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Ahmad Zendedel and Parisa Ahmadi Ghotbi

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Convergence in Fractional Probability Space and 0-1 Kolmogorov Theorem

Ahmad Zendedel [a], Parisa Ahmadi Ghotbi [b]
[a] *Department of statistics, The Islamic Azad University Neyshaboor; Iran*
[b] *Department of statistics, Shiraz University, Iran*

About the authors

Ahmad Zendedel received his BSc in Statistics from Ferdowsi University, Mashhad, Iran in 1991 and MSc in Statistics from the Shahid Beheshti University of Tehran, Iran in 1995 and PhD in Applied Statistics from Islamic Azad University, Tehran, Iran in 2004. Since 1995, he is working as a member of Full-time Mathematics and Statistics Department, Neyshabour, Iran. At present, He is an assistant professor in Islamic Azad University (IAU). He is interested in the Applied Statistics, especially: Financial Statistics. Parisa Ahmadi Ghotbi has Bachelor of Science in Statistics from Azad University of Mashhad, Iran in 2005 and Masters of Science in Statistics from Azad University of Tehran, Iran in 2009. At present, she is studying PhD of Science in Probability at International Unit of Shiraz University, Iran from 2011. Her Doctoral Thesis is "Steiltjes sample characteristic function estimation procedure". And, her second research area is probability calculus of fractional order.

Public interest statement

Fractional calculus including the subjects, derivative, integration and Teylor's series of fractional order plays an essential role in a number of fields of application such as the stochastic mechanics and fractional differential and integral equations. Probability density of fractional order has been raised by Guy Jumarie in 2007, which has application in the stochastic differential equation. Thus, expanding the mentioned concept by Guy Jumarie to the other fractional probability concepts such as fractional probability space, measure and convergence was so interesting. In this paper, we discussed some important definitions and theorems of the fractional probability theory and compared them with the ones of the classical probability theory.

Abstract

In this study, we define the fractional random variable. The concept of convergence in fractional probability, almost surely convergence and some related theorems and examples are studied with the purpose of expanding the fractional probability theory parallel to the classical one. It is shown that almost surely convergence in the fractional probability space does not lead to the convergence in fractional probability. And, some valuable features related to fractional probability theory such as Cauchy function in fractional probability are discussed. We proved that a fractional random variable converges in fractional probability if it is Cauchy in fractional probability.

Finally, the well-known 0-1 kolmogorov theorem is proved in a fractional probability space.

Keywords: *Almost sure, Cauchy, Convergence, Fractional order, Kolmogorov, Probability measure.*

1. Introduction

We try to continue the concept of fractional probability calculus, based on the study by Guy Jumarie (2007) which defines probability density of fractional order and fractional moments by using fractional calculus [1]. Our paper is in the continuation of the paper by Mostafaei and Ahmadi Ghotbi (2010) in which the fractional probability space (Ω, F, P_α), the fractional probability measure $P_\alpha : F \to [0,1]$, $0 < \alpha < 1$ and invalidity of the classical probability measure continuity theorem, $P(\lim_{n\to\infty} X_n(\omega)) = \lim_{n\to\infty} P(X_n(\omega))$ for the fractional probability measure P_α have been explained [11]. There is a great literature on fractional calculus, fractional derivatives and fractional integration [2, 3, 4, 5, 6, 7, 8]. Fractional probability distribution is defined by the measure $\mu\{dx\} = p(x)(dx)^\alpha$. Combining this with the definition of the fractional Taylor`s series $f(x+h) = E_\alpha(D_x^\alpha h^\alpha) f(x)$, obtained by modified fractional Riemann-Liouville derivatives, leads to the definition of the probability density of fractional order α. Using fractional calculus, Guy Jumarie (2007) defined the probability density of fractional order α, $P_\alpha(x)$ as the following

