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ON ASYMPTOTIC STABILITY OF A CLASS OF TIME-DELAY SYSTEMS

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ABSTRACT In this paper, we give some new necessary and sufficient conditions for the asymptotic stability of a class of time-delay systems of the form

$$x'(t) + (1-a)x(t) + A(x(t-k) + x(t-l)) = 0, \quad t \geq 0,$$

where a is a real number, A is a 2×2 real constant matrix, and k, l are positive numbers such that $k > l$.

1. INTRODUCTION

In this paper, we study the asymptotic stability of the solutions of time delay systems of the form

$$(1.1) \quad x'(t) + (1-a)x(t) + A(x(t-k) + x(t-l)) = 0, \quad t \geq 0,$$

where a is a real number, A is a 2×2 real constant matrix, and k, l are positive numbers such that $k > l$. Time-delay systems are a type of differential equations in which the derivative of the unknown function at a certain time is given in terms of the values of the function at previous times. Also, they are called delay differential equations, retarded differential equations or differential-difference equations. On the other hand, since asymptotic stability is an interdisciplinary material, the asymptotic stability of these systems has a wide range of applications as biology, physics and medicine. For instance, Naresh et al. (2009) formulated a nonlinear mathematical model to study the framework of disease Epidemiology. As another example Ikeda and Watanabe (2014) were investigated the stochastic differential equations and diffusion processes in their study about physic. Kruthika et al. (2017) were studied stability analysis of nonlinear time-delayed systems with application to biological models. They analysed the local stability of a gene-regulatory network and immunotherapy for cancer modelled as nonlinear time-delay systems.

Many authors have also focused on the asymptotic stability of time delay systems. Some important studies about related subject can be examined from the below authors. Bellman and Cooke (1963), Cooke and Grossman (1982), Cooke and Van den Driessche (1986), Stépán (1989), Freedman and Kuang (1991), Kuang (1993), Ruan and Wei (2003), Elaydi (2005), Matsunaga (2008), Smith (2010), Gray et al. (2011), Khokhlova et al. (2011), Xu (2012),

Hrabalova (2013), Nakajima (2014), Liu et al. (2016), Li et al. (2018). Also, an equation which is the special case of our system which is investigated by Kuang (1993) and he demonstrated that the zero solution of the delay differential equation with two delays of the form

$$(1.2) \quad x'(t) + q(x(t-k) + x(t-l)) = 0, \quad t \geq 0,$$

where $k \geq 0$, $l \geq 0$ and q is positive constant, is asymptotically stable if and only if

$$(1.3) \quad 2q(k+l) \cos\left(\frac{k-l}{k+l} \frac{\pi}{2}\right) < \pi.$$

2. PRELIMINARIES

It is known that for the time delay equations, an equation is asymptotically stable if and only if all roots of the associated characteristic equation have negative real parts. Stability analysis, however, does not require the exact calculation of the characteristic roots; only the sign of the real part of the critical root must be determined. This analysis can be performed by D-subdivision method (see, e.g., Stépán [1998], Insperger and Stépán [2011]), which gives a necessary and sufficient condition for stability based on the coefficients of the characteristic equation. The aim of this paper is to obtain new results for the asymptotic stability of zero solution of system (1.1), while the characteristic equation of system (1.1) has roots on the imaginary axis when A is a constant matrix. If we get $x(t) = Py(t)$ for a regular matrix P in (1.1), then we have the following system;

$$y'(t) + (1-a)y(t) + P^{-1}AP(y(t-k) + y(t-l)) = 0, \quad t \geq 0.$$

Thus, matrix A can be given one of the following two matrices in Jordan form [7]:

$$(I) A = \begin{pmatrix} q_1 & r \\ 0 & q_2 \end{pmatrix}, \quad q_1, q_2 \text{ and } r \text{ are real constants,}$$

$$(II) A = q \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad q, \theta \text{ are real constants and } |\theta| \leq \frac{\pi}{2}.$$

