cogent mathematics & statistics

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Abuage Mariam and Kilicman Adem

Publisher: Cogent OA

Journal: *Cogent Mathematics & Statistics*

DOI: http://dx.doi.org/10.1080/25742558.2018.1479218

Functions and $w\nu$-lindelöf with respect to a hereditary class H

Mariam Abuage [1] **and A. Kiliçman** [2]

[1] *Institute for Mathematical Research, University Putra Malaysia, 43400 UPM, Serdang, Selangor, Malaysia*
[2] *Department of Mathematics, University Putra Malaysia, 43400 UPM, Serdang, Selangor, Malaysia*

Abstract

A collection $\mathcal{H}$ of a nonempty subsets of $\mathcal{X}_{\mathcal{G}}$ is called hereditary class if it is closed under hereditary property. In this work, we define and study the notion of some generalizations of ν-Lindelöf generalized topological spaces with respect to a hereditary class $\mathcal{H}$, namely; $w\nu\mathcal{H}$-Lindelöf hereditary generalized topological spaces. Moreover, investigate basic properties of the concepts, its relation to known concepts and its preservation by functions properties.

2010 Mathematics Subject Classifications: 54A05, 54A08, 54D20.
Key words: Generalized topological spaces, $w\nu$-Lindelöf, generalized continuous function.

1 About the Author

Mariam M. Abuage is a lecturer at the Department of Mathematics, Fuculty of Science Sabrata University/ Libya. She received her B.Sc. degree in 2004 from the Department of Mathematics, Al Zawia University/ Libya, and obtained her M.Sc. degree in 2010 from Al Zawia University. In August 2014, she registered as a post-graduate student at Institute for Mathematical Research University Putra Malaysia UPM, to pursue her Doctor of Philosophy (PhD) in field of Pure Mathematics (General Topology). Her research interest involves with generalizations of ν-Lindelöf in generalized topological spaces.

2 Public Interest Statement

Lindelöfness and its generalizations are important and interesting concepts in topology. Further, Lindelöf and its generalizations have been done to generalized topological spaces, the earlier generalizations to generalized covering properties is ν-Lindelöf. Recently, the concept of ν-Lindelöfness with respect to hereditary class has been introduced. This paper will introduce and define one of generalizations of ν-Lindelöf; namely, $w\nu$-Lindelöf with respect to hereditary class. Some properties and counterexamples are showed. Functions properties are investigated, and we proved that the image of a $w\nu\mathcal{H}$-Lindelöf under an almost (ν,μ)-continuous function is $w\mu\mathcal{H}$-Lindelöf.

3 Introduction and preliminaries

A lot of attention has been made to study properties of covering in topological spaces, which include open and different kind of generalized open sets. Further, several authors have introduced the generalization of Lindelöf space separately for many reasons and according to the sets that they are interested in. In this work, we use the notions of generalized topology and hereditary classes introduced by (Császár, 2002), (Császár, 2005) and (Császár, 2006), respectively. in order to define some of generalizations of $\nu\mathcal{H}$-Lindelöf (Qahis, AlJarrah, Noiri, et al., 2016), namely; $w\nu\mathcal{H}$-Lindelöf hereditary generalized topological spaces. In literature, (Sarsak, 2012) introduced and studied ν-Lindelöf sets in generalized topological spaces. Recently, (Abuage & Kiliçman, 2017) introduced $w\nu$-Lindelöf generalized topological spaces. The notion ν-Lindelöfness in term of a hereditary classes was studied by (Qahis et al., 2016).
The strategy of using generalized topologies and hereditary classes to extend classical topological concepts have been used by many authors such as (Császár, 2006; Zahran, El-Saady, & Ghareeb, 2012; Kim & Min, 2012; Ramasamy, Rajamani, & Inthumathi, 2012), among others. Realization of generalized continuous function was introduced by (Császár, 2002), interesting types of functions in generalized topological spaces have been introduced by many mathematicians, such as ((Al-Omari & Noiri, 2012), (Al-Omari & Noiri, 2013)), ((Min, 2009), (Min, 2010a), (Min, 2010b)). The purpose of this paper is to study the effect of functions on $w\nu\mathcal{H}$-Lindelöf generalized topological spaces. We also show that some functions preserve this property. The main result is that the image of a $w\nu\mathcal{H}$-Lindelöf under an almost (ν,μ)-continuous function is $w\mu\mathcal{H}$-Lindelöf.

Suppose a non-empty set $\mathcal{X}_{\mathcal{G}}$, $P(\mathcal{X}_{\mathcal{G}})$ denotes the power set of $\mathcal{X}_{\mathcal{G}}$ and ν be a non-empty family of $P(\mathcal{X}_{\mathcal{G}})$. The symbol ν implies a generalized topology (briefly. $\mathcal{GT}$) on $\mathcal{X}_{\mathcal{G}}$ (Császár, 2002) if the empty set $\varnothing\in\nu$ and $\mathcal{U}_\gamma\in\nu$ where $\gamma\in\Omega$ implies $\bigcup_{\gamma\in\Omega}\mathcal{U}_\gamma\in\nu$. The pair $(\mathcal{X}_{\mathcal{G}},\nu)$ is called generalized topological space (briefly. $\mathcal{GTS}$) and we always denote it by $\mathcal{GTS}$ $(\mathcal{X}_{\mathcal{G}},\nu)$ or $\mathcal{X}_{\mathcal{G}}$. Each element of $\mathcal{GT}$ ν is said to be ν-open set and the complement of ν-open set is called ν-closed set. Let $\mathcal{A}$ be a subset of a $\mathcal{GTS}$ $(\mathcal{X}_{\mathcal{G}},\nu)$, then $i_\nu(\mathcal{A})$ (resp. $c_\nu(\mathcal{A})$) denotes the union of all ν-open sets contained in $\mathcal{A}$ (resp. denotes the intersection of all $\nu-$closed sets containing $\mathcal{A}$), and $\mathcal{X}_{\mathcal{G}}\setminus\mathcal{A}$ denotes the complement of $\mathcal{A}$, $c_\nu(\mathcal{X}_{\mathcal{G}}\setminus\mathcal{A})=\mathcal{X}_{\mathcal{G}}\setminus(i_\nu\mathcal{A})$. Moreover, $\mathcal{A}$ is said to be ν-regular open (resp. ν-regular closed) iff $\mathcal{A}=i_\nu c_\nu(\mathcal{A})$ (resp. $\mathcal{A}=c_\nu i_\nu(\mathcal{A})$) (Császár, 2008). If a set $\mathcal{X}_{\mathcal{G}}\in\nu$, then a $\mathcal{GTS}$ $(\mathcal{X}_{\mathcal{G}},\nu)$ is called ν-space (Noiri, 2006), and will be denoted by a ν-space $(\mathcal{X}_{\mathcal{G}},\nu)$ or a ν-space $\mathcal{X}_{\mathcal{G}}$. $\mathcal{X}_{\mathcal{G}}$ is said to be quasi-topological space (Császár, 2006), if the finite intersection of ν-open sets of ν belongs to ν and denoted by $\mathcal{QTS}$ $(\mathcal{X}_{\mathcal{G}},\nu)$. If $\beta\subseteq P(\mathcal{X}_{\mathcal{G}})$ and $\varnothing\in\beta$. Then β is called a ν-base (Császár, n.d.) for ν if $\{\cup\beta^{'}:\beta^{'}\subseteq\beta\}=\nu$, and we say that ν is generated by $\mathcal{B}$. A $\mathcal{GTS}$ $(\mathcal{X}_{\mathcal{G}},\nu)$ is said to be ν-extremally disconnected (Császár, n.d.) if the ν-closure of every ν-open set is ν-open. Moreover, a subset $\mathcal{A}$ of a $\mathcal{GTS}$

$(\mathcal{X}_{\mathcal{G}},\nu)$ is called ν-clopen if it is both ν-open and ν-closed.

Let $(\mathcal{X}_{\mathcal{G}},\nu)$ be a $\mathcal{GTS}$, a cover $\mathfrak{U}$ of a subsets of $\mathcal{X}_{\mathcal{G}}$ is called ν-open cover if the elements of $\mathfrak{U}$ are ν-open subsets of $\mathcal{X}_{\mathcal{G}}$ (Thomas & John, 2012). A $\mathcal{GTS}$ $(\mathcal{X}_{\mathcal{G}},\nu)$ is said to be ν-Lindelöf (Sarsak, 2012) (resp. $w\nu$-Lindelöf (Abuage & Kiliçman, 2017)) if for each ν-open cover $\mathfrak{U}=\{\mathcal{U}_{\gamma}:\gamma\in\Omega\}$ of Λ_{ν} admits a countable sub-collection $\{\mathcal{U}_{\gamma_n}:n\in\mathbb{N}\}$ such that

$$\Lambda_{\nu}=\bigcup_{n\in\mathbb{N}}\mathcal{U}_{\gamma_n}\ (resp.\ \Lambda_{\nu}=c_{\nu}(\bigcup_{n\in\mathbb{N}}\mathcal{U}_{\gamma_n})),$$

where Λ_{ν} is the union of all ν-open set in $\mathcal{X}_{\mathcal{G}}$.