Definition 1 (Probability density of fractional order α) let X denote a real-valued random variable with the probability density $P_\alpha(x)$, where $P_\alpha(x) \geq 0$. X is referred to as a random variable with fractional probability density of order $\alpha, 0 < \alpha < 1$, whenever one has

$$P\{x < X \leq x'\} = F(x, x') = \int_x^{x'} P_\alpha(\xi)(d\xi)^\alpha, \ (1)$$

with normalizing condition $\int_{-\infty}^{+\infty} P_\alpha(x)(dx)^\alpha = 1$. Also, he introduced the kth fractional moment as

$$m_{k\alpha} = E\{X^{k\alpha}\} = \int_R x^{k\alpha} P_\alpha(x)(dx)^\alpha \ (2). \ [1]$$

These definitions can be considered as the first step in expanding a fractional probability theory. Furthermore, Mostafaei and Ahmadi Ghotbi (2010) introduced the fractional probability measure P_α and the fractional probability space (Ω, F, P_α) as the following

Definition 2 (Fractional probability principles) given a sample space Ω and an associated σ-field F, the fractional probability measure of order $\alpha, 0 < \alpha < 1$, is a set function $P_\alpha : F \to [0,1]$, $0 < \alpha < 1$ that satisfies

1. $P_\alpha(A) \geq 0 \quad for\ all\ A \in F$
2. $P_\alpha(\Omega) = 1$
3. $for\ all\ A_i \in F,\ even\ if\ A_i s\ are\ pairwise\ disjo\text{int}\ ,\ then\ P_\alpha(\bigcup_{i=1} A_i) \leq \sum_{i=1} P_\alpha(A_i)$

Definition 3 (Fractional probability space). A fractional probability space is a triple (Ω, F, P_α) where

- Ω is the sample space corresponding to outcomes of some of the experiments.
- F is the σ-algebra of subsets of Ω. These subsets are called events.
- $P_\alpha : F \to [0,1]$, $0 < \alpha < 1$ is a fractional probability measure.

Theorem 1 (Fractional probability measure properties). Let (Ω, F, P_α) be a fractional probability space, then one has

a) $p_\alpha(\varnothing) = 0$

b) If A, B are two events that $A \subset B$, then $P_\alpha(A) \leq P_\alpha(B)$

c) $1 - P_\alpha(A^c) \leq P_\alpha(A) \leq 1$. [11]

Now, according to the difference between fractional and classical probability space, it would be interesting to know what happen to the essential probabilistic concepts such as convergence. It is desirable to know whether major theorems in classical probability theory are satisfied in fractional probability space. These would be first steps to expand a fractional probability theory which has application in the fractional statistical mechanics and the fractional diffusion equation [12, 13, 14], parallel to the classical probability theory.

In this paper, the definition of fractional random variable is denoted and some well-known convergence theorems or lemmas such as almost surely convergence $X_n \xrightarrow{a.s.} X$ are verified in fractional probability space. Furthermore, regarding the classical definition of the convergence in probability $X_n \xrightarrow{P} X$ the convergence in fractional probability $X_n \xrightarrow{P_\alpha} X$ is defined. Also the relation between the almost surely convergence $X_n \xrightarrow{a.s.} X$ in fractional probability space and the convergence in fractional probability $X_n \xrightarrow{P_\alpha} X$ is verified. Finally, the validity of the important 0-1 kolmogorov theorem in the fractional probability space (Ω, F, P_α) is proved.

2. Fractional random variable

A random variable X is a measurable function form the sample space Ω to R;

$$X : \Omega \to R,$$

That is the inverse image of any Borel set F-measurable:

$$X^{-1}(\beta) = \{\omega \in \Omega : X(\omega) \in \mathrm{B}\} \in F \quad for\ all\ A \in R.\ [9, 10]$$

In other words, random variable is a function by which inverse projection of any Borel set is an event. But, being random variable is not a natural feature of functions, so a fractional random variable is only expressed when a fractional probability measure is defined. That is, after defining fractional probability measure for a random variable, that random variable is called a fractional random variable. Properties of random variables are satisfied for fractional random variables. For example, let X, Y denote two fractional random variables on (Ω, F). Then, by using transformation of fractional probability density, fractional probability density function of the random variable $X+Y$ is obtained. Therefore, $X+Y$ is a fractional random variable.

