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## APPLIED & INTERDISCIPLINARY MATHEMATICS | RESEARCH ARTICLE

# Hyers-Ulam stability of elliptic Möbius difference equation

Young Woo Nam<sup>1\*</sup>

**Abstract:** The linear fractional map  $f(z) = \frac{az+b}{cz+d}$  on the Riemann sphere with complex coefficients  $ad - bc \neq 0$  is called Möbius map. If  $f$  satisfies  $ad - bc = 1$  and  $-2 < a + d < 2$ , then  $f$  is called *elliptic* Möbius map. Let  $\{b_n\}_{n \in \mathbb{N}_0}$  be the solution of the elliptic Möbius difference equation  $b_{n+1} = f(b_n)$  for every  $n \in \mathbb{N}_0$ . We show that the sequence  $\{b_n\}_{n \in \mathbb{N}_0}$  on the complex plane as well as on the real numbers has no Hyers-Ulam stability by conjugation method.

**Subjects:** Science; Mathematics & Statistics; Advanced Mathematics; Analysis - Mathematics; Complex Variables; Sequences & Series

**Keywords:** difference equation; rational difference equation; stability; Hyers- Ulam stability; Möbius map

### 1. Introduction

The first order difference equation is defined as the solution of  $b_{n+1} = F(n, b_n)$  for  $n \in \mathbb{N}_0$  with the initial point  $b_0$ . An interesting non-linear difference equation is the rational difference equation. For instance, Pielou logistic difference equation (Pielou, 1974) or Beverton-Holt equation (Bohner & Warth, 2007; Sen, 2008) are the first order rational difference equation as a model for population dynamics with constraint. These equations are understood as the iteration of a kind of Möbius transformation on the real line. For the introduction and examples of difference equation defined on the real line, see (Elaydi, 2005).

In this paper, we investigate the Hyers-Ulam stability of another kind of Möbius transformation which does not appear in population dynamics and extend the result to the complex plane. Hyers-Ulam stability raised from Ulam's question (Ulam, 1960) about the stability of approximate homomorphism between metric groups. The first answer to this question was given by Hyers (Hyers, 1941) for Cauchy additive equation in Banach space. Later, the theory of Hyers-Ulam stability is developed in the area of functional equation and differential equation by many authors. The theory of Hyers-Ulam stability for difference equation appears in relatively recent decades and is mainly searched for linear difference equations, for example, see (Jung, 2015; Jung & Nam,

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### PUBLIC INTEREST STATEMENT

Difference equation is a field of mathematics which may describe the discrete dynamical systems. It has been successfully used to describing some phenomenon, for instance, population dynamics. In this article, a certain type of difference equations has no control of errors even though the amount of error of each term is arbitrarily small. The equation in the article is the elliptic linear fractional map and it is a kind of rotation where the map is defined on the set of complex numbers.

2016; Popa, 2015; Xu & Brzdek, 2015). Denote the set of natural numbers by  $\mathbb{N}$  and denote the set  $\mathbb{N} \cup \{\infty\}$  by  $\mathbb{N}_0$ . The set of real numbers and complex numbers by  $\mathbb{R}$  and  $\mathbb{C}$ , respectively. Denote the unit circle by  $\mathbb{S}^1$ .

Suppose that the complex valued sequence  $\{a_n\}_{n \in \mathbb{N}}$  satisfies the inequality

$$|a_{n+1} - F(n, a_n)| \leq \varepsilon$$

for a  $\varepsilon > 0$  and for all  $n \in \mathbb{N}_0$ , where  $|\cdot|$  is the absolute value of complex number. If there exists a sequence  $\{b_n\}_{n \in \mathbb{N}}$  which satisfies that

$$b_{n+1} = F(n, b_n) \tag{1.1}$$

for each  $n \in \mathbb{N}_0$  and  $|a_n - b_n| \leq G(\varepsilon)$  for all  $n \in \mathbb{N}_0$ , where the positive number  $G(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . Then we say that the difference Equation (1.1) has *Hyers-Ulam stability*.

