( \hat{\lambda}_{21} \) is the unbiased estimator of variance based on \( (X_{(1)})^2 \), \( \hat{\lambda}_{21} = \frac{n^2 (X_{(1)})^2}{2} \), \( \hat{\lambda}_{22} \) is the unbiased estimator of variance based on \( (X(n) - \bar{X})^2 \), \( \hat{\lambda}_{22} = \frac{(X(n) - \bar{X})^2}{\mu_2(n)} \), \( \mu_2(n), \mu_4(n) \) are the coefficients of \( \hat{\lambda}_2 \) and \( \hat{\lambda}_2 \) from \( E(X(n) - \bar{X})^2 \) and \( E(X(n) - \bar{X})^4 \), respectively.

**Proof.** \( \hat{\lambda}_{21} = \frac{2(X_{(1)})^2}{n} \), \( E\hat{\lambda}_{21} = \lambda \), \( D\hat{\lambda}_{21} = 5 \lambda^2 \), similar to Theorem 4.2, we only need to compute the expectation, second moment and variance of \( (X(n) - \bar{X})^2 \):

\[
f_{X(n) - \bar{X}}(w) = \left[ \sum_{0}^{\infty} (-1)^k \sum_{k=1}^{n} \frac{(n - 1 - k)^{n-2}(n - 1)}{n! \alpha} \exp\left(-\frac{(n - 1)(k + 1)x}{(n - 1 - k)\alpha}\right) \right] x > 0.
\]

\[
E(X(n) - \bar{X})^2 = \mu_2(n) = 2 \left[ \sum_{0}^{\infty} (-1)^k \sum_{k=1}^{n} \frac{(n - 1 - k)^{n-2}(n - 1)}{n! \alpha} \exp\left(-\frac{(n - 1)(k + 1)x}{(n - 1 - k)\alpha}\right) \right] \lambda, \text{ let } \mu_2(n) \text{ denote the coefficient of } \lambda,
\]

then \( E\hat{\lambda}_{22} = \frac{E(X(n) - \bar{X})^2}{\mu_2(n)} = \lambda \).

Similarly, \( E(X(n) - \bar{X})^4 = 2 \left[ \sum_{0}^{\infty} (-1)^k \sum_{k=1}^{n} \frac{(n - 1 - k)^{n-2}(n - 1)}{n! \alpha} \exp\left(-\frac{(n - 1)(k + 1)x}{(n - 1 - k)\alpha}\right) \right] \lambda^2, \text{ let } \mu_4(n) \text{ denote the coefficient of } \lambda^2, \)

then \( D\hat{\lambda}_{22} = \frac{D(X(n) - \bar{X})^2}{\mu_2(n)} = \frac{[\mu_4(n) - \mu_2(n)]^2}{\mu_2(n)} \),

\[
d_2 = \frac{5 \lambda^2}{5} + \frac{[\mu_4(n) - \mu_2(n)]^2}{\mu_2(n)} = \frac{\mu_4(n) - \mu_2(n))^2}{\mu_2(n)} + 4(\mu_2(n))^2,
\]

when \( d_2 = \frac{\mu_4(n) - \mu_2(n)]^2}{\mu_2(n) + 4(\mu_2(n))^2} \), \( \hat{\lambda}_2 \) is the local minimum variance unbiased estimator of variance based on \( (X_{(1)})^2 \) and \( (X(n) - \bar{X})^2 \).

### 5. Efficiency comparison of three local minimum variance unbiased estimators of expectation and variance

**Remark 5.1.** The efficiency comparison of three local minimum variance unbiased estimators is the comparison of variances.

\[
D\hat{\lambda}_0 = \frac{\alpha^2}{n},
\]

\[
D\hat{\lambda}_1 = \left[ \frac{\sum_{0}^{\infty} (-1)^k C_{n-1}^{k - 1} \sum_{0}^{\infty} (-1)^k C_{n-1}^{k - 1} \alpha^2}{\sum_{0}^{\infty} (-1)^k C_{n-1}^{k - 1}} \right] \lambda^2,
\]

\[
D\hat{\lambda}_2 = \left[ \frac{\sum_{0}^{\infty} (-1)^k C_{n-1}^{k - 1} \sum_{0}^{\infty} (-1)^k C_{n-1}^{k - 1} \alpha^2}{\sum_{0}^{\infty} (-1)^k C_{n-1}^{k - 1}} \right] \lambda^2,
\]

\[
D\hat{\lambda}_0 = \frac{5}{5n^2 - n + 2},
\]

\[
D\hat{\lambda}_1 = \frac{5}{5} \left[ \sum_{0}^{\infty} (-1)^k C_{n-1}^{k - 1} \sum_{0}^{\infty} (-1)^k C_{n-1}^{k - 1} \frac{\alpha^2}{\lambda} \right] \lambda^2,
\]

\[
D\hat{\lambda}_2 = \frac{5}{5} \left( \sum_{0}^{\infty} (-1)^k C_{n-1}^{k - 1} \sum_{0}^{\infty} (-1)^k C_{n-1}^{k - 1} \frac{\alpha^2}{\lambda} \right) \lambda^2 + 6 \sum_{0}^{\infty} (-1)^k C_{n-1}^{k - 1} \left( \sum_{0}^{\infty} (-1)^k C_{n-1}^{k - 1} \right)^2.
\]
\[ \hat{\lambda}_2 = \frac{5}{20} \left[ 24 \left( \sum_{i=0}^{n-2-1} \binom{n-1}{k} C_i \right) - \left( \sum_{i=0}^{n-2-1} \binom{n-1}{k} C_i \right)^2 \right] \lambda^2 \]

Let \( \alpha = 1 \), here is the variance comparison of three local minimum variance unbiased estimators of expectation and variance when sample size is 2 to 56.