3. Almost surely convergence in fractional probability space

Assume that Q is a specific property of sample space. However, there is a possibility that all members of sample space Ω do not have this property. Then, it is said that this property is almost surely satisfied when, first, those points that do not have the property are events, and second, the probability of the event is zero. In other words, we can state that Q is almost surely satisfied if $P\{\omega \in \Omega :\sim Q(\omega)\} = 0$. [9, 10]

Now we can claim in the fractional case Q is almost surely satisfied if $P_\alpha\{\omega \in \Omega :\sim Q(\omega)\} = 0$. That is, fractional probability of the event $\{\omega \in \Omega :\sim Q(\omega)\}$ (the points that do not have the property Q) is zero. For example, suppose we are given a $\Omega = \{1,2,3,4\}$ and $F = P(\Omega)$, $P_\alpha(\{4\}) = \frac{1}{2}$, $P_\alpha(\{2\}) = P_\alpha(\{3\}) = \frac{1}{4}$, $P_\alpha(\{4\}) = 0$.

Now, if fractional random variable X on Ω is set as

$$X(1) = 5,\ X(2) = X(3) = X(4) = 0.$$

Then, the fractional random variable X in all points of Ω, except the point $\omega = 1$, is zero. According to the mathematical analysis concepts, we cannot say the function X is zero. On the other hand, fractional probability of the point, in which the function X is not zero, is zero, $P_\alpha(\{1\}) = 0$. Then $X = 0$, $a.s.$. Since fractional random variables are a specific branch of the classical random variables and they are classified as functions, therefore, we can discuss their convergence.

Definition 5: pointwise convergence. Suppose that $< X_n >$ is a sequence of fractional random variable on fractional probability space (Ω, F, P_α). The sequence $< X_n >$ of random variables is called pointwise convergence when numerical sequence $< X_n(\omega) >$ is convergence for any $\omega \in \Omega$. Certainly, value of $\lim_{n\to\infty} X_n(\omega)$ depends on ω, so the value of $\lim_{n\to\infty} X_n(\omega)$, for any $\omega \in \Omega$, is obtained in R. Therefore, we have a function such as X from Ω to R ($X : \Omega \to R$) that is defined in the point $\omega \in \Omega$ by the relation $X(\omega) = \lim_{n\to\infty} X_n(\omega)$, which itself is a fractional random variable.

If X is the pointwise limit of $< X_n >$, then we have $X = \lim_{n\to\infty} X_n$, $pointwise$. Or, $X_n \overset{p.w}{\to} X$. So, the definition of pointwise convergence for fractional random variables is exactly the same as the classical one in the probability theory. If for the set of points in which $< X_n >$ (a sequence of fractional random variables on fractional probability space (Ω, F, P_α)) is not convergent, the fractional probability is zero, then we can state that $< X_n >$ is almost surely convergent to X and we have $X = \lim_{n\to\infty} X_n$, $a.s.$ or $X_n \overset{a.s.}{\to} X$. As we already know, if $X_n \overset{a.s.}{\to} X$, then $P(\omega : X_n(\omega) \to X(\omega)) = 1$, hence, almost surely convergence is also called convergence with a probability of one. Now in fractional probability space if $X_n \overset{a.s.}{\to} X$, then $P_\alpha(\omega : X_n(\omega) \not\to X(\omega)) = 0$. Therefore, almost surely convergence, in a fractional probability space, is also called convergence with fractional probability one, because as we know $1 \le P_\alpha(A) + P_\alpha(A^C)$, so if we define $A = (\omega : X_n(\omega) \to X(\omega))$, then

$$1 \le P_\alpha(\omega : X_n(\omega) \to X(\omega)) + P_\alpha(\omega : X_n(\omega) \not\to X(\omega)),$$

where $P_\alpha(\omega : X_n(\omega) \not\to X(\omega)) = 0$.