Here we consider the case (II), the other case should be considered similarly. The characteristic equation of system (1.1) is given as

$$(2.1) \quad F(\lambda) := \det(\lambda I_2 + (1-a)I_2 + A(e^{-\lambda k} + e^{-\lambda l})) = 0,$$

where I_2 is the 2×2 identity matrix. Using (2.1), we obtain

$$\begin{aligned} F(\lambda) &= (\lambda + (1-a) + q \cos \theta (e^{-\lambda k} + e^{-\lambda l}))^2 + (q \sin \theta (e^{-\lambda k} + e^{-\lambda l}))^2 \\ &= (\lambda + (1-a) + q \cos |\theta| (e^{-\lambda k} + e^{-\lambda l}))^2 - (iq \sin |\theta| (e^{-\lambda k} + e^{-\lambda l}))^2 \\ &= (\lambda + (1-a) + qe^{i|\theta|} (e^{-\lambda k} + e^{-\lambda l}))(\lambda + (1-a) + qe^{-i|\theta|} (e^{-\lambda k} + e^{-\lambda l})). \end{aligned}$$

If we let

$$(2.2) \quad f(\lambda) = (\lambda + (1-a) + q(e^{-\lambda k + i|\theta|} + e^{-\lambda l + i|\theta|})),$$

then we have

$$F(\lambda) = f(\lambda) \overline{f(\bar{\lambda})},$$

where $\bar{\lambda}$ is the complex conjugate of any complex λ . Note that $\overline{f(\bar{\lambda})} = 0$ implies $f(\bar{\lambda}) = 0$.

3. SOME AUXILIARY LEMMAS

In this section, we will investigate the distribution of the zeros of the characteristic equation of system (1.1). Thus, we state and prove some basic results on the roots of the characteristic equation of system (1.1).

Lemma 1 (Stepan, 1989) *The zero solution of (1.1) is asymptotically stable if and only if all the roots of equation*

$$(3.1) \quad f(\lambda, k, l) = \lambda + (1-a) + q(e^{-\lambda k + i|\theta|} + e^{-\lambda l + i|\theta|}) = 0,$$

lie in the left half of the complex plane.

Since f is an analytic function of λ , k and l for the fixed numbers a, q and θ , one can regard the root $\lambda = \lambda(k, l)$ of (3.1) as a continuous function of k and l . The next lemma plays very important role for the main theorem.

Lemma 2 (Cooke and Grossman, 1982) *As k and l vary, the sum of the multiplicities of the roots of (3.1) in the open right half-plane can change only if a root appears on or crosses the imaginary axis.*

Consequently, we claim that (3.1) has only imaginary roots $\pm i\omega$. We will determine how the value of k and l change as equation (3.1) has roots on the imaginary axis. Now, we can write the characteristic equation (3.1) as follows;

$$(3.2) \quad \lambda + (1-a) + q(e^{-\lambda k + i\theta} + e^{-\lambda l + i\theta}) = 0.$$

At the same time, we take $\lambda = i\omega$ such that $\omega \in \mathbb{R}$. Firstly, since $f(0) = (1-a) + 2qe^{i\theta} \neq 0$, we see that $\omega \neq 0$. For $\omega \neq 0$, we get

$$(3.3) \quad f(i\omega) = i\omega + (1-a) + q(e^{-i\omega k + i\theta} + e^{-i\omega l + i\theta}) = 0.$$

Using the real part and the imaginary part of (3.3)

$$(3.4) \quad \omega = q(\sin(\omega k - \theta) + \sin(\omega l - \theta)),$$

$$(3.5) \quad a - 1 = q(\cos(\omega k - \theta) + \cos(\omega l - \theta)),$$

which is equivalent to

$$(3.6) \quad \omega = 2q \sin\left(\frac{\omega(k+l)}{2} - \theta\right) \cos\left(\frac{\omega(k-l)}{2}\right),$$

$$(3.7) \quad a-1 = 2q \cos\left(\frac{\omega(k+l)}{2} - \theta\right) \cos\left(\frac{\omega(k-l)}{2}\right),$$

is obtained.

Lemma 3. Suppose that, $q > 0$ and $0 < \theta \leq \frac{\pi}{2}$. Let $\lambda = i\omega$ be a root (3.1) where $\omega \in \left(\frac{-\pi}{k-l}, \frac{\pi}{k-l}\right) - \left\{\frac{-n\pi+2\theta}{k+l}, \frac{n\pi+2\theta}{k+l}\right\}$ for $n = 0, 1, 2, \dots$, then the following conditions hold:

(i) If $\left(2q \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a-1)^2 \leq 0$, then there exists no real number ω .