A non-empty family $\mathcal{H}$ of subsets of $\mathcal{X}_{\mathcal{G}}$ is called a hereditary class (Császár, 2006) if $\mathcal{A}\in\mathcal{H}$ and $\mathcal{B}\subset\mathcal{A}$ imply that $\mathcal{B}\in\mathcal{H}$ (Kuratowski., 1933). Given a generalized topological space $(\mathcal{X}_{\mathcal{G}},\nu)$ with a hereditary class $\mathcal{H}$, for a subset $\mathcal{A}$ of $\mathcal{X}_{\mathcal{G}}$, the generalized local function of $\mathcal{A}$ with respect to $\mathcal{H}$ and ν (Császár, 2006) is defined as follows: $\mathcal{A}^{\square}=\{x\in\mathcal{X}_{\mathcal{G}}\quad \mathcal{U}\cap\mathcal{A}\notin\mathcal{H}\text{ for all }\mathcal{U}\in\nu_x\}$, where $\nu_x=\{\mathcal{U}:x\in\mathcal{U}\text{ and }\mathcal{U}\in\nu\}$; and the following are defined: $c_{\nu}^{*}=\mathcal{A}\cup\mathcal{A}^{*}$ and the family $\nu^{*}=\{\mathcal{A}\subset\mathcal{X}_{\mathcal{G}}:\mathcal{X}_{\mathcal{G}}\setminus\mathcal{A}=c_{\nu}^{*}(\mathcal{X}_{\mathcal{G}}\setminus\mathcal{A})\}$ is a $\mathcal{GT}$ on $\mathcal{X}_{\mathcal{G}}$. The elements of ν^{*} are called ν^{*}-open and the complement of a ν^{*}-open set is called ν^{*}-closed set. It is clear that a subset $\mathcal{A}$ is ν^{*}-closed if and only if $\mathcal{A}^{\square}\subset\mathcal{A}$. If the hereditary class $\mathcal{H}$ satisfies the additional condition: if $\mathcal{A},\mathcal{B}\in\mathcal{H}$ implies $\mathcal{A}\cup\mathcal{B}\in\mathcal{H}$, then $\mathcal{H}$ is called an ideal on $\mathcal{X}_{\mathcal{G}}$ (Kuratowski., 1933). We call $(\mathcal{X}_{\mathcal{G}},\nu,\mathcal{H})$ a hereditary generalized topological space and denoted by $\mathcal{HGTS}$ $\mathcal{X}_{\mathcal{G}}$ or simply $\mathcal{X}_{\mathcal{G}}$. Let a $\mathcal{GTS}$ $(\mathcal{X}_{\mathcal{G}},\nu)$, we denoted by $\mathcal{H}_{\mathbf{c}}$ the hereditary class of countable subsets of $\mathcal{X}_{\mathcal{G}}$.

Definition 3.1 *(Sarsak, 2012) Let* $(\mathcal{X}_{\mathcal{G}},\nu)$ *and* $\mathcal{A}\subseteq\mathcal{X}_{\mathcal{G}}$. *Then a collection* $\{\mathcal{U}\cap\mathcal{A}:\mathcal{U}\in\nu\}$ *is said to be generalized topology on* $\mathcal{A}$, *and denote by* $\nu(\mathcal{A})$. *A* $\mathcal{GT}$ $\nu(\mathcal{A})$ *on* $\mathcal{A}$ *forms a generalized topological subspace of* $\mathcal{X}_{\mathcal{G}}$, *denoted by* $(\mathcal{A},\nu(\mathcal{A}))$.

Let $(\mathcal{X}_{\mathcal{G}},\nu,\mathcal{H})$ be a $\mathcal{HGTS}$ and $\mathcal{A}\subseteq\mathcal{X}_{\mathcal{G}}$, $\mathcal{A}\neq\varnothing$. We denoted by $\mathcal{H}_{\mathcal{A}}$ the collection $\{H\cap(\mathcal{A}\cap\Lambda_{\nu}):H\in\mathcal{H}\}$ and by $(\mathcal{A},\nu(\mathcal{A}))$ the subspace of $(\mathcal{X}_{\mathcal{G}},\nu)$ on $\mathcal{A}$.

Definition 3.2 *(Qahis et al., 2016) Let* $(\mathcal{X}_{\mathcal{G}},\nu)$ *be a* $\mathcal{GTS}$ *and* $\mathcal{H}$ *be a hereditary class on* $\mathcal{X}_{\mathcal{G}}$. *A* $\mathcal{HGTS}$ $(\mathcal{X}_{\mathcal{G}},\nu,\mathcal{H})$ *is called* $\nu\mathcal{H}$*-Lindelöf or* ν*-Lindelöf respect to a hereditary class on* $\mathcal{X}_{\mathcal{G}}$ *if each* ν*-open cover* $\{\mathcal{U}_{\gamma}:\gamma\in\Omega\}$ *of* Λ_{ν} *has a countable subcollection* $\{\mathcal{U}_{\gamma_n}:n\in\mathbb{N}\}$ *such that* $\Lambda_{\nu}\setminus\bigcup_{n\in\mathbb{N}}\mathcal{U}_{\gamma_n}\in\mathcal{H}$.

Lemma 3.3 *(Császár, 2008)*

(a) If $\mathcal{F}$ is ν-closed set then $i_{\nu}(\mathcal{F})$ is ν-regular open.

(b) If $\mathcal{U}$ is ν-open set then $c_{\nu}(\mathcal{U})$ is ν-regular closed.

Theorem 3.4 *(Császár, 2006) Let* $(\mathcal{X}_{\mathcal{G}},\nu)$ *be a* $\mathcal{GTS}$ *and* $\mathcal{H}$ *be a hereditary class on* $\mathcal{X}_{\mathcal{G}}$

(i) A $\mathcal{GT}$ ν^{*} finer than ν,

(ii) If $\mathcal{A}$ be a subset of $\mathcal{X}_{\mathcal{G}}$, then $\mathcal{A}^* \subseteq c_\nu(\mathcal{A})$.

Theorem 3.5 *(Császár, 2006) Let $(\mathcal{X}_{\mathcal{G}}, \nu)$ be a $\mathcal{GTS}$ and $\mathcal{H}$ be a hereditary class on $\mathcal{X}_{\mathcal{G}}$ and $\mathcal{U}$ be a subset of $\mathcal{X}_{\mathcal{G}}$, If $\mathcal{U}$ is ν^*-open, then for each $x \in \mathcal{U}^*$ there is $\mathcal{U} \in \nu_x$ and $H \in \mathcal{H}$ such that $x \in \mathcal{U} \setminus H \subset \mathcal{U}^*$.*

Lemma 3.6 *(Carpintero, Rosas, Salas-Brown, & Sanabria, 2016) Let a function $g : (\mathcal{X}_{\mathcal{G}}, \nu) \to (\mathcal{Y}_{\mathcal{G}}, \mu)$. If $\mathcal{H}$ is a hereditary class on $\mathcal{X}_{\mathcal{G}}$, then $g(\mathcal{H}) = \{g(H) : H \in \mathcal{H}\}$ is a hereditary class on $\mathcal{Y}_{\mathcal{G}}$.*

4 $w\nu$-Lindelöf with respect to a hereditary class $\mathcal{H}$

The following concepts give a characterization of $w\nu\mathcal{H}$-Lindelöf.