So,

$$1 \le P_\alpha(\omega : X_n(\omega) \to X(\omega)).$$

According to the first condition of fractional probability principle,

$$P_\alpha(\omega : X_n(\omega) \to X(\omega)) \le 1.$$

So,

$$1 \le P_\alpha(\omega : X_n(\omega) \to X(\omega)) \le 1.$$

Therefore,

$$P_\alpha(\omega : X_n(\omega) \to X(\omega)) = 1.$$

Remark 1 Suppose that we are given the fractional probability space (Ω, F, P_α). If $X_n \overset{p.w}{\to} X$, then $X_n \overset{a.s.}{\to} X$.

Example 2 Let $([0,1], \mathrm{B}([0,1]), P_\alpha)$ be a fractional probability space in which fractional probability function is defined by the expression

$$P_\alpha(x \in [0,1]) = \frac{1}{(1-0)^\alpha} = 1,$$

where $P_\alpha(x \in [a,b]) = \frac{1}{(b-a)^\alpha}$ is the uniform probability density of fractional order α on $[a,b]$. The sequence of functions X_n on $[0,1]$ is defined as

$$X_n(\omega) = \omega^n, \ \omega \in [0,1].$$

For any n, X_n is a random variable and also it is a fractional random variable. Because this random variable is defined in the fractional probability space (Ω, F, P_α), we have

$$\lim_{n\to\infty} X_n(\omega) = \begin{cases} 0 & 0 \le \omega < 1 \\ 1 & \omega = 1 \end{cases}$$

Because X_n does not tend to zero only in $\omega = 1$ and the fractional probability of this point is zero, so $\lim_{n\to\infty} X_n = 0 \ (a.s.)$.

In the classical probability calculus, it has been proved as a lemma that $X_n \xrightarrow{a.s.} X$ if and only if for any $\varepsilon > 0$, $P(|X_n - X| > \varepsilon : i.o.) = 0$. Now by giving an example it is illustrated that this lemma is not satisfied in the fractional probability calculus. That means we cannot claim that $X_n \xrightarrow{a.s.} X$ if and only if for any $\varepsilon > 0$, $P_\alpha(|X_n - X| > \varepsilon : i.o.) = 0$.

Example 3 Suppose $([0,a], \mathrm{B}([0,a]), P_\alpha)$ is a fractional probability space and we have $P_\alpha = (l([0,a]))^{-\alpha} = a^{-\alpha}, a > 1$. The sequence of functions X_n on $[0,a]$ is defined as following

$$X_n(\omega) = (\frac{\omega}{a})^n \quad , \omega \in [0,a].$$

So for $n, n \in N$, X_n is a fractional random variable and we have

$$\lim_{n\to\infty} X_n(\omega) = \begin{cases} 0 & 0 \le \omega < a \\ 1 & \omega = a \end{cases}$$

Since X_n only in $\omega = a$ does not tend to zero and the fractional probability of this point is zero, $\lim_{n\to\infty} X_n = 0 \ (a.s.)$ and $X_n \xrightarrow{a.s.} X = 0$.