### 1.1. Classification of Möbius transformation

Denote the Riemann sphere by  $\hat{\mathbb{C}}$ , which is the one point compactification of the complex plane, namely,  $\mathbb{C} \cup \{\infty\}$ . Similarly, we define the *extended real line* as  $\mathbb{R} \cup \{\infty\}$  and denote it by  $\hat{\mathbb{R}}$ . Möbius transformation (or Möbius map) is the linear fractional map defined on  $\hat{\mathbb{C}}$  as follows

$$g(z) = \frac{az + b}{cz + d} \tag{1.2}$$

where  $a, b, c$  and  $d$  are complex numbers and  $ad - bc \neq 0$ . Define  $g(-\frac{d}{c}) = \infty$  and  $g(\infty) = \frac{a}{c}$ . If  $c = 0$ , then  $g$  is the linear function. Thus we assume that  $c \neq 0$  throughout this article. The Möbius map which preserves  $\hat{\mathbb{R}}$  is called the *real Möbius map*. A Möbius map is real if and only if the coefficients of the map  $a, b, c$  and  $d$  are real numbers.

The Möbius map has two fixed points counting with multiplicity. Denote these points by  $\alpha$  and  $\beta$ . The real Möbius maps are classified to the three different cases using fixed points.

- If  $\alpha$  and  $\beta$  are real distinct numbers, the map is called *real hyperbolic* Möbius map,
- If  $\alpha = \beta$ , then the map is called *real parabolic* Möbius map, and
- If  $\alpha$  and  $\beta$  are two distinct non-real complex numbers, then the map is called *real elliptic* Möbius map.

Möbius map  $x \mapsto \frac{ax+b}{cx+d}$  is the same as  $x \mapsto \frac{pax+pb}{pcx+pd}$  for all numbers  $p \neq 0$ . Thus we may assume that  $ad - bc = 1$  when we choose  $p = \sqrt{ad - bc}$ . Moreover, Möbius map has the matrix representation  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  under the condition  $ad - bc = 1$ . Denote the matrix representation of the Möbius map  $g$ , by also  $g$  and its trace by  $\text{tr}(g)$ , which means  $a + d$ . In the complex analysis or hyperbolic geometry, Möbius maps with complex coefficients can be classified similarly with different method. For instance, see (Beardon, 1983). Möbius transformation in (1.2) (with real or complex coefficients) for  $ad - bc = 1$  is classified as follows

- If  $\text{tr}(g) \in \mathbb{R} \setminus [-2, 2]$ , then  $g$  is called *hyperbolic*,
- If  $\text{tr}(g) = \pm 2$ , then  $g$  is called *parabolic*,
- If  $\text{tr}(g) \in (-2, 2)$ , then  $g$  is called *elliptic* and
- If  $\text{tr}(g) \in \mathbb{C} \setminus \mathbb{R}$ , then  $g$  is called *purely loxodromic*.

The real Möbius maps are also classified by the above notions. In this article, we investigate Hyers-Ulam stability of elliptic Möbius transformations. Other cases would appear in the forthcoming articles.

## 2. No Hyers-Ulam stability with dense subset

In this section we prove non-stability in the sense of Hyers-Ulam, which is not only for the elliptic Möbius transformation but also for any function satisfying the assumption of the following Theorem 2.1.

**Theorem 2.1.** Let  $\{b_n\}_{n \in \mathbb{N}_0}$  be the sequence in  $\mathbb{R}$  satisfying  $b_{n+1} = F(b_n)$  with a map  $F$  for  $n \in \mathbb{N}_0$ . Suppose that there exists a dense subset  $A$  of  $\mathbb{R}$  such that if  $b_0 \in A$ , then the sequence  $\{b_n\}_{n \in \mathbb{N}_0}$  is dense in  $\mathbb{R}$ . Suppose also that  $\{b_n\}_{n \in \mathbb{N}_0}$  has no periodic point. Then the sequence  $\{b_n\}_{n \in \mathbb{N}_0}$  has no Hyers-Ulam stability.

*Proof.* For any  $a_0 \in \mathbb{R}$  and  $\varepsilon > 0$ , choose the sequence  $\{a_n\}_{n \in \mathbb{N}_0}$  as follows

- (1)  $a_0 \in \mathbb{R}$  is arbitrary,
- (2)  $a_1$  satisfies that  $|F(a_0) - a_1| \leq \varepsilon$  and  $a_1 \in A$ , that is, the sequence  $\{F^n(a_1)\}_{n \in \mathbb{N}_0}$  is dense in  $\mathbb{R}$ .

Let  $n_k$  for  $k \geq 1$  be the positive numbers  $n_1 < n_2 < \dots < n_k < \dots$  such that  $F^{n_k}(a_1)$  are points in the ball of which center is  $a_1$  and diameter is  $\varepsilon$ .

- (1)  $a_{n+1} = F^n(a_1)$  for  $n = 0, 1, 2, \dots, n_1 - 1$ ,
- (2)  $a_{n_1+1} = a_1$  and  $a_k = a_{k+n_1+1}$  for every  $k \in \mathbb{N}$ .