Definition 4.1 *Let $(\mathcal{X}_{\mathcal{G}}, \nu)$ be a $\mathcal{GTS}$ and $\mathcal{H}$ be a hereditary class on $\mathcal{X}_{\mathcal{G}}$. A $\mathcal{HGTS}$ $(\mathcal{X}_{\mathcal{G}}, \nu, \mathcal{H})$ is said to be $w\nu\mathcal{H}$-Lindelöf or $n\nu$-Lindelöf respect to a hereditary class on $\mathcal{X}_{\mathcal{G}}$ if each ν-open cover $\{\mathcal{U}_\gamma : \gamma \in \Omega\}$ of Λ_ν has a countable subcollection $\{\mathcal{U}_{\gamma_n} : n \in \mathbb{N}\}$ such that*

$$\Lambda_\nu \setminus (c_\nu(\bigcup_{n\in\mathbb{N}} \mathcal{U}_{\gamma_n})) \in \mathcal{H}.$$

Proposition 4.1 *A $\mathcal{HGTS}$ $(\mathcal{X}_{\mathcal{G}}, \nu, \mathcal{H})$ is $w\nu\mathcal{H}$-Lindelöf if and only if every collection $\{\mathcal{F}_\gamma : \gamma \in \Omega\}$ of ν-closed sets of $\mathcal{X}_{\mathcal{G}}$ such that $(\bigcap_{\gamma\in\Omega} \mathcal{F}_\gamma) \cap \Lambda_\nu = \varnothing$ admits a countable sub-collection $\{\mathcal{F}_{\gamma_n} : n \in \mathbb{N}\}$ such that $i_\nu(\bigcap_{n\in\mathbb{N}} \mathcal{F}_{\gamma_n}) \cap \Lambda_\nu \in \mathcal{H}$.*

Proof. Necessity. Let $\{\mathcal{F}_\gamma : \gamma \in \Omega\}$ be a collection of ν-closed sets of $\mathcal{X}_{\mathcal{G}}$ such that $(\bigcap_{\gamma\in\Omega} \mathcal{F}_\gamma) \cap \Lambda_\nu = \varnothing$. Then $\Lambda_\nu = \mathcal{X}_{\mathcal{G}} \setminus (\bigcap_{\gamma\in\Omega} \mathcal{F}_\gamma) = \bigcup_{\gamma\in\Omega} (\mathcal{X}_{\mathcal{G}} \setminus \mathcal{F}_\gamma)$, i.e., the collection $\{\mathcal{X}_{\mathcal{G}} \setminus \mathcal{F}_\gamma : \gamma \in \Omega\}$ is a ν-open cover of Λ_ν. Since $\mathcal{X}_{\mathcal{G}}$ is $w\nu\mathcal{H}$-Lindelöf, there is a countable sub-collection $\{\mathcal{X}_{\mathcal{G}} \setminus \mathcal{F}_{\gamma_n} : n \in \mathbb{N}\}$ such that

$$\Lambda_\nu \setminus (c_\nu \bigcup_{n\in\mathbb{N}} (\mathcal{X}_{\mathcal{G}} \setminus \mathcal{F}_{\gamma_n}))) \in \mathcal{H}.$$

Thus

$$\Lambda_\nu \setminus (c_\nu \bigcup_{n\in\mathbb{N}} (\mathcal{X}_{\mathcal{G}} \setminus \mathcal{F}_{\gamma_n})) = \Lambda_\nu \setminus (c_\nu (\mathcal{X}_{\mathcal{G}} \setminus (\bigcap_{n\in\mathbb{N}} \mathcal{F}_{\gamma_n}))) = \Lambda_\nu \setminus (\mathcal{X}_{\mathcal{G}} \setminus i_\nu (\bigcap_{n\in\mathbb{N}} \mathcal{F}_{\gamma_n})) \in \mathcal{H}.$$

It is obviously to show that;

$$\Lambda_\nu \cap i_\nu (\bigcap_{n\in\mathbb{N}} \mathcal{F}_{\gamma_n}) = \Lambda_\nu \setminus (\mathcal{X}_{\mathcal{G}} \setminus i_\nu (\bigcap_{n\in\mathbb{N}} \mathcal{F}_{\gamma_n})) \in \mathcal{H}.$$

Sufficiency, Suppose $\{\mathcal{U}_\gamma : \gamma \in \Omega\}$ be a ν-open cover of Λ_ν, then $\Lambda_\nu = \bigcup_{\gamma\in\Omega}\mathcal{U}_\gamma$ and $\{\mathcal{X}_\mathcal{G} \setminus \mathcal{U}_\gamma : \gamma \in \Omega\}$ is a collection of ν-closed sets of $\mathcal{X}_\mathcal{G}$. Thus $(\mathcal{X}_\mathcal{G} \setminus \bigcup_{\gamma\in\Omega}\mathcal{U}_\gamma) \cap \Lambda_\nu = \varnothing$, i.e., $\bigcap_{\gamma\in\Omega}(\mathcal{X}_\mathcal{G} \setminus \mathcal{U}_\gamma) \cap \Lambda_\nu = \varnothing$. By hypothesis, there is a countable sub-collection $\{\mathcal{X}_\mathcal{G} \setminus \mathcal{U}_{\gamma_n} : n \in \mathbb{N}\}$ such that $i_\nu(\bigcap_{n\in\mathbb{N}}(\mathcal{X}_\mathcal{G} \setminus \mathcal{U}_{\gamma_n})) \cap \Lambda_\nu \in \mathcal{H}$. Since,

$$i_\nu(\bigcap_{n\in\mathbb{N}}(\mathcal{X}_\mathcal{G} \setminus \mathcal{U}_{\gamma_n})) \cap \Lambda_\nu = \Lambda_\nu \setminus (\mathcal{X}_\mathcal{G} \setminus (i_\nu(\bigcap_{n\in\mathbb{N}}(\mathcal{X}_\mathcal{G} \setminus \mathcal{U}_{\gamma_n})))).$$

Then,

$$\Lambda_\nu \setminus (\mathcal{X}_\mathcal{G} \setminus (i_\nu(\bigcap_{n\in\mathbb{N}}(\mathcal{X}_\mathcal{G} \setminus \mathcal{U}_{\gamma_n}))) = \Lambda_\nu \setminus (c_\nu(\bigcup_{n\in\mathbb{N}}(\mathcal{U}_{\gamma_n})) \in \mathcal{H}.$$

Which implies that a $\mathcal{HGTS}$ $(\mathcal{X}_\mathcal{G}, \nu, \mathcal{H})$ is a $w\nu\mathcal{H}$-Lindelöf.

Proposition 4.2 *Let $(\mathcal{X}_\mathcal{G}, \nu)$ be a $\mathcal{GTS}$ with a hereditary class $\mathcal{H}$, then $(\mathcal{X}_\mathcal{G}, \nu)$ is $w\nu$-Lindelöf if and only if $(\mathcal{X}_\mathcal{G}, \nu, \mathcal{H}_\mathbf{c})$ is $w\nu\mathcal{H}_\mathbf{c}$-Lindelöf $\mathcal{HGTS}$.*

Proof. The necessity is obvious. Sufficiency, suppose $(\mathcal{X}_\mathcal{G}, \nu, \mathcal{H}_\mathbf{c})$ is $w\nu\mathcal{H}_\mathbf{c}$-Lindelöf $\mathcal{HGTS}$. Let $\{\mathcal{U}_\gamma : \gamma \in \Omega\}$ be a ν-open cover of Λ_ν. Then by hypothesis, there is a countable sub-collection $\{\mathcal{U}_n : n \in \mathbb{N}\}$ such that

$$\Lambda_\nu \setminus (c_\nu(\bigcup_{n\in\mathbb{N}}\mathcal{U}_n)) \in \mathcal{H}_\mathbf{c}.$$

Assume, $\Lambda_\nu \setminus (c_\nu(\bigcup_{n\in\mathbb{N}}\mathcal{U}_n)) = \{x_i : i \in \mathbb{N}\}$, pick out $\mathcal{U}_{\gamma_i}$ such that $x_i \in \mathcal{U}_{\gamma_i}$ for each $i \in \mathbb{N}$. Thus,

$$\Lambda_\nu = (c_\nu(\bigcup_{n\in\mathbb{N}}\mathcal{U}_n)) \cup (c_\nu(\bigcup_{i\in\mathbb{N}}\mathcal{U}_{\gamma_i})).$$

Which implies that a $\mathcal{X}_\mathcal{G}$ is $w\nu$-Lindelöf $\mathcal{GTS}$.

By Proposition above, it is clear that a $\mathcal{GTS}$ $(\mathcal{X}_\mathcal{G}, \nu)$ is $w\nu$-Lindelöf if and only if $(\mathcal{X}_\mathcal{G}, \nu, \{\varnothing\})$ is $w\nu\{\varnothing\}$-Lindelöf $\mathcal{HGTS}$.