Now we prove that $P_\alpha(|X_n - X| > \varepsilon : i.o.) \ne 0$, for any $\varepsilon > 0$. As we know $P_\alpha(A_n : i.o) = P_\alpha(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k) = P_\alpha(\lim_{n\to\infty}\bigcup_{k=n}^{\infty} A_k)$, so

$$\begin{aligned} P_\alpha(|X_n - 0| > \varepsilon : i.o.) &= P_\alpha(\bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty}\{\omega : |X_n(\omega)| \ge \varepsilon\}) \\ &= P_\alpha(\lim_{N\to\infty}\bigcup_{n=N}^{\infty}\{\omega : |X_n(\omega)| \ge \varepsilon\}) \\ &= P_\alpha(\lim_{N\to\infty}\bigcup_{n=N}^{\infty}\left\{\omega : (\frac{\omega}{a})^n \ge \varepsilon\right\}) \end{aligned}$$

According to the fractional probability function $P_\alpha = (l([0,a]))^{-\alpha} = a^{-\alpha}$, $a > 1$, $\lim_{N\to\infty} P_\alpha(\bigcup_{n=N}^{\infty}\left\{\omega : (\frac{\omega}{a})^n \geq \varepsilon\right\})$ is obtained as following

$$\lim_{N\to\infty} P_\alpha(\bigcup_{n=N}^{\infty}\left\{\omega : (\frac{\omega}{a})^n \geq \varepsilon\right\}) = \lim_{N\to\infty} P_\alpha(\bigcup_{n=N}^{\infty}\left\{\omega : \frac{\omega}{a} \geq \varepsilon^{\frac{1}{n}}\right\})$$

Whereas variation amplitude of ω is the interval $[0,a]$, $\frac{\omega}{a}$ varies between 0 and 1. That is $\frac{\omega}{a} \in [0,1]$. So, for obtaining $\bigcup_{n=N}^{\infty}\left\{\omega : \frac{\omega}{a} \geq \varepsilon^{\frac{1}{n}}\right\}$, for any $n \in [N, \infty]$, we have

$$n = N : \left\{\omega : (\frac{\omega}{a}) \in [\varepsilon^{\frac{1}{N}}, 1]\right\}$$

$$n = N+1 : \left\{\omega : (\frac{\omega}{a}) \in [\varepsilon^{\frac{1}{N+1}}, 1]\right\},$$

and so on. Therefore, we have

$$\begin{aligned}
P_\alpha(\bigcup_{n=N}^{\infty}\left\{\omega : (\frac{\omega}{a})^n \geq \varepsilon\right\}) &= P_\alpha(\bigcup_{n=N}^{\infty}\left\{\omega : \frac{\omega}{a} \in [\varepsilon^{\frac{1}{n}}, 1]\right\}) \\
&= P_\alpha(\left\{\omega : \frac{\omega}{a} \in [\varepsilon^{\frac{1}{N}}, 1]\right\}) \\
&= P_\alpha(\left\{\omega \in [a\varepsilon^{\frac{1}{N}}, a]\right\})
\end{aligned}$$

$$\begin{aligned}
P_\alpha(\left\{\omega : \omega \in [a\varepsilon^{\frac{1}{N}}, a]\right\}) &= \int_{a\varepsilon^{\frac{1}{N}}}^{a} a^{-\alpha} (dx)^\alpha \\
&= \int_0^{1-\varepsilon^{\frac{1}{N}}} (du)^\alpha, \; u = \frac{x}{a} - \varepsilon^{\frac{1}{N}} \\
&= \alpha \int_0^{1-\varepsilon^{\frac{1}{N}}} (1 - \varepsilon^{\frac{1}{N}} - u)^{\alpha-1} du \\
&= -(1 - \varepsilon^{\frac{1}{N}} - u)^\alpha = (1 - \varepsilon^{\frac{1}{N}})^\alpha.
\end{aligned}$$

Therefore,

$$\lim_{N\to\infty} P_\alpha(\bigcup_{n=N}^{\infty}\left\{\omega : (\frac{\omega}{a})^n \geq \varepsilon\right\}) = \lim_{N\to\infty} (1 - \varepsilon^{\frac{1}{N}})^\alpha = 0$$