Then the sequence  $\{a_n\}_{n \in \mathbb{N}_0}$  satisfies that  $|a_{n+1} - F(a_n)| \leq \varepsilon$  for all  $n \in \mathbb{N}_0$ . Moreover, since the sequence  $\{a_n\}_{n \in \mathbb{N}}$  is periodic,  $\{a_n\}_{n \in \mathbb{N}_0}$  is the finite set and it is bounded. However, the fact that the sequence  $\{b_n\}_{n \in \mathbb{N}_0}$  is dense in  $\mathbb{R}$  implies that  $|a_n - b_n|$  is unbounded for  $n \in \mathbb{N}$ . Hence, the sequence  $\{b_n\}_{n \in \mathbb{N}_0}$  does not have Hyers-Ulam stability. •

**Remark 2.2.** Theorem 2.1 can be generalized to any metric space only if the definition of Hyers-Ulam stability is modified suitably. For instance, if  $F$  is the map from the metric space  $X$  to itself and  $|\cdot|$  is changed to the distance  $\text{dist}(\cdot, \cdot)$  from the metric on  $X$ , then we can define Hyers-Ulam stability on the metric space and Theorem 2.1 is applied to it. For example, the unit circle  $\mathbb{S}^1$  is the metric space of which distance between two points defined from the minimal arc length connecting these two points. Then Hyers-Ulam stability on  $\mathbb{S}^1$  can be defined.

## 3. Real elliptic Möbius transformation

**Lemma 3.1** Let  $g(x) = \frac{ax+b}{cx+d}$  be the linear fractional map where  $a, b, c$  and  $d$  are real numbers,  $c \neq 0$  and  $ad - bc = 1$ . Then  $g$  has fixed points which are non-real complex numbers if and only if  $g$  is real elliptic Möbius map, that is,  $-2 < a + d < 2$ .

*Proof.* The equation  $g(x) = x$  implies that

$$x = \frac{a - d \pm \sqrt{(a + d)^2 - 4}}{2c}.$$

Then the fixed points are non-real complex numbers if and only if  $-2 < a + d < 2$ . •

**Lemma 3.2** Let  $g$  be the map defined on  $\hat{\mathbb{C}}$  in Lemma 3.1 and two complex numbers  $\alpha$  and its complex conjugate  $\bar{\alpha}$  be the fixed points of  $g$ . Let  $h$  be the map defined as  $h(x) = \frac{x-\alpha}{x-\bar{\alpha}}$ . If  $g$  is the elliptic Möbius map, that is,  $-2 < a + d < 2$ , then

$$h \circ g \circ h^{-1}(x) = \frac{x}{(c\alpha + d)^2}.$$

for  $x \in \hat{\mathbb{C}}$ . Moreover,  $|g'(\alpha)| = |c\alpha + d| = 1$ .

*Proof.* The map  $h \circ g \circ h^{-1}$  has the fixed points 0 and  $\infty$ . Since both  $g$  and  $h$  are linear fractional maps, so is  $h \circ g \circ h^{-1}$ . Then  $h \circ g \circ h^{-1}(x) = kx$  for some  $k \in \mathbb{C}$ . The equation  $h \circ g(x) = kh(x)$  implies that  $h'(g(x))g'(x) = kh'(x)$ . Thus

$$h'(g(\alpha))g'(\alpha) = h'(\alpha)g'(\alpha) = kh'(\alpha)$$

Then  $k = g'(\alpha) = \frac{1}{(c\alpha + d)^2}$ . Moreover,  $|g'(\alpha)| = 1$  if and only if  $|c\alpha + d| = 1$ . Recall that both  $\alpha$  and  $\bar{\alpha}$  are roots of the equation,  $cx^2 - (a - d)x - b = 0$ . Then

$$\begin{aligned} |c\alpha + d|^2 &= (c\alpha + d)(c\bar{\alpha} + d) \\ &= c^2\alpha\bar{\alpha} + cd(\alpha + \bar{\alpha}) + d^2 \\ &= c^2\left(-\frac{b}{c}\right) + cd\frac{a-d}{c} + d^2 \\ &= -bc + ad - d^2 + d^2 \\ &= ad - bc \\ &= 1. \end{aligned}$$