Proposition 4.3 *A $\mathcal{HGTS}$ $(\mathcal{X}_\mathcal{G}, \nu, \mathcal{H})$ is $\nu\mathcal{H}$-Lindelöf then it is $w\nu\mathcal{H}$-Lindelöf $\mathcal{HGTS}$.*

Proof. Let $\{\mathcal{U}_\gamma : \gamma \in \Omega\}$ be a ν-open cover of Λ_ν. Since a $\mathcal{HGTS}$ $(\mathcal{X}_\mathcal{G}, \nu, \mathcal{H})$ is $\nu\mathcal{H}$-Lindelöf then there is a countable sub-collection $\{\mathcal{U}_n : n \in \mathbb{N}\}$ such that

$$\Lambda_\nu \setminus \bigcup_{n\in\mathbb{N}}(\mathcal{U}_n) \in \mathcal{H}.$$

But, $\Lambda_\nu \setminus (c_\nu(\bigcup_{n\in\mathbb{N}}\mathcal{U}_n)) \subseteq \Lambda_\nu \setminus \bigcup_{n\in\mathbb{N}}(\mathcal{U}_n)$. So, $\Lambda_\nu \setminus (c_\nu(\bigcup_{n\in\mathbb{N}}\mathcal{U}_n)) \in \mathcal{H}$, and the proof is completed.

The converse of above Proposition is not true as the following example shows:

Example 4.1 *Let $\mathbb{R}$ be the real set, choose $a \in R$, $\beta = \{\{a, x\} : x \in \mathbb{R}, a \neq x\}$ and a hereditary class $\mathcal{H} = \{\varnothing, \mathbb{R}\}$. If the $\mathcal{GT}$ $\nu(\beta)$ generated on $\mathbb{R}$ by the ν-base β. Then $(\mathbb{R}, \nu(\beta), \mathcal{H})$ is a $\mathcal{HGTS}$, and for each non-empty ν-open set $\mathcal{U}$ of $\mathbb{R}$, we have $c_\nu\mathcal{U} = \mathbb{R}$. So, each ν-open cover*

$\{\mathcal{U}_\gamma : \gamma \in \Omega\}$ *of* $\mathbb{R}$, *there is a countable sub-collection* $\{\mathcal{U}_{\gamma_n} : n \in \mathbb{N}\}$ *such that*

$$\mathbb{R} \setminus (c_\nu(\bigcup_{n\in\mathbb{N}} \mathcal{U}_{\gamma_n})) \in \mathcal{H}.$$

Thus $\mathcal{HGTS}$ $(\mathbb{R}, \nu(\beta), \mathcal{H})$ is $w\nu\mathcal{H}$-Lindelöf. Now, $\mathfrak{U} = \{\{0, x\} : x \in \mathbb{R}\}$ is a ν-open cover of $\mathbb{R}$ and let $\{\{0, x_n\} : n \in \mathbb{N}\}$ be a countable sub-collection of $\mathfrak{U}$, it follows that $\mathbb{R} \setminus (\bigcup_{n\in\mathbb{N}} \{0, x_n\}) \notin \mathcal{H}$. Therefore, a $\mathcal{HGTS}$ $(\mathbb{R}, \nu(\beta), \mathcal{H})$ is not $\nu\mathcal{H}$-Lindelöf.

In the following proposition, we will show that a ν-Lindelöf with respect to hereditary classes is special case of $w\nu$-Lindelöf $\mathcal{GTS}$.

Proposition 4.4 *Let a* $\mathcal{GTS}$ $(\mathcal{X}_\mathcal{G}, \nu)$.

(i) $(\mathcal{X}_\mathcal{G}, \nu)$ is $w\nu$-Lindelöf if and only if $(\mathcal{X}_\mathcal{G}, \nu, \mathcal{H}_n)$ is $\nu\mathcal{H}_n$-Lindelöf.

(ii) $(\mathcal{X}_\mathcal{G}, \nu)$ is $w\nu$-Lindelöf if and only if $(\mathcal{X}_\mathcal{G}, \nu, \mathcal{H})$ is $\nu\mathcal{H}$-Lindelöf with a ν-codense hereditary class $\mathcal{H}$.

Proof. (i) $(\Rightarrow)$ Let $(\mathcal{X}_\mathcal{G}, \nu)$ is $w\nu$-Lindelöf and $\{\mathcal{U}_\gamma : \gamma \in \Omega\}$ be a ν-open cover of Λ_ν. Thus there is a countable sub-collection $\{\mathcal{U}_{\gamma n} : n \in \mathbb{N}\}$ such that $\Lambda_\nu = c_\nu(\bigcup_{n\in\mathbb{N}} \mathcal{U}_{\gamma n})$. Which implies that,

$$\Lambda_\nu \setminus c_\nu(\bigcup_{n\in\mathbb{N}} \mathcal{U}_{\gamma n}) = \varnothing.$$

So, $i_\nu(\Lambda_\nu \setminus \bigcup_{n\in\mathbb{N}} \mathcal{U}_{\gamma n}) = \varnothing$, and hence $\Lambda_\nu \setminus \bigcup_{n\in\mathbb{N}} \mathcal{U}_{\gamma n} \in \mathcal{H}_n$. Which proves that a $\mathcal{HGTS}$ $(\mathcal{X}_\mathcal{G}, \nu, \mathcal{H}_n)$ is $\nu\mathcal{H}_n$-Lindelöf.

$(\Leftarrow)$ Suppose $(\mathcal{X}_\mathcal{G}, \nu, \mathcal{H}_n)$ is $\nu\mathcal{H}_n$-Lindelöf and let $\{\mathcal{U}_\gamma : \gamma \in \Omega\}$ be a ν-open cover of Λ_ν. Thus there is a countable sub-collection $\{\mathcal{U}_{\gamma n} : n \in \mathbb{N}\}$ such that

$$\Lambda_\nu \setminus \bigcup_{n\in\mathbb{N}} \boxplus U_{\gamma n}) \in \mathcal{H}_n.$$

This implis that $i_\nu(\Lambda_\nu \setminus \bigcup_{n\in\mathbb{N}} \mathcal{U}_{\gamma n}) = \varnothing$, then $\Lambda_\nu \setminus c_\nu(\bigcup_{n\in\mathbb{N}} \mathcal{U}_{\gamma n}) = \varnothing$. Thus the proof is completed.

(ii) $(\Rightarrow)$ From (i) $\mathcal{H}$ is a ν-codense hereditary class.

$(\Leftarrow)$ Let $(\mathcal{X}_\mathcal{G}, \nu, \mathcal{H})$ is $\nu\mathcal{H}$-Lindelöf and $\{\mathcal{U}_\gamma : \gamma \in \Omega\}$ be a ν-open cover of Λ_ν. Thus there is a countable sub-collection $\{\mathcal{U}_{\gamma n} : n \in \mathbb{N}\}$ such that

$$\Lambda_\nu \setminus \bigcup_{n\in\mathbb{N}} \left(\mathcal{U}_{\gamma n}\right) \in \mathcal{H}.$$

Since a hereditary class $\mathcal{H}$ is ν-codense on $\mathcal{X}_\mathcal{G}$, $\Lambda_\nu \setminus \bigcup_{n\in\mathbb{N}} \left(\mathcal{U}_{\gamma n}\right)$ has empty ν-interior, then $\Lambda_\nu = c_\nu(\bigcup_{n\in\mathbb{N}} \mathcal{U}_{\gamma n})$. Which implies that $(\mathcal{X}_\mathcal{G}, \nu)$ is $w\nu$-Lindelöf $\mathcal{GTS}$.