According to the continuity of fractional probability functions [11], the equality $P_\alpha(\lim_{n\to\infty} A_n) = \lim_{n\to\infty} P_\alpha(A_n)$ is not satisfied. So, in spite of $\lim_{N\to\infty} P_\alpha(\bigcup_{n=N}^{\infty}\left\{\omega : (\frac{\omega}{a})^n \geq \varepsilon\right\}) = 0$, we cannot claim that

$$P_\alpha(\lim_{N\to\infty} \bigcup_{n=N}^{\infty}\{\omega : |X_n(\omega)| \geq \varepsilon\}) = 0$$
$$P_\alpha(|X_n - 0| > \varepsilon : i.o.) = 0$$

Therefore, it is concluded that the lemma, $X_n \xrightarrow{a.s.} X$ if and only if for any $\varepsilon > 0$, $P(|X_n - X| > \varepsilon : i.o.) = 0$, is not satisfied in fractional probability space.

4. Convergence in fractional probability

Definition 6 The sequence $< X_n >$ of random α-variables converges to a random α-variable X in fractional probability, written $X_n \xrightarrow{P_\alpha} X$, if for any $\varepsilon > 0$

$$\lim_{n\to\infty} P_\alpha(|X_n - X| > \varepsilon) = 0 .$$

Almost sure convergence in fractional probability space does not imply convergence in fractional probability. By giving the following example we demonstrate that in the fractional probability space we cannot prove that if $X_n \xrightarrow{a.s.} X$, then $X_n \xrightarrow{P_\alpha} X$.

Example 4 Suppose $([0,n], \mathrm{B}([0,n]), P_\alpha)$ is a fractional probability space and

$$P_\alpha = (l([0,n]))^{-\alpha} = n^{-\alpha}, n \geq 1 .$$

The sequence of functions X_n on $[0,n]$ is defined as the following

$$X_n(\omega) = \frac{\omega}{n}, \ \ \omega \in [0,n]$$

So that for n, $n \in N$, X_n is a random α-variable and we have

$$\lim_{n\to\infty} X_n(\omega) = \begin{cases} 0 & 0 \leq \omega < n \\ 1 & \omega = n \end{cases}$$

$\lim_{n\to\infty} X_n = 0 \ \ (a.s.)$.

Now we need to show that $\lim_{n\to\infty} P_\alpha(|X_n - X| > \varepsilon) \neq 0$. By using fractional calculus and transformation of fractional probability density, the fractional probability density of $X_n(\omega) = \frac{\omega}{n}, \ \omega \in [0,n]$ is obtained as the following. The sequence of functions X_n on $[0,n]$ is set as $X_n(\omega) = \frac{\omega}{n}$, $\omega \in [0,n]$. For any n, X_n is a fractional random variable or random α-variable. As follows

$$\lim_{n\to\infty} X_n(\omega) = 0, \ 0 \leq \omega \leq n$$

$$P_\alpha(\lim_{n\to\infty} X_n(\omega)) = P_\alpha(0) = P_\alpha(0 \le \omega \le n) = n^{-\alpha}$$

$$P_\alpha(X_n(\omega)) = P_\alpha(\frac{\omega}{n})$$

Based on the fractional probability density of ω, $P_\alpha(\omega)$ and using transformation of fractional probability density, fractional probability function (fractional probability density) of the random α-variable X_n, $P_\alpha(\frac{\omega}{n})$, is calculated as bellow

$$P_\alpha(\omega) = n^{-\alpha},\ \ 0 \le \omega \le n$$

$$\int_0^n n^{-\alpha}(d\omega)^\alpha = 1$$

$$d(X_n(\omega)) = \frac{d\omega}{n},\ \ d\omega = nd(X_n)$$

$$\int_0^{\frac{n}{n}=1} P_\alpha(nX_n)(ndX_n)^\alpha = 1,\ \ 0 \le X_n(\omega) \le \frac{n}{n} = 1.$$

Since $P_\alpha(nX_n) = n^{-\alpha}$, one has

$$\int_0^1 (dX_n)^\alpha = 1,\ \ 0 \le X_n(\omega) \le 1$$

So, $P_\alpha(X_n(\omega)) = n^\alpha n^{-\alpha} = 1$, which leads to $\lim_{n\to\infty} P_\alpha(X_n(\omega)) = \lim_{n\to\infty} \frac{n^\alpha}{n^\alpha} = 1$.