(ii) If $\left(2q \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a-1)^2 > 0$, then the real numbers ω , q and the delays $(k+l)$ are as follows:

$$\omega = \pm\gamma = \pm\sqrt{\left(2q \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a-1)^2}, \quad q = \frac{a-1}{2 \cos\left(\frac{\omega(k+l)}{2} - \theta\right) \cos\left(\frac{\omega(k-l)}{2}\right)} \text{ and}$$

$k+l = (k_n + l_n)^\pm$ for $n = 0, 1, 2, \dots$ where $(k_n + l_n)^+$ and $(k_n + l_n)^-$;

$$(k_n + l_n)^+ = \frac{2}{\gamma} \left\{ 2n\pi + \arccos\left(\frac{a-1}{2q \cos\left(\frac{\omega(k-l)}{2}\right)}\right) + \theta \right\}$$

$$(k_n + l_n)^- = \frac{2}{\gamma} \left\{ 2n\pi + \arccos\left(\frac{a-1}{2q \cos\left(\frac{\omega(k-l)}{2}\right)}\right) - \theta \right\},$$

and $i\gamma$ or $-i\gamma$ is a root of (3.1) for the sum of delays $(k_n + l_n)^+$ or $(k_n + l_n)^-$.

Proof. By squaring both sides of (3.6) and (3.7), and adding them together, we get

$$(3.8) \quad \omega^2 + (a-1)^2 = \left(2q \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2.$$

If $\left(2q \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a-1)^2 \leq 0$, then statement (3.8) implies $\omega^2 < 0$, contradicts with $\omega^2 > 0$. Since $\omega \neq 0$, condition (i) is verified; that is, (3.1) has no root on the imaginary axis

for all $k > l > 0$.

On the other hand, if $\left(2q \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a-1)^2 > 0$, statement (3.8) implies that

$$(3.9) \quad \omega = \pm \sqrt{\left(2q \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a-1)^2}.$$

If we let

$$\gamma = \sqrt{\left(2q \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a-1)^2},$$

then we can write $\omega = \pm\gamma$. By (3.7), we have

$$q = \frac{a-1}{\cos\left(\frac{\omega(k+l)}{2} - \theta\right) \cos\left(\frac{\omega(k-l)}{2}\right)}.$$

Now, we will show that $i\gamma$ is a root of (3.1).

In case of $\omega = \gamma$, since $\cos\left(\frac{\omega(k-l)}{2}\right) > 0$ for $\omega \in \left(0, \frac{\pi}{k-l}\right) - \left\{\frac{n\pi+2\theta}{k+l}\right\}$ and using (3.6)

$$(3.10) \quad \sin\left(\frac{\omega(k+l)}{2} - \theta\right) > 0,$$

is obtained. Thus, we use (3.6) and (3.7), we have

$$\frac{\omega(k+l)}{2} - \theta = 2n\pi + \arccos\left(\frac{a-1}{2q \cos\left(\frac{\omega(k-l)}{2}\right)}\right) \quad n = 0, 1, 2, \dots,$$

which yields $(k_n + l_n)^+$. Also, from (3.6) and (3.7), we get

$$\sin\left(\arccos\left(\frac{a-1}{2q \cos\left(\frac{\omega(k-l)}{2}\right)}\right)\right) = \frac{\gamma}{2q \cos\left(\frac{\omega(k-l)}{2}\right)}$$

because of

$$\arccos\left(\frac{a-1}{2q \cos\left(\frac{\omega(k-l)}{2}\right)}\right) = \begin{cases} \arcsin\left(\frac{\gamma}{2q \cos\left(\frac{\omega(k-l)}{2}\right)}\right) & \text{if } a-1 \geq 0 \\ \pi - \arcsin\left(\frac{\gamma}{2q \cos\left(\frac{\omega(k-l)}{2}\right)}\right) & \text{if } a-1 < 0. \end{cases}$$