Hence,  $|g'(\alpha)| = \frac{1}{|c\alpha + d|^2} = 1.$  •

By Lemma 3.2, the map  $h \circ g \circ h^{-1}$  is a rotation on  $\mathbb{S}^1$ . Since  $h^{-1}$  is bijective from  $\mathbb{S}^1 \setminus \{1\}$  to  $\mathbb{R}$ ,  $x$  is a periodic point under  $h \circ g \circ h^{-1}$  in  $\mathbb{S}^1 \setminus \{1\}$  with period  $p$  if and only if  $h^{-1}(x)$  is periodic in  $\mathbb{R}$  with the same period. Recall that  $h^{-1}(1) = \infty$ . Thus when we investigate Hyers-Ulam stability of the sequence  $\{b_n\}_{n \in \mathbb{N}_0}$  as the solution of the elliptic linear fractional map  $g$ , we have to choose carefully the initial point  $b_0 \in \mathbb{R}$  satisfying  $g^k(b_0) \neq \infty$  for all  $k \in \mathbb{N}$ .

**Proposition 3.3.** *Let  $g$  be the elliptic linear fractional map defined in Lemma 3.1 on  $\hat{\mathbb{R}}$ . Suppose that there exists  $x \in \mathbb{R}$  such that  $g^k(x) \neq x$  for all  $k \in \mathbb{N}$ . Then the sequence  $\{g^k(x)\}_{k \in \mathbb{N}}$  is dense in  $\mathbb{R}$  where  $x \in \mathbb{R} \setminus \{g^{-k}(\infty)\}_{k \in \mathbb{N}}$ .*

*Proof.* Lemma 3.2 implies that  $h \circ g \circ h^{-1}(x) = e^{i\theta}x$  for some  $\theta \in \mathbb{R}$ . If  $2\pi\theta$  is a rational number  $\frac{q}{p}$ , then  $h \circ g^p \circ h^{-1}(x) = x$  for all  $x \in \mathbb{C}$ . Thus  $g^p(x) = x$  for all  $x \in \mathbb{C}$ . Then  $2\pi\theta$  is an irrational number. Since  $x \mapsto e^{i\theta}x$  is an irrational rotation on  $\mathbb{S}^1$ , the sequence  $\{e^{ik\theta}x\}_{k \in \mathbb{N}}$  is dense in  $\mathbb{S}^1$  for every  $x \in \mathbb{S}^1$ . Moreover, when  $x = 1$  is chosen, the set  $\mathbb{S}^1 \setminus \{e^{-ik\theta}\}_{k \in \mathbb{N}_0}$  is also a dense subset of  $\mathbb{S}^1$ .

The direct calculation implies that  $h^{-1}(x) = \frac{\bar{\alpha}x - \alpha}{x - 1}$  and then  $h^{-1}(1) = \infty$ . Choose two points  $p$  and  $p'$  close enough to each other in  $\mathbb{S}^1 \setminus \{e^{-ik\theta}\}_{k \in \mathbb{N}_0}$ . Then

$$\begin{aligned} |h^{-1}(p) - h^{-1}(p')| &= \left| \frac{\bar{\alpha}p - \alpha}{p - 1} - \frac{\bar{\alpha}p' - \alpha}{p' - 1} \right| \\ &= \left| \frac{(\bar{\alpha}p - \alpha)(p' - 1) - (\bar{\alpha}p' - \alpha)(p - 1)}{(p - 1)(p' - 1)} \right| \\ &= \left| \frac{(\alpha - \bar{\alpha})(p - p')}{(p - 1)(p' - 1)} \right| \\ &= \left| \frac{\alpha - \bar{\alpha}}{(p - 1)(p' - 1)} \right| \cdot |p - p'|. \end{aligned} \tag{3.1}$$

We choose the sequence  $\{p_{n_k}\}_{k \in \mathbb{N}_0}$  for some  $p_0 \in \mathbb{S}^1 \setminus \{e^{-ik\theta}\}_{k \in \mathbb{N}_0}$  which satisfies that  $h \circ g^{n_k} \circ h^{-1}(p_0) = p_{n_k}$  and  $p_{n_k} \rightarrow p$  as  $k \rightarrow \infty$  for different numbers  $n_1 < n_2 < \dots < n_k < \dots$ . Since  $h^{-1}$  is a bijection from  $\mathbb{S}^1 \setminus \{1\}$  to  $\mathbb{R}$ , by the Equation (3.1) we obtain that

$$|h^{-1}(p) - g^{n_k} \circ h^{-1}(p_0)| = |h^{-1}(p) - h^{-1}(p_{n_k})|$$

$$\leq \left( \left| \frac{\alpha - \bar{\alpha}}{(\rho - 1)^2} \right| + \delta \right) |p - p_{n_k}|$$

for some  $\delta > 0$ . Then any point  $h^{-1}(p)$  in  $\mathbb{R}$  is an accumulation point in the sequence  $\{g^k(x)\}_{k \in \mathbb{N}_0}$  where  $x \in h^{-1}(\mathbb{S}^1 \setminus \{e^{-ik\theta}\}_{k \in \mathbb{N}_0})$ , which is a dense subset of  $\mathbb{R}$ .