So, almost sure convergence in (Ω, F, P_α) does not imply convergence in fractional probability.

To show that the convergence in fractional probability does not imply almost sure convergence in fractional probability space we have the following example

Example 5 Suppose $\Omega = [0,1]$, $F = \mathrm{B}([0,1])$ and P_α is the fractional probability measure that is denoted by $P_\alpha = (lenght\ of\ a\ sub\,\mathrm{int}\,erval\ of\ [0,1]\,)^\alpha$. The sequence of random variables $X_{n,i}$, $n \ge 1$, $1 \le i \le n$ is defined as

$X_{1,1} = I_{[0,1]}$

$X_{2,1} = I_{[\circ,\frac{1}{2}]}$ $\quad X_{2,2} = I_{[\frac{1}{2},1]}$

$X_{3,1} = I_{[\circ,\frac{1}{3}]}$ $\quad X_{3,2} = I_{[\frac{1}{3},\frac{2}{3}]}$ $\quad X_{3,3} = I_{[\frac{2}{3},1]}$

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$X_{n,1} = I_{[0,\frac{1}{n}]}$ $\quad X_{n,2} = I_{[\frac{1}{n},\frac{2}{n}]}$ $\quad \ldots \quad X_{n,n} = I_{[\frac{n-1}{n},1]}$.

And the sequence of random variables Y_n is defined as

$$Y_1 = X_{1,1} = \begin{cases} 0 & X \notin [0,1] \\ 1 & X \in [0,1] \end{cases}$$

$$Y_2 = X_{2,1} \quad , \quad Y_3 = X_{2,2}$$

$$Y_4 = X_{3,1} \quad , \quad Y_5 = X_{3,2} \quad , \quad Y_6 = X_{3,3} \ldots$$

It is going to be outlined that $< Y_n >$ in fractional probability is convergent to zero ($Y_n \xrightarrow{P_\alpha} Y = 0$) but $< Y_n >$ is not almost surely convergent. First it is noticed that Y_ns have the values zero or one. So suppose that $n \in N$ is arbitrary then $X_{n,i}$ on one of the subintervals $[\frac{n-1}{n},1],\ldots,[0,\frac{1}{n}]$ is 1. Therefore, for any n there is a point in the sample space in which $Y_n = 1$. So $\overline{\lim} Y_n = 1$ and $\underline{\lim} Y_n = 0$. However the fractional probability of this point in which $Y_n = 1$ is not zero but is $P_\alpha(Y_n = 1) = (\frac{1}{n})^\alpha$.

On the other hand, Y_n`s index is in the form of $\frac{n(n-1)}{2} + k$ that $n = 1, 2, \ldots, 1 \le k \le n$. If $m = \frac{n(n-1)}{2} + k$, then $P_\alpha(Y_m = 1) = (\frac{1}{n})^\alpha$. As a result,

$$P_\alpha(|Y_m - 0| > \varepsilon) = P_\alpha(Y_m = 1) = (\frac{1}{n})^\alpha .$$

Therefore,

$$\lim_{n\to\infty} P_\alpha(|Y_n| > \varepsilon) = \lim_{n\to\infty} (\frac{1}{n})^\alpha = 0 .$$

So, we have $Y_n \xrightarrow{P_\alpha} 0$. Thus, convergence in fractional probability does not imply almost sure convergence in (Ω, F, P_α).