Proposition 4.5 *Let* $(\mathcal{X}_\mathcal{G}, \nu, \mathcal{H})$ *be a* $w\nu\mathcal{H}$*-Lindelöf* $\mathcal{HGTS}$ *and* $\mathcal{A}$ *be a* ν*-clopen subset of* $\mathcal{X}_\mathcal{G}$. *Then* $(\mathcal{A}, \nu(\mathcal{A}), \mathcal{H}_A)$ *is* $w\nu(\mathcal{A})\mathcal{H}_A$*-Lindelöf.*

Proof. Let $\mathcal{A}$ be a ν-clopen subset of $\mathcal{X}_{\mathcal{G}}$. If $\{\mathcal{O}_\gamma = \mathcal{U}_\gamma \cap \mathcal{A} : \mathcal{U}_\gamma \in \nu \text{ } for\ each\ \gamma \in \Omega\}$ be a $\nu(\mathcal{A})$-open cover of $\mathcal{A} \cap \Lambda_\nu = \mathcal{A}$. Hence the family $\{\mathcal{U}_\gamma : \gamma \in \Omega\} \cup (\mathcal{X}_{\mathcal{G}} \setminus \mathcal{A})$ forms a ν-open cover of Λ_ν. Since $\mathcal{X}_{\mathcal{G}}$ is an $w\nu\mathcal{H}$-Lindelöf space, then there is a countable subfamily $\{\mathcal{U}_{\gamma_n} : n \in \mathbb{N}\} \cup (\mathcal{X}_{\mathcal{G}} \setminus \mathcal{A})$ such that

$$\Lambda_\nu \setminus [c_\nu(\bigcup_{n\in\mathbb{N}}(\mathcal{U}_{\gamma_n})) \cup (\mathcal{X}_{\mathcal{G}} \setminus \mathcal{A})] = H \in \mathcal{H}.$$

Now,

$$\begin{aligned}
\mathcal{A} \cap H &= \mathcal{A} \cap (\Lambda_\nu \setminus [c_\nu(\bigcup_{n\in\mathbb{N}}\mathcal{U}_{\gamma_n}) \cup (\mathcal{X}_{\mathcal{G}} \setminus \mathcal{A})]) \\
&= \mathcal{A} \cap (\Lambda_\nu \setminus (c_\nu(\bigcup_{n\in\mathbb{N}}\mathcal{U}_{\gamma_n})) \cap (\Lambda_\nu \setminus (\mathcal{X}_{\mathcal{G}} \setminus \mathcal{A})) \\
&= \mathcal{A} \cap (\Lambda_\nu \setminus (c_\nu(\bigcup_{n\in\mathbb{N}}\mathcal{U}_{\gamma_n})) \cap (\Lambda_\nu \cap \mathcal{A}) \\
&= \mathcal{A} \cap (\Lambda_\nu \setminus (c_\nu(\bigcup_{n\in\mathbb{N}}\mathcal{U}_{\gamma_n})) \cap \mathcal{A} = \mathcal{A} \cap (\Lambda_\nu \setminus (c_\nu(\bigcup_{n\in\mathbb{N}}\mathcal{U}_{\gamma_n})) \\
&= \mathcal{A} \cap (\Lambda_\nu \cap (\mathcal{X}_{\mathcal{G}} \setminus (c_\nu(\bigcup_{n\in\mathbb{N}}\mathcal{U}_{\gamma_n})) = (\mathcal{A} \cap \Lambda_\nu) \cap (\mathcal{X}_{\mathcal{G}} \setminus (c_\nu(\bigcup_{n\in\mathbb{N}}\mathcal{U}_{\gamma_n})) \\
&= \mathcal{A} \cap (\mathcal{X}_{\mathcal{G}} \setminus (c_\nu(\bigcup_{n\in\mathbb{N}}\mathcal{U}_{\gamma_n})) = \mathcal{A} \setminus (c_\nu(\bigcup_{n\in\mathbb{N}}\mathcal{U}_{\gamma_n})) = \mathcal{A} \setminus (\mathcal{A} \cap (c_\nu(\bigcup_{n\in\mathbb{N}}\mathcal{U}_{\gamma_n})))
\end{aligned}$$

However, $c_\nu(\bigcup_{n\in\mathbb{N}}\mathcal{U}_{\gamma_n}) \cap \mathcal{A} = c_{\nu(\mathcal{A})}(\bigcup_{n\in\mathbb{N}}\mathcal{V}_{\gamma_n})$. Therefore, we have

$$A \cap H = \mathcal{A} \setminus (c_{\nu(\mathcal{A})}(\bigcup_{n\in\mathbb{N}}\mathcal{V}_{\gamma_n})) \in \mathcal{H}_A.$$

This proves that a subset $\mathcal{A}$ is an $w\nu(\mathcal{A})\mathcal{H}_A$-Lindelöf .

Proposition 4.6 *Let $(\mathcal{X}_{\mathcal{G}}, \nu)$ be a ν-space and $\mathcal{H}$ be a hereditary class on $\mathcal{X}_{\mathcal{G}}$. If $(\mathcal{X}_{\mathcal{G}}, \nu^*, \mathcal{H})$ is $w\nu^*\mathcal{H}$-Lindelöf then $(\mathcal{X}_{\mathcal{G}}, \nu, \mathcal{H})$ is $w\nu\mathcal{H}$-Lindelöf $\mathcal{HGTS}$.*

Proof. The proof follows from Theorem 3.4 and Proposition 4.5, since every ν-closed (ν-open) set is ν^*-closed (ν^*-open) set. Thus every ν-clopen set is ν^*-clopen set.

In the following example, we show that the converse of Proposition 4.6 is not true.

Example 4.2 *Let $\mathbb{R}$ be the set of real numbers and $\nu = \{\mathcal{U} \subset \mathbb{R} : \mathcal{U} \text{ is uncountable}\} \cup \{\varnothing\}$ be a $\mathcal{GT}$ on $\mathbb{R}$. Suppose $\mathcal{H} = \{\mathbb{R} \setminus \mathcal{U} : \mathcal{U} \in \nu\}$ be a hereditary class on $\mathbb{R}$, observe that $\mathcal{H}$ is not closed under countable union. A ν-space $\mathbb{R}$ is $\nu\mathcal{H}$-Lindelöf (see. (Qahis et al., 2016)) so it is $w\nu\mathcal{H}$-Lindelöf. Now, for each $x \in \mathbb{R}$, $\{x\}$ is ν^*-open. Further, $\{x\}$ is ν-closed set so it is ν^*-closed, and hence $c_{\nu^*}(\{x\}) = \{x\}$. Further, $\{\{x\} : x \in \mathbb{R}\}$ is a ν^*-open cover of a ν^*-space $\mathbb{R}$. Assume that there is a countable collection $\{\{x_i\} : i \in \mathbb{N}\}$ such that $\mathbb{R} \setminus (c_{\nu^*}(\bigcup_{i\in\mathbb{N}}\{x_i\}) \in \mathcal{H}$. And this is not possible. Therefore, a ν^*-space $\mathbb{R}$ is not $w\nu^*\mathcal{H}$-Lindelöf.*

The converse of Proposition 4.6 will be hold if a hereditary class $\mathcal{H}$ is closed under countable union as the following:

Proposition 4.7 *Let* $(\mathcal{X}_{\mathcal{G}},\nu)$ *be a* ν*-space and a hereditary class* $\mathcal{H}$ *on* $\mathcal{X}_{\mathcal{G}}$ *is closed under countable union, then* $(\mathcal{X}_{\mathcal{G}},\nu^*,\mathcal{H})$ *is* $w\nu^*\mathcal{H}$*-Lindelöf if and only if* $(\mathcal{X}_{\mathcal{G}},\nu,\mathcal{H})$ *is* $w\nu\mathcal{H}$*-Lindelöf* $\mathcal{HGTS}$.

Proof. The necessity is obviously by Proposition 4.6. For sufficiency, suppose a $(\mathcal{X}_{\mathcal{G}},\nu,\mathcal{H})$ is $w\nu\mathcal{H}$-Lindelöf and $\mathcal{H}$ is closed under countable union. Given $\{\mathcal{U}^*_\gamma : \gamma\in\Omega\}$ a ν^*-open cover of $\mathcal{X}_{\mathcal{G}}$, then for each $x\in\mathcal{X}_{\mathcal{G}}$, $x\in\mathcal{U}^*_{\gamma_x}$ for some $\gamma_x\in\Omega$. By Theorem 3.5, there is $\mathcal{U}_{\gamma_x}\in\nu_x$ and $H_{\gamma_x}\in\mathcal{H}$ such that $x\in\mathcal{U}_{\gamma_x}\setminus H_{\gamma_x}\subset\mathcal{U}^*_{\gamma_x}$. Since the collection $\{\mathcal{U}_{\gamma x}:\gamma\in\Omega\}$ is a ν-open cover of $\mathcal{X}_{\mathcal{G}}$, then there exists a countable sub-collection $\{\mathcal{U}_{\gamma x_n}:n\in\mathbb{N}\}$ such that

$$\mathcal{X}_{\mathcal{G}}\setminus(c_\nu(\bigcup_{n\in N}\mathcal{U}_{\gamma x_n}))=H\in\mathcal{H}.$$

Since $\mathcal{H}$ is closed under countable union, then $\cup\{H_{\gamma_{x_n}}:n\in\mathbb{N}\}\in\mathcal{H}$. Then, $H\cup[\cup\{H_{\gamma_{x_n}}:n\in\mathbb{N}\}]\in\mathcal{H}$. Note that $\mathcal{X}_{\mathcal{G}}\setminus(c_{\nu^*}(\bigcup_{n\in\mathbb{N}}\mathcal{U}^*_{\gamma_n}))\subset H\cup[\cup\{H_{\gamma_{x_n}}:n\in\mathbb{N}\}]\in\mathcal{H}$. So,

$$\mathcal{X}_{\mathcal{G}}\setminus(c_{\nu^*}(\bigcup_{n\in\mathbb{N}}\mathcal{U}^*_{\gamma_n}))\in\mathcal{H}.$$

Therefore, $(\mathcal{X}_{\mathcal{G}},\nu^*,\mathcal{H})$ is $w\nu^*\mathcal{H}$-Lindelöf $\mathcal{HGTS}$.