Theorem 2.1 and Proposition 3.3 imply that the real elliptic linear fractional map does not have Hyers-Ulam stability.

**Corollary 3.4** Let  $g(x) = \frac{ax+b}{cx+d}$  be the linear fractional map on  $\hat{\mathbb{R}}$  where  $a, b, c$  and  $d$  are real numbers,  $c \neq 0$  and  $ad - bc = 1$ . Suppose that  $-2 < a + d < 2$  and  $h \circ g \circ h^{-1}$  is an irrational rotation on the unit circle where  $h(x) = \frac{x-a}{x-a}$ . If the sequence  $\{b_n\}_{n \in \mathbb{N}_0}$  in  $\mathbb{R}$  is the solution of  $b_{n+1} = g(b_n)$  for  $n \in \mathbb{N}_0$ , then either  $g^k(b_0) = \infty$  for some  $k \in \mathbb{N}$  or  $\{b_n\}_{n \in \mathbb{N}_0}$  has no Hyers-Ulam stability.

**Example 3.5** The linear fractional map  $g$  is as follows

$$g(x) = \frac{x-4}{4x-3} = \frac{\frac{1}{\sqrt{13}}x - \frac{4}{\sqrt{13}}}{\frac{4}{\sqrt{13}}x - \frac{3}{\sqrt{13}}}$$

Since number  $\frac{1}{\sqrt{13}} - \frac{3}{\sqrt{13}} = -\frac{2}{\sqrt{13}}$  is between  $-2$  and  $2$ , the map  $g$  is a real elliptic Möbius map. Moreover, the fixed points of  $g$  are  $\frac{1 \pm \sqrt{3}i}{2}$ . Define  $h$  as the map  $h(x) = \frac{x - \frac{1+\sqrt{3}i}{2}}{x - \frac{1-\sqrt{3}i}{2}}$ . Then  $h \circ g \circ h^{-1}(x) = \frac{-11+4\sqrt{13}i}{13}x$ . If we denote  $\frac{-11+4\sqrt{13}i}{13}$  by  $e^{i\theta}$ , then  $\theta = \cos^{-1}(\frac{11}{13})$ . Then  $2\pi\theta$  is an irrational number. Then for any  $x \in \mathbb{R}$ , either  $g^n(x) = \infty$  for some  $n \in \mathbb{N}$  or the sequence  $\{g^k(x)\}_{k \in \mathbb{N}}$  is dense in  $\mathbb{R}$  by Proposition 3.3

#### 4. Extension of non stability to complex plane

In this section, we extend no Hyers-Ulam stability of the real elliptic linear fractional map to the elliptic Möbius transformation with complex coefficients. Let  $\ell$  be the straight line in the complex plane. Define the extended line as  $\ell \cup \{\infty\}$  and denote it by  $\hat{\ell}$ . Interior of the circle  $C$  in  $\mathbb{C}$  means that the bounded region of the set  $\mathbb{C} \setminus C$ .

**Lemma 4.1.** Let  $f(z) = \frac{az+b}{cz+d}$  be the Möbius map with complex coefficients  $a, b, c$  and  $d$  for  $c \neq 0$  and  $ad - bc = 1$ . If  $f$  is elliptic, that is,  $-2 < a + d < 2$ , then there exists the unique extended line which is invariant under  $g$  in  $\hat{\mathbb{C}}$ .

*Proof.* Let  $\alpha$  and  $\beta$  be the fixed points of  $f$ . In particular, the fixed points of  $g$  are as follows

$$\alpha = \frac{a-d + \sqrt{(a+d)^2 - 4}}{2c}, \quad \beta = \frac{a-d - \sqrt{(a+d)^2 - 4}}{2c}$$

Denote the straight line,  $\{z \in \mathbb{C} : |z - \alpha| = |z - \beta|\}$  by  $\ell$  in  $\mathbb{C}$  and denote the extended line  $\ell \cup \{\infty\}$  by  $\hat{\ell}$ . We prove that  $\hat{\ell}$  is the unique invariant extended line under  $f$ .