Theorem 2 Suppose that $\{X_n, X, n \ge 1\}$ are random α-variables. $\{X_n\}$ converges in fractional probability if $\{X_n\}$ is Cauchy in fractional probability. Cauchy in fractional probability means $X_n - X_m \xrightarrow{P_\alpha} 0 \;\; as \;\; n,m \to \infty$ or more precisely, given any $\varepsilon > 0, \delta > 0$, there exists $n_\circ = n_\circ(\varepsilon, \delta)$ such that for all $r, s \ge n_\circ$ we have

$$P_\alpha[|X_r - X_s| > \varepsilon] < \delta.$$

Proof It is illustrated that if $X_n \xrightarrow{P_\alpha} X$ then $\{X_n\}$ is Cauchy in fractional probability. For any $\varepsilon > 0$,

$$[|X_r - X_s| > \varepsilon] \subset \left[|X_r - X| > \frac{\varepsilon}{2}\right] \bigcup \left[|X_s - X| > \frac{\varepsilon}{2}\right]$$

Thus, taking fractional probabilities, we have

$$P_\alpha[|X_r - X_s| > \varepsilon] \le P_\alpha\left[|X_r - X| > \frac{\varepsilon}{2}\right] + P_\alpha\left[|X_s - X| > \frac{\varepsilon}{2}\right].$$

If $P_\alpha\left[\left|X_n - X\right| > \varepsilon\right] \le \dfrac{\delta}{2}$, for any $n_\circ \ge n_\circ(\varepsilon,\delta)$. Then, $P_\alpha\left[\left|X_r - X_s\right| > \varepsilon\right] \le \delta$, for any $r,s \ge n_\circ$.

5. 0-1 kolmogrove theorem in fractional probability space

Suppose $< X_n >$ is a sequence of fractional random variables. We define the following σ-algebras

$$\mathrm{A}_n = \sigma\{X_{n+1}, X_{n+2}, \ldots\},\ \mathrm{A} = \bigcap_{n=1}^{\infty} \mathrm{A}_n$$

Theorem 3 (0-1 kolmogrove theorem in fractional probability space). Suppose $< X_n >$ is a sequence of independent variables. So, for any $A \in \mathrm{A}$, we have $P_\alpha(A) = 0$ Or $P_\alpha(A) = 1$.

Proof Let $D_n = \sigma\{X_1, X_2, \ldots X_n\}$ for any $n \ge 2$. So, for any n and for any $A \in \mathrm{A}_n$ and for any $B \in D_n$, we have $P_\alpha(A \bigcap B) = P_\alpha(A)P_\alpha(B)$. This equality is also satisfied for any n and for any $A \in \mathrm{A}_n$ and for any $B \in \bigcup_{n=1}^{\infty} D_n$. We assume $D = \bigcup_{n=1}^{\infty} D_n$. So, according to the classical probability theory, D has the following properties

- D is closed under finite intersection,
- Precisely D consists of Ω.

D also has another class of properties as following

- D Consists of D,
- D is closed under difference operation,
- D is closed under increasing and countable union of their members.

According to these properties and some theorems of classical probability theory, it is proved that $\sigma(D) = D$. So, for any $A \in \mathrm{A}_n$ and for any $B \in D$, we have

$$P_\alpha(A \bigcap B) = P_\alpha(A)P_\alpha(B).$$

But, $A \subset D$. Therefore, by assuming $A = B$ in above equality, we would have

$$\forall A \in \mathrm{A} \quad P_\alpha(A \bigcap A) = P_\alpha(A)P_\alpha(A).$$

And, it is concluded that any member of A has the probability equal to zero or one.

6. Concluding remarks

In this study we have proposed a new section in fractional probability theory named as "convergences in the fractional probability space (Ω, F, P_α)" to expand a probability theory of fractional order completely parallel to the classical probability theory. It would also be of interest to study some other probabilistic concepts and theorems in the fractional probability space such as Hilbert space and L^p convergence for the fractional probability measure P_α.

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