4.1 Function Properties on $w\nu$-Lindelöf with respect to a hereditary class $\mathcal{H}$

The notions of continuous functions in generalized topological spaces was introduced by (Császár, 2002). Let ν and μ be generalized topologies on $\mathcal{X}_{\mathcal{G}}$ and $\mathcal{Y}_{\mathcal{G}}$, respectively. Then a function $g:(\mathcal{X}_{\mathcal{G}},\nu)\to(\mathcal{Y}_{\mathcal{G}},\mu)$ from a ν-space $(\mathcal{X}_{\mathcal{G}},\nu)$ into a μ-space $(\mathcal{Y}_{\mathcal{G}},\mu)$ is called (ν,μ)-continuous iff $\mathcal{U}\in\mu$ implies that $g^{-1}(\mathcal{U})\in\nu$.

Definition 4.2 *Let* $\mathcal{A}$ *be a subset of* $\mathcal{GTS}$ $(\mathcal{X}_{\mathcal{G}},\nu)$, *then* $\mathcal{A}$ *is called* ν*-preopen (resp.* $\nu-\beta$*-open) (Császár, 2005) if* $\mathcal{A}\subseteq i_\nu c_\nu(\mathcal{A})$ *(resp.* $\mathcal{A}\subseteq c_\nu i_\nu c_\nu(\mathcal{A})$*)*

The complement of ν*-preopen (resp.* $\nu-\beta$*-open) is said to be* ν*-preclosed (resp.* $\nu-\beta$*-closed), we denote by* π *the class of all* ν*-preopen sets in* $\mathcal{X}_{\mathcal{G}}$, *by* β *the class of all* $\nu-\beta$*-open sets in* $\mathcal{X}_{\mathcal{G}}$.

Definition 4.3 *A function* $g:(\mathcal{X}_{\mathcal{G}},\nu)\to(\mathcal{Y}_{\mathcal{G}},\mu)$ *is called:*

(1) almost (ν,μ)*- continuous (Min, 2009), if for each* $t\in\mathcal{X}_{\mathcal{G}}$ *and each* μ*-open set* $\mathcal{U}$ *containing* $g(t)$, *there is a* ν*-open set* $\mathcal{V}$ *with* $t\in\mathcal{V}$ *such that* $g(\mathcal{V})\subseteq i_\mu c_\mu(\mathcal{U})$.

(2) almost (π,μ) *- continuous (resp. almost* (β,μ)*- continuous) [,*

KilimanSarsakAbuage.] if for each $t\in\mathcal{X}_{\mathcal{G}}$ *and each* μ*-regular open set* $\mathcal{U}$ *in* $\mathcal{Y}_{\mathcal{G}}$ *containing* $g(t)$*, there is a* ν*-preopen (resp.* $\nu-\beta$*-open) set* $\mathcal{V}$ *containing* t *such that* $g(\mathcal{V})\subseteq\mathcal{U}$.

Remark 4.1 *Let* $g:(\mathcal{X}_{\mathcal{G}},\nu)\to(\mathcal{Y}_{\mathcal{G}},\mu)$ *be a function between* $\mathcal{GTS}$ *'s* $(\mathcal{X}_{\mathcal{G}},\nu)$ *and* $(\mathcal{Y}_{\mathcal{G}},\mu)$*. Then we have the following implications but the reverse relations may not be true in general:*

almost (ν,μ) - continuous $\Rightarrow$ almost (π,μ) - continuous $\Rightarrow$ almost (β,μ) - continuous

Example 4.3 *Let* $\mathcal{X}_{\mathcal{G}}=\{a,b,c\}$ *and* $\nu=\{\varnothing,\{a,b\}\}$ *be a* $\mathcal{GT}$ *on* $\mathcal{X}_{\mathcal{G}}$*. Then* $\pi=\nu\cup\{\{a\},\{b\}\}$*. Define a function* $g:(\mathcal{X}_{\mathcal{G}},\nu)\to(\mathcal{X}_{\mathcal{G}},\nu)$ *as follows:* $g(a)=a$*,* $g(b)=g(c)=c$*. Then* g *is almost* (π,μ) *- continuous function but not almost* (ν,μ) *- continuous.*

Example 4.4 *Let* $\mathcal{X}_{\mathcal{G}}=\{a,b,c\}$ *and* $\nu=\{\varnothing,\{a\},\{b\},\{a,b\}\}$ *be a* $\mathcal{GT}$ *on* $\mathcal{X}_{\mathcal{G}}$*. Then* $\pi=\nu$ *and* $\beta=\nu\cup\{\{a,b\},\{a,c\},\mathcal{X}_{\mathcal{G}}\}$*. Consider a function* $g:(\mathcal{X}_{\mathcal{G}},\nu)\to(\mathcal{X}_{\mathcal{G}},\nu)$ *defined by* $g(a)=g(b)=b$*,* $g(c)=a$*. Then* g *is almost* (β,μ) *- continuous function without begin almost* (π,μ) *- continuous.*

Proposition 4.8 *Let* $g:(\mathcal{X}_{\mathcal{G}},\nu)\to(\mathcal{Y}_{\mathcal{G}},\mu)$ *be an almost* (ν,μ) *- continuous surjection from a* ν*-space* $(\mathcal{X}_{\mathcal{G}},\nu)$ *into a* μ*-space* $(\mathcal{Y}_{\mathcal{G}},\mu)$*, and* $\mathcal{H}$ *be a hereditary class on* $\mathcal{X}_{\mathcal{G}}$*. If* $\mathcal{X}_{\mathcal{G}}$ *is* $w\nu\mathcal{H}$*-Lindelöf then* $\mathcal{Y}_{\mathcal{G}}$ *so is.*

Proof. Let $\{\mathcal{U}_\gamma:\gamma\in\Omega\}$ be a μ-open cover of $\mathcal{Y}_{\mathcal{G}}$, Since g is almost (ν,μ)-continuous, that means $g^{-1}(i_\mu c_\mu(\mathcal{U}_\gamma))$ is a ν-open in $\mathcal{X}_{\mathcal{G}}$. Thus $\{g^{-1}(i_\mu c_\mu(\mathcal{U}_\gamma)):\gamma\in\Omega\}$ is a ν-open cover of $\mathcal{X}_{\mathcal{G}}$, then there is a countable sub-collection $\{g^{-1}(i_\mu c_\mu(\mathcal{U}_{\gamma_n})):n\in\mathbb{N}\}$ such that

$$\mathcal{X}_{\mathcal{G}}\setminus c_\nu(\bigcup_{n\in\mathbb{N}}g^{-1}(i_\mu c_\mu(\mathcal{U}_{\gamma_n})))\in\mathcal{H}.$$

Now;

$$\begin{aligned}\mathcal{X}_{\mathcal{G}}\setminus(c_\nu(g^{-1}(c_\mu(\bigcup_{n\in\mathbb{N}}(\mathcal{U}_{\gamma_n})))))&\subseteq\mathcal{X}_{\mathcal{G}}\setminus(c_\nu(g^{-1}(\bigcup_{n\in\mathbb{N}}(c_\mu\mathcal{U}_{\gamma_n}))))\\&=\mathcal{X}_{\mathcal{G}}\setminus(c_\nu(\bigcup_{n\in\mathbb{N}}g^{-1}(c_\mu(\mathcal{U}_{\gamma_n}))))\\&\subseteq\mathcal{X}_{\mathcal{G}}\setminus c_\nu(\bigcup_{n\in\mathbb{N}}g^{-1}(i_\mu c_\mu(\mathcal{U}_{\gamma_n})))\in\mathcal{H}.\end{aligned}$$

By Lemma 3.3 $c_\mu(\bigcup_{n\in\mathbb{N}}(\mathcal{U}_{\gamma_n}))$ is μ-regular closed in $\mathcal{Y}_{\mathcal{G}}$ and g is an almost (ν,μ) - continuous, we have $g^{-1}(c_\mu(\bigcup_{n\in\mathbb{N}}(\mathcal{U}_{\gamma_n})))$ is ν-closed in $\mathcal{X}_{\mathcal{G}}$. Thus

$$\mathcal{X}_{\mathcal{G}}\setminus(g^{-1}(c_\mu(\bigcup_{n\in\mathbb{N}}(\mathcal{U}_{\gamma_n}))))=\mathcal{X}_{\mathcal{G}}\setminus(c_\nu(g^{-1}(c_\mu(\bigcup_{n\in\mathbb{N}}(\mathcal{U}_{\gamma_n})))))\in\mathcal{H}.$$

By Lemma 3.6,

$$g(\mathcal{X}_{\mathcal{G}} \setminus (g^{-1}(c_\mu(\bigcup_{n\in\mathbb{N}}(\mathcal{U}_{\gamma_n}))))) = g(\mathcal{X}_{\mathcal{G}}) \setminus (g(g^{-1}(c_\mu(\bigcup_{n\in\mathbb{N}}(\mathcal{U}_{\gamma_n})))))$$
$$= \mathcal{Y}_{\mathcal{G}} \setminus (c_\mu(\bigcup_{n\in\mathbb{N}}(\mathcal{U}_{\gamma_n}))) \in g(\mathcal{H}).$$

Which proves that a $\mathcal{HGTS}$ $\mathcal{Y}_{\mathcal{G}}$ is $w\mu g(\mathcal{H})$ -Lindelöf.