*Claim:* The points,  $-\frac{d}{c}$  and  $\frac{a}{c}$  are in  $\hat{\ell}$ .

$$\alpha + \frac{d}{c} = \frac{a+d + \sqrt{(a+d)^2 - 4}}{2c}, \quad \beta + \frac{d}{c} = \frac{a+d - \sqrt{(a+d)^2 - 4}}{2c}$$

The fact that  $a + d$  is a real number and  $\sqrt{(a + d)^2 - 4}$  is a purely imaginary number implies that

$$\left| \alpha + \frac{d}{c} \right| = \left| \beta + \frac{d}{c} \right| = \frac{1}{|c|}. \tag{4.1}$$

Then  $-\frac{d}{c}$  is in  $\hat{\ell}$ . By similar calculation,  $\frac{a}{c}$  is also in  $\hat{\ell}$ . The proof the claim is complete.

For the invariance of  $\hat{\ell}$ , it suffice to show that if  $z \in \ell \setminus \{-\frac{d}{c}\}$ , then  $f(z) \in \ell$ . For any  $z \in \ell \setminus \{-\frac{d}{c}\}$ , we have that

$$f(z) - \alpha = f(z) - f(\alpha) = \frac{az + b}{cz + d} - \frac{a\alpha + b}{c\alpha + d} = \frac{(ad - bc)(z - \alpha)}{(cz + d)(c\alpha + d)} = \frac{z - \alpha}{(cz + d)(c\alpha + d)} \tag{4.2}$$

By similar calculation, we obtain that

$$f(z) - \beta = \frac{z - \beta}{(cz + d)(c\beta + d)}. \tag{4.3}$$

The Equation (4.1) in the claim implies that  $|c\alpha + d| = |c\beta + d|$ . The definition of  $\ell$  implies that  $cz + d \neq 0$  for  $z \in \ell \setminus \{-\frac{d}{c}\}$  and  $|z - \alpha| = |z - \beta|$ . The equation  $|f(z) - \alpha| = |f(z) - \beta|$  holds by comparing the Equation (4.2) with (4.3). Hence, by this result and the above claim, the extended line  $\hat{\ell}$  is invariant under  $f$ . There is the unique straight line which connects  $-\frac{d}{c}$  and  $\frac{a}{c}$  in  $\mathbb{C}$ . Hence,  $\hat{\ell}$  is the unique invariant extended line in  $\hat{\mathbb{C}}$  under  $f$ . •

**Remark 4.2.** Define the map  $h$  as  $h(z) = \frac{z - \beta}{z - \alpha}$ . Then by the straightforward calculation we obtain  $h \circ f \circ h^{-1}(z) = f'(z)z$  and  $|f'(z)| = 1$ . Let  $C_r$  be the circle of radius  $r > 0$  of which center is the origin in  $\mathbb{C}$ . Observe that a fixed point of  $f$  is contained in the interior of  $h^{-1}(C_r)$  for all  $r > 0$ . If  $h \circ f \circ h^{-1}$  is an irrational rotation in  $\mathbb{C}$ , then every concentric circles  $C_r$  are invariant under  $h \circ f \circ h^{-1}$ . Then  $h^{-1}(C_r)$  is an invariant circle under  $f$  for every  $r > 0$ . However, since  $h(\infty) = 1$ , the unique invariant extended line under  $f$  is  $h^{-1}(\mathbb{S}^1)$ . Moreover,  $h^{-1}(C_{1/r})$  is the reflected image of  $h^{-1}(C_r)$  to the invariant extended line.

Due to the existence of the extended line which is invariant under  $g$ , no Hyers-Ulam stability of the sequence  $\{g^n(z)\}_{n \in \mathbb{N}_0}$  of the real elliptic linear fractional map is extendible to that of the elliptic Möbius transformation.

**Proposition 4.3.** Let  $f$  be the elliptic Möbius transformation on  $\hat{\mathbb{C}}$  and  $\hat{\ell}$  be the extended line invariant under  $f$ . Suppose that there exists  $z \in \hat{\ell}$  such that  $f^k(z) \neq z$  for all  $k \in \mathbb{N}$ . Then the sequence  $\{f^k(z)\}_{k \in \mathbb{N}}$  is dense in  $\ell = \hat{\ell} \setminus \{\infty\}$  where  $z \in \ell \setminus \{f^{-k}(\infty)\}_{k \in \mathbb{N}}$ .

*Proof.* Lemma 4.1 implies that there exists the invariant extended line under  $f$ . In the proof of Proposition 3.3, replace  $\bar{\alpha}$  by  $\beta$  and apply the proof of Proposition 3.3. Then the similar calculation completes the proof. •

Then Proposition 4.3 and Theorem 2.1 implies that no Hyers-Ulam stability of  $\{f^n(z)\}_{n \in \mathbb{N}_0}$  where  $f$  is the elliptic Möbius map for  $z \in \mathbb{C}$ .