**Scatter plot with regression line 1:**
\[ e^{\hat{\alpha}_1} = a_1 \ast n^{b_1} + c_1, \hat{a}_1 = 2.939, \hat{b}_1 = -0.3144, \hat{c}_1 = 0.4314 \]

**Scatter plot with regression line 2:**
\[ e^{\hat{\alpha}_2} = a_2 \ast n^{b_2} + c_2, \hat{a}_2 = 2.733, \hat{b}_2 = -0.7192, \hat{c}_2 = 1.045 \]

**Scatter plot with regression line 3:**
\[ e^{\hat{\alpha}_3} = a \ast n^{b} + c, a = 0.7955, b = -1.338, c = 0.6916 \]
Comment 5.1. If \( X \sim E(\alpha) \), \( X_1, \ldots, X_n \) is the sample from \( X \sim E(\alpha) \) with sample size \( n \), \( X_{(1)}, \ldots, X_{(n)} \) are the order statistics, then the efficiency comparison of three local minimum variance unbiased estimator of expectation is that: \( D_{\alpha_0} < D_{\alpha_1} < D_{\alpha_2} \).

Proof. Because \( \hat{\alpha}_0 = \bar{X} \) is the UMVUE of the expectation, we have \( D_{\alpha_0} < D_{\alpha_1}, D_{\alpha_0} < D_{\alpha_2} \). Based on comparison among scatter plot with regression lines 1 or 2 as well as 3, we can obtain \( D_{\alpha_1} < D_{\alpha_2} \). Hence, \( D_{\alpha_0} < D_{\alpha_1} < D_{\alpha_2} \).

Scatter plot with regression line 4:
\[
e^{-\frac{\hat{\alpha}_0}{\lambda}} = a_3 \cdot n^{b_3} + c_3, \quad \hat{a}_3 = 2.665, \quad \hat{b}_3 = -0.5197, \quad \hat{c}_3 = 0.8951
\]

Scatter plot with regression line 5:
\[
e^{-\frac{\hat{\alpha}_2}{\lambda}} = a_4 \cdot n^{b_4} + c_4, \quad \hat{a}_4 = 3.089, \quad \hat{b}_4 = -0.971, \quad \hat{c}_4 = 1.104
\]

Scatter plot with regression line 6:
\[
e^{-\frac{\hat{\alpha}_1}{\lambda}} = a \cdot n^b + c, \quad a = 1.02, \quad b = -1.614, \quad c = 0.6719
\]
Comment 5.2. If \( X \sim E(\sqrt{\lambda}) \), \( X_1, \ldots, X_n \) is the sample from \( X \sim E(\sqrt{\lambda}) \) with sample size \( n \), \( X_{(1)}, \ldots, X_{(n)} \) are the order statistics, then the efficiency comparison of three local minimum variance unbiased estimator of variance is that: \( D_{\lambda_0} < D_{\lambda_1} < D_{\lambda_2} \).

Proof. Based on comparison among scatter plot with regression line 4 or 5 as well as 6, we can obtain \( D_{\lambda_0} < D_{\lambda_1} < D_{\lambda_2} \).

6. Discussion and conclusion
This article continues the works of references, to improve and perfect the sampling theorem of the exponential distribution. As natural corollary of the sampling theorem of the exponential distribution, one can obtain three local minimum variance unbiased estimators of mean and variance of the exponential distribution, respectively. We know that the sample mean is the UMVUE of expectation and \( \frac{1}{n} \sum_{i=1}^{n} X_i^2 \) is the UMVUE of variance. Therefore, three local minimum variance unbiased estimators of mean and variance are not substituted for uniformly minimum variance unbiased estimators of mean and variance, respectively, but only rich in natural estimators. From Tables 1–2 and scatter plots with regression lines 1–6, we can draw a conclusion that \( D_{\lambda_i}(i = 0, 1, 2) \) and \( D_{\lambda_i}(i = 0, 1, 2) \) are strictly monotonous decreasing as n increases; moreover, they are all convergent to zero, hence, they are all consistent.

Remark 6.1. The advantages of those estimators are as follows: If sample is not complete or the record value of the sample mean is not given, and the record value of the difference between sample maximum and mean and the sample minimum are known, then the local minimum variance unbiased estimator of expectation, which is based on \( X_{(1)} \) and \( (X_{(n)} - X) \) is a practical estimator; similarly, if sample is not complete or the record value of the sample mean is not given, and the record value of the sample maximum and the sample minimum are known, then the local minimum variance unbiased estimator of expectation, which is based on \( X_{(1)} \) and \( (X_{(n)} - X_{(1)}) \) is a recommendable estimator. If sample is not complete or the record value of the sample mean is

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not given, and the record value of \((\bar{X} - X_{(1)})^2\) and \((X_{(1)})^2\) are known, then the local minimum variance unbiased estimator of variance, which is based on \((X_{(1)})^2\) and \((\bar{X} - X_{(1)})^2\) is a practical estimator, similarly, under different sample condition, If sample is not complete or the record value of the sample mean is not given, and the record value of \((X_{(n)} - X)^2\)and \((X_{(1)})^2\) are known, or the record value of \((X_{(n)} - X_{(1)})^2\) and \((X_{(1)})^2\) are known, then the local minimum variance unbiased estimator of variance, which is based on \((X_{(n)} - X)^2\) and \((X_{(1)})^2\) or which is based on \((X_{(n)} - X_{(1)})^2\) and \((X_{(1)})^2\) is a recommendable estimator, respectively.

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References