If $\mathcal{H} = \{\varnothing\}$ in the above Proposition, then we have the following result:

Corollary 4.1 *(Abuage & Kiliçman, 2017) Let $g:(\mathcal{X}_{\mathcal{G}},\nu) \to (\mathcal{Y}_{\mathcal{G}},\mu)$ be an almost (ν,μ)-continuous surjection from a ν-space $(\mathcal{X}_{\mathcal{G}},\nu)$ into a μ-space $(\mathcal{Y}_{\mathcal{G}},\mu)$, if a ν-space $\mathcal{X}_{\mathcal{G}}$ is $w\nu$-Lindelöf then a μ-space $\mathcal{Y}_{\mathcal{G}}$ so is.*

Theorem 4.4 *(Ekici, 2012) Let $(\mathcal{X}_{\mathcal{G}},\nu)$ be a $\mathcal{GTS}$ where $c_\nu(\varnothing) = \varnothing$. Then $(\mathcal{X}_{\mathcal{G}},\nu)$ is a submaximal and extremally disconnected $\mathcal{GTS}$ if any subset of $(\mathcal{X}_{\mathcal{G}},\nu)$ is $\nu-\beta$-open if and only if it is ν-open.*

Obviously, if $\mathcal{X}_{\mathcal{G}} \in \nu$ in $\mathcal{GTS}(\mathcal{X}_{\mathcal{G}},\nu)$ then $c_\nu(\varnothing) = \varnothing$, so the following proposition proves immediately by Theorem 4.4.

Proposition 4.9 *Let $(\mathcal{X}_{\mathcal{G}},\nu)$ be a submaximal and ν-extremally disconnected ν-space. Then a function $g:(\mathcal{X}_{\mathcal{G}},\nu) \to (\mathcal{Y}_{\mathcal{G}},\mu)$ is an almost (ν,μ)-continuous if and only if it is almost (β,μ)-continuous.*

Corollary 4.2 *Let $g:(\mathcal{X}_{\mathcal{G}},\nu) \to (\mathcal{Y}_{\mathcal{G}},\mu)$ be an almost (β,μ)-continuous surjection, and $\mathcal{H}$ be a hereditary class on $\mathcal{X}_{\mathcal{G}}$. If $\mathcal{X}_{\mathcal{G}}$ is submaximal, ν-extremally disconnected and $w\nu\mathcal{H}$-Lindelöf ν-space. Then $\mathcal{Y}_{\mathcal{G}}$ is $w\mu g(\mathcal{H})$-Lindelöf.*

Proof. The proof follows directly from Proposition 4.8 and Proposition 4.9.

Lemma 4.5 *Let a $(\mathcal{X}_{\mathcal{G}},\nu)$ be a submaximal $\mathcal{QTS}$ then every ν-preopen set is ν-open.*

Proof. Assume, a subset $\mathcal{V}$ is a ν-preopen, then by Proposition 3.11 (Sarsak, 2013) $\mathcal{V} = \mathcal{U} \cap \mathcal{A}$ for some ν-regular open set $\mathcal{U}$ and ν-dense set $\mathcal{A}$ of $\mathcal{X}_{\mathcal{G}}$. Since $(\mathcal{X}_{\mathcal{G}},\nu)$ is submaximal $\mathcal{QTS}$, so $\mathcal{A}$ is ν-open set of $\mathcal{X}_{\mathcal{G}}$ and thus $\mathcal{V}$ is ν-open set of $\mathcal{X}_{\mathcal{G}}$.

Next proposition proves directly, by Lemma 4.5, so the proof omitted.

Proposition 4.10 *Let $(\mathcal{X}_{\mathcal{G}},\nu)$ be a submaximal $\mathcal{QTS}$ then a function $g:(\mathcal{X}_{\mathcal{G}},\nu) \to (\mathcal{Y}_{\mathcal{G}},\mu)$ is an almost (ν,μ)-continuous if and only if it is almost (π,μ)-continuous.*

By Proposition 4.8 and Proposition 4.10 the following corollary concluded:

Corollary 4.3 *Let $g:(\mathcal{X}_{\mathcal{G}},\tau) \to (\mathcal{Y}_{\mathcal{G}},\mu)$ be an almost (π,μ)-continuous surjection, and $\mathcal{H}$ be a hereditary class on $\mathcal{X}_{\mathcal{G}}$. If a space $\mathcal{X}_{\mathcal{G}}$ is submaximal and weakly Lindelöf then $\mathcal{Y}_{\mathcal{G}}$ is $w\mu g(\mathcal{H})$-*

Lindelöf.

Definition 4.6 *(Al-Omari & Noiri, 2012) A function* $g:(\mathcal{X}_{\mathcal{G}},\nu)\to(\mathcal{Y}_{\mathcal{G}},\mu)$ *is said to be*

(a) almost (ν,μ)*-open if* $g(\mathcal{V})\subseteq i_{\mu}c_{\mu}(g(\mathcal{V}))$ *for each* ν*-open set* $\mathcal{V}$ *in* $\mathcal{X}_{\mathcal{G}}$*,*

(b) contra (ν,μ)*- continuous if* $g^{-1}(\mathcal{U})$ *is* ν*-closed in* $\mathcal{X}_{\mathcal{G}}$ *for every* μ*-open set* $\mathcal{U}$ *in* $\mathcal{Y}_{\mathcal{G}}$*.*

(Al-Omari & Noiri, 2012), showed that if a function g from a ν-space $(\mathcal{X}_{\mathcal{G}},\nu)$ into a μ-space $(\mathcal{Y}_{\mathcal{G}},\mu)$ is an almost (ν,μ)-open and contra (ν,μ)- continuous, then g is almost (ν,μ) - continuous Moreover, if g is a contra (ν,μ)- continuous and a μ-space $\mathcal{Y}_{\mathcal{G}}$ is μ-extremally disconnected, then g is almost (ν,μ)- continuous On using Proposition 4.8 above, we conclude the following corollaries:

Corollary 4.4 *Let* $g:(\mathcal{X}_{\mathcal{G}},\nu)\to(\mathcal{Y}_{\mathcal{G}},\mu)$ *be an an almost* (ν,μ)*-open and contra* (ν,μ)*-continuous surjection from a* ν*-space* $(\mathcal{X}_{\mathcal{G}},\nu)$ *into a* μ*-space* $(\mathcal{Y}_{\mathcal{G}},\mu)$*, with a hereditary class* $\mathcal{H}$ *on* $\mathcal{X}_{\mathcal{G}}$*. If* $\mathcal{X}_{\mathcal{G}}$ *is* $w\nu\mathcal{H}$*-Lindelöf then* $\mathcal{Y}_{\mathcal{G}}$ *so is.*

Corollary 4.5 *Let* $g:(\mathcal{X}_{\mathcal{G}},\nu)\to(\mathcal{Y}_{\mathcal{G}},\mu)$ *be a contra* (ν,μ)*-continuous and a* μ*-space* $\mathcal{Y}_{\mathcal{G}}$ *is* μ*-extremally disconnected from a* ν*-space* $(\mathcal{X}_{\mathcal{G}},\nu)$ *into a* μ*-space* $(\mathcal{Y}_{\mathcal{G}},\mu)$*, with a hereditary class on* $\mathcal{X}_{\mathcal{G}}$*. If* $\mathcal{X}_{\mathcal{G}}$ *is* $w\nu\mathcal{H}$*-Lindelöf then* $\mathcal{Y}_{\mathcal{G}}$ *so is.*