**Corollary 4.4.** Let  $f(z) = \frac{az+b}{cz+d}$  be the Möbius map on  $\hat{\mathbb{C}}$  where  $c \neq 0$  and  $ad - bc = 1$ . If  $f$  is elliptic, that is,  $-2 < a + d < 2$ , then the sequence  $\{b_n\}_{n \in \mathbb{N}_0}$  satisfying  $b_{n+1} = f(b_n)$  for  $n \in \mathbb{N}_0$  has no Hyers-Ulam stability on  $\mathbb{C}$ .

*Proof.* The sequence  $\{a_n\}_{n \in \mathbb{N}_0}$  is in the invariant extended line  $\hat{\ell}$  under  $f$  satisfying

$$|a_{n+1} - f(a_n)| \leq \varepsilon$$

for  $n \in \mathbb{N}_0$ . Choose  $\{a_n\}_{n \in \mathbb{N}_0}$  is the finite sequence which satisfies the properties in the proof of Theorem 2.1 in  $\ell$ . Then the similar proof in Section 3 implies that any sequence  $\{b_n\}_{n \in \mathbb{N}_0}$  satisfying  $b_{n+1} = f(b_n)$  for every  $n \in \mathbb{N}_0$  in  $\ell$  has no Hyers-Ulam stability.

Let  $h$  be the map  $h(z) = \frac{z-\alpha}{z-\beta}$ . Observe that  $h \circ f \circ h^{-1}$  is an irrational rotation on  $\mathbb{C}$ . If  $\{b_n\}_{n \in \mathbb{N}_0}$  is contained in  $\mathbb{C} \setminus \hat{\ell}$ , then  $\{b_n\}_{n \in \mathbb{N}_0}$  is contained in the circle disjoint from  $\ell$  and this circle is  $h^{-1}(C_r)$  for some  $r > 0$ . Observe that if  $r = 0$ , then  $C_r = \{\alpha\}$  and if  $r = 1$ , then  $C_r = \hat{\ell}$ . Since  $h \circ f \circ h^{-1}$  is an irrational rotation,  $\{f^n(h^{-1}(x))\}_{n \in \mathbb{N}_0}$  is the dense subset of  $h^{-1}(C_r)$ . We may assume that  $0 < r < 1$  and the fixed point  $\alpha$  is contained in the interior of  $C_r$  by Remark 4.2. Denote the (minimal) distance between the point  $\alpha$  and the line  $\ell$  by  $L$ . Then the density of  $\{b_n\}_{n \in \mathbb{N}_0}$  on the circle  $C_r$  and the finiteness of  $\{a_n\}_{n \in \mathbb{N}_0}$  imply that  $|a_n - b_n| \geq L$  for infinitely many  $n \in \mathbb{N}$  for all  $\varepsilon < L$ . Hence,  $\{b_n\}_{n \in \mathbb{N}_0}$  has no Hyers-Ulam stability. •

### 5. Non stability of periodic sequence

Let the sequence  $\{b_n\}_{n \in \mathbb{N}_0}$  satisfying  $b_{n+p} = b_n$  for every  $n \in \mathbb{N}_0$  for some  $p \in \mathbb{N}$  be *periodic sequence*. The least positive number  $p$  satisfying the above equation is called the *period* of sequence. If  $p = 1$ , then it is called *constant sequence*.

**Lemma 5.1.** *Let the sequence  $\{b_n\}_{n \in \mathbb{N}_0}$  in  $\mathbb{R}$  be a periodic sequence with period  $p$ . Then  $\{b_n\}_{n \in \mathbb{N}_0}$  has no Hyers-Ulam stability.*

*Proof.* Any periodic sequence has constant subsequence. For example, let  $\{c_n\}_{n \in \mathbb{N}_0}$  be the sequence satisfying  $c_n = b_{pn}$  for every  $n \in \mathbb{N}_0$ . Thus  $\{c_n\}_{n \in \mathbb{N}_0}$  is the constant sequence  $\{c_0\}_{n \in \mathbb{N}_0}$ . It suffice to show that the constant sequence has no Hyers-Ulam stability. For any small enough  $\varepsilon > 0$ , define the sequence  $\{d_n\}_{n \in \mathbb{N}_0}$  as follows

- (1)  $d_0$  is arbitrary and
- (2)  $d_n = d_0 + n\varepsilon$  for  $n \in \mathbb{N}$

Then  $|d_{n+1} - d_n| \leq \varepsilon$  for all  $n \in \mathbb{N}_0$ . However, for any constant sequence  $\{c_0\}_{n \in \mathbb{N}_0}$  satisfying  $|c_0 - d_0| \leq \varepsilon$  such that

$$|c_k - d_k| = |c_0 - d_k| = |c_0 - d_0 - n\varepsilon| \geq -|c_0 - d_0| + n\varepsilon \geq (n-1)\varepsilon$$

for all  $n \geq 2$ . Since  $|c_k - d_k|$  is unbounded for  $n \in \mathbb{N}$ , the constant sequence  $\{c_n\}_{n \in \mathbb{N}_0}$  has no Hyers-Ulam stability. Hence, the periodic sequence  $\{b_n\}_{n \in \mathbb{N}_0}$  has no Hyers-Ulam stability either. •

**Example 5.2.** *There are non-linear Möbius transformations, of which finitely many composition is the identity map. For example, see the following maps*

$$p(z) = \frac{\sqrt{3}z - 2}{2z - \sqrt{3}}, \quad q(z) = \frac{-1}{z - \sqrt{3}}, \quad r(z) = \frac{-z - 1}{z}.$$

*Thus  $p \circ p(z) = p^2(z) = z$  for all  $z \in \mathbb{C}$  and the trace of  $p$  is that  $\text{tr}(p) = \sqrt{3} + (-\sqrt{3}) = 0$ . The map  $q$  satisfies that  $q^3 = p$  by the direct calculation and then  $q^6(z) = z$  for all  $z \in \mathbb{C}$ . The trace of  $q$  is that  $\text{tr}(q) = 0 + (-\sqrt{3}) = -\sqrt{3}$ . Finally,  $r^3(z) = z$  for all  $z \in \mathbb{C}$  and  $\text{tr}(r) = -1 + 0 = -1$ . All of the traces of  $p, q$  and  $r$  are between  $-2$  and  $2$ .*

Remark 4.2 implies that  $h \circ g \circ h^{-1}(x) = e^{i\theta}x$  for every elliptic Möbius map  $g$ . If  $\theta = \frac{q}{p}$ , then  $g^p(x) = x$  for all  $x \in \mathbb{C}$ . Thus the sequence  $\{g^n(x)\}_{n \in \mathbb{N}_0}$  is periodic. Then Corollary 3.4 and Lemma 5.1 implies the following theorem.



**Theorem 5.3.** Let  $g(x) = \frac{ax+b}{cx+d}$  be the linear fractional map on  $\hat{\mathbb{R}}$  for  $c \neq 0$ ,  $ad - bc = 1$ . Suppose that  $g$  is the elliptic linear fractional map, that is,  $-2 < a + d < 2$ . Then the sequence  $\{b_n\}_{n \in \mathbb{N}_0}$  in  $\mathbb{R}$  satisfying  $b_{n+1} = g(b_n)$  for  $n \in \mathbb{N}_0$  either satisfies that  $g^k(b_0) = \infty$  for some  $k \in \mathbb{N}$  or it has no Hyers-Ulam stability.

If the sequence  $\{f^n(z)\}_{n \in \mathbb{N}_0}$  is periodic, then Corollary 4.4 and Lemma 5.1 implies the following theorem.

**Theorem 5.4.** Let  $f(z) = \frac{az+b}{cz+d}$  be the Möbius map on  $\hat{\mathbb{C}}$  for  $ad - bc = 1$ ,  $c \neq 0$ . Suppose that  $f$  is the elliptic Möbius map, that is,  $-2 < a + d < 2$ . Then the sequence  $\{b_n\}_{n \in \mathbb{N}_0}$  in  $\mathbb{C}$  satisfying  $b_{n+1} = f(b_n)$  for  $n \in \mathbb{N}_0$  either satisfies that  $g^k(b_0) = \infty$  for some  $k \in \mathbb{N}$  or it has no Hyers-Ulam stability.

## 6. Conclusion and further research

In this paper, we show that the difference equation from the linear fractional map of elliptic type has no Hyers-Ulam stability. Using conjugation, this type of linear fractional map is actually a kind of rotation around the fixed points. The irrational rotation has dense orbits on every invariant circles and the rational rotation has all points which is periodic. Any of both maps does not have Hyers-Ulam stability. In the future, we investigate Hyers-Ulam stability of the difference equation from the linear fractional map of other types—parabolic, hyperbolic and loxodromic equations.

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