Proposition 4.11 *Let* $g:(\mathcal{X}_{\mathcal{G}},\nu,\mathcal{H})\to(\mathcal{Y}_{\mathcal{G}},\mu)$ *be an almost* (ν,μ)*-continuous surjection, if* $(\mathcal{X}_{\mathcal{G}},\nu,\mathcal{H})$ *is* $w\nu\mathcal{H}$*-Lindelöf and* μ*-space* $\mathcal{Y}_{\mathcal{G}}$ *is countable, then* $\mathcal{Y}_{\mathcal{G}}$ *is* $w\mu$*-Lindelöf.*

Proof. Suppose $(\mathcal{X}_{\mathcal{G}},\nu,\mathcal{H})$ be a $w\nu\mathcal{H}$-Lindelöf and g be an almost (ν,μ)-continuous surjection, by Proposition 4.8 $\mathcal{Y}_{\mathcal{G}}$ is $n\mu g(\mathcal{H})$-Lindelöf. Since $\mathcal{Y}_{\mathcal{G}}$ is countable so a hereditary class $\mathcal{H}$ is countable, by applying Proposition 4.2 the proof is completed.

Proposition 4.12 *Let* $g:(\mathcal{X}_{\mathcal{G}},\nu,\mathcal{H})\to(\mathcal{Y}_{\mathcal{G}},\mu)$ *be an almost* (ν,μ)*-continuous surjection, if* $(\mathcal{X}_{\mathcal{G}},\nu,\mathcal{H})$ *is* $w\nu\mathcal{H}$*-Lindelöf and* μ*-space* $\mathcal{Y}_{\mathcal{G}}$ *is countable, then* $(\mathcal{Y}_{\mathcal{G}},\mu,g(\mathcal{H}_n))$ *is* $\mu g(\mathcal{H}_n)$*-Lindelöf.*

Proof. The proof follows immediately by Proposition 4.8, Proposition 4.2 and Proposition 4.4 (i).

Proposition 4.13 *Let* $g:(\mathcal{X}_{\mathcal{G}},\nu,\mathcal{H})\to(\mathcal{Y}_{\mathcal{G}},\mu)$ *be an almost* (ν,μ)*-continuous surjection, if* $(\mathcal{X}_{\mathcal{G}},\nu,\mathcal{H})$ *is* $w\nu\mathcal{H}$*-Lindelöf and* μ*-space* $\mathcal{Y}_{\mathcal{G}}$ *is countable, then* $(\mathcal{Y}_{\mathcal{G}},\mu,g(\mathcal{H}))$ *is* $\mu g(\mathcal{H})$*-*

Lindelöf with a μ-codense hereditary class $\mathcal{H}$.

Proof. The proof follows immediately by Proposition 4.8, Proposition 4.2 and Proposition 4.4 (ii).

Proposition 4.14 *Let $g:(\mathcal{X}_{\mathcal{G}},\nu)\to(\mathcal{Y}_{\mathcal{G}},\mu)$ be an almost (ν,μ)-continuous surjection, $\mathcal{H}$ be a hereditary class on $\mathcal{X}_{\mathcal{G}}$ which is closed under countable union, if $(\mathcal{X}_{\mathcal{G}},\nu^*,\mathcal{H})$ is $n\nu^*\mathcal{H}$-Lindelöf then $(\mathcal{Y}_{\mathcal{G}},\mu^*,g(\mathcal{H}))$ is $n\mu^*g(\mathcal{H})$-Lindelöf.*

Proof. By Proposition 4.7, $(\mathcal{X}_{\mathcal{G}},\nu^*,\mathcal{H})$ is $n\nu\mathcal{H}$-Lindelöf. Since g is almst (ν,μ)-continuous surjection, then $(\mathcal{Y}_{\mathcal{G}},\mu,g(\mathcal{H}))$ is $n\mu g(\mathcal{H})$-Lindelöf. But $\mathcal{H}$ is closed under countable union thus $g(\mathcal{H})$ is closed under countable union on $\mathcal{Y}_{\mathcal{G}}$. Again by applying Proposition 4.7 the proof is completed.

5 Conclusion

Our work aims to define and study the notion of weakly ν-Lindelöf with respect to a hereditary class $\mathcal{H}$: $w\nu\mathcal{H}$-Lindelöf, its properties and its relation to known concepts are showed.

Corresponding author:

Email addresses: slaam.salem@yahoo.com [1], akilicman@yahoo.com [2]

References

Abuage, M., & Kiliçman, A. (2017). Some properties and mappings on weakly ν-Lindelof generalized topological spacesSome properties and mappings on weakly ν-lindelof generalized topological spaces. *Journal of Nonlinear Sciences and Applications*, 10(8), 4150–4161.

Abuage, M., Kiliman, A., & Sarsak, M. (n.d.). $n\nu$-Lindelöfness. *Malaysian Journal of Mathematical Sciences*, 11 (S).

Al-Omari, A., & Noiri, T. (2012). A unified theory of contra-(μ, λ)-continuous func- tions in generalized topological spaces. *Acta Mathematica Hungarica*, *135*(1-2), 31–41.

Al-Omari, A., & Noiri, T. (2013). A unified theory of weakly contra-(μ, λ)- continuous functions in generalized topological spaces. *Studia Universitatis Babes-Bolyai*, *Mathematica*, *58*(1), 107–117.

Carpintero, C., Rosas, E., Salas-Brown, M., & Sanabria, J. (2016). μ-compactness with respect to a hereditary class. *Boletim da Sociedade Paranaense de Matemática*, *34* (2), 231–236.

Császár, Á. (n.d.). Extremally disconnected generalized topologies.(2004). *Annales Univ. Budapest, Sectio Math*, *17*, 151–165.

Császár, Á. (2002). Generalized topology, generized continuity. *Acta Mathematica Hungarica*, *96*(4), 351–357.

Császár, Á. (2005). Generalized open sets in generalized topologies. *Acta Mathematica Hungarica*, *106*(1-2), 53–66.

Császár, A. (2006). Further remarks on the formula of γ-interior. *Acta Mathematica Hungarica*, *113* , 325–332.Császár, Á . (2006). Modification of generalized topologies via hereditary classes. *Acta Mathematica Hungarica*, *115*(1-2), 29–36.

Császár, A. (2008). δ-and θ-modifications of generalized topologies. *Acta Mathematica Hungarica*, *120*(3), 275–279.

Ekici, E. (2012). Generalized submaximal spaces. *Acta Mathematica Hungarica*, *134* (1-2), 132–138.

Kim, Y. K., & Min, W. K. (2012). On operations induced by hereditary classes on generalized topological spaces. *Acta Mathematica Hungarica*, *137* (1-2), 130–138. Kuratowski., K. (1933). Topologies i. *Warszawa*.

Min, W. K. (2009). Almost continuity on generalized topological spaces. *Acta Mathematica Hungarica*, *125*(1-2), 121–125.

Min, W. K. (2010a). (δ, δ')-continuity on generalized topological spaces. *Acta Mathematica Hungarica*, *129*(4), 350–356.

Min, W. K. (2010b). Generalized continuous functions defined by generalized open sets on generalized topological spaces. *Acta Mathematica Hungarica*, *128*(4), 299–306.

Noiri, T. (2006). Unified characterizations for modifications of r_0 and r_1 topological spaces. *Rend. Circ. Mat. Palermo.*, *55*(2), 29–42.

Qahis, A., AlJarrah, H. H., Noiri, T., et al. (2016). μ-lindelöfness in terms of a hereditary class. *Missouri Journal of Mathematical Sciences*, *28*(1), 15–24.

Ramasamy, R., Rajamani, M., & Inthumathi, V. (2012). Some new general- ized topologies via hereditary classes. *Boletim da Sociedade Paranaense de Matemática*, *30* (2), 71–77.

Sarsak, M. S. (2012). On μ-compact sets in μ-spaces. *Questions Answers Gen. Topology*, *31*, 49–57.

Sarsak, M. S. (2013). On some properties of generalized open sets in generalized topological spaces. *Demonstratio Mathematica*, *46*(2), 415–427.

Thomas, J., & John, S. J. (2012). μ-compactness in generalized topological spaces. *Journal of Advanced Studies in Topology*, *3*(3), 18–22.

Zahran, A. M., El-Saady, K., & Ghareeb, A. (2012). Modification of weak structures via hereditary classes. *Applied Mathematics Letters*, *25*(5), 869–872.