Hence, for the case $k+l = (k_n + l_n)^+$, $i\gamma$ is a root of (3.1). Indeed,

$$\begin{aligned} f(i\omega) &= i\omega + (1-a) + q\left(e^{i(\omega k - \theta)} + e^{i(\omega l - \theta)}\right). \\ &= i\sqrt{\left(2q \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a-1)^2} + (1-a) + 2q \cos\left(\frac{\omega(k-l)}{2}\right) e^{-i\left(\frac{\omega(k+l)}{2} - \theta\right)}, \\ &= i\sqrt{\left(2q \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a-1)^2} + (1-a) + \\ &\quad + 2q \cos\left(\frac{\omega(k-l)}{2}\right) e^{-i\left(2n\pi + \arccos\left(\frac{a-1}{2q \cos\left(\frac{\omega(k-l)}{2}\right)}\right)\right)}, \\ &= i\sqrt{\left(2q \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a-1)^2} + (1-a) + 2q \cos\left(\frac{\omega(k-l)}{2}\right) \times \\ &\quad \times \left\{ \cos\left(\arccos\left(\frac{a-1}{2q \cos\left(\frac{\omega(k-l)}{2}\right)}\right)\right) - i \sin\left(\arccos\left(\frac{a-1}{2q \cos\left(\frac{\omega(k-l)}{2}\right)}\right)\right) \right\}, \\ &= i\sqrt{\left(2q \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a-1)^2} - i\sqrt{\left(2q \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a-1)^2}, \\ &= 0. \end{aligned}$$

Thus, this implies that $i\gamma$ is a root of (3.1). Similarly, in case $\omega < 0$, it can be shown that $-i\gamma$ is a root of (3.1) for the sum of delays $(k_n + l_n)^-$. The proof is completed. \square

When $q > 0$, we have the following analogous result.

Lemma 4. Suppose that, $q < 0$ and $0 < \theta \leq \frac{\pi}{2}$. Let $\lambda = i\omega$ be a root of (3.1) where

$\omega \in \left(\frac{-\pi}{k-l}, \frac{\pi}{k-l}\right) - \left\{\frac{-n\pi+2\theta}{k+l}, \frac{n\pi+2\theta}{k+l}\right\}$ for $n = 0, 1, 2, \dots$, then the following conditions hold:

- (i) If $\left(2q \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a-1)^2 \leq 0$, then there exists no real number ω .

(ii) If $\left(2q \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a-1)^2 > 0$, then the real numbers ω , q and the delays $(k+l)$ are as follows:

$$\omega = \pm \gamma = \pm \sqrt{\left(2q \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a-1)^2}, \quad q = \frac{a-1}{2 \cos\left(\frac{\omega(k+l)}{2} - \theta\right) \cos\left(\frac{\omega(k-l)}{2}\right)}$$

and $k+l = (\alpha_n + \beta_n)^\pm$ for $n = 0, 1, 2, \dots$, where $(\alpha_n + \beta_n)^+$ and $(\alpha_n + \beta_n)^-$;

$$(\alpha_n + \beta_n)^+ = \frac{2}{\gamma} \left\{ 2n\pi - \arccos\left(\frac{a-1}{2q \cos\left(\frac{\omega(k-l)}{2}\right)}\right) + \theta \right\}$$

$$(\alpha_n + \beta_n)^- = \frac{2}{\gamma} \left\{ (2n+2)\pi - \arccos\left(\frac{a-1}{2q \cos\left(\frac{\omega(k-l)}{2}\right)}\right) - \theta \right\},$$

and $i\gamma$ or $-i\gamma$ is a root of (3.1) for the sum of delays $(\alpha_n + \beta_n)^+$ or $(\alpha_n + \beta_n)^-$.

Proof. The proof is similar to the lemma 3.

Remark 1. For $q > 0$, from the definitions of $(k_n + l_n)^\pm$;

$$\min\{(k_n + l_n)^\pm : n = 0, 1, \dots\} = \begin{cases} (k_0 + l_0)^- & \text{if } (a-1) - 2q \cos \theta \cos\left(\frac{\omega(k-l)}{2}\right) < 0 \\ (k_0 + l_0)^+ & \text{if } (a-1) - 2q \cos \theta \cos\left(\frac{\omega(k-l)}{2}\right) \geq 0, \end{cases}$$

is obtained. Similarly, for $q < 0$, we get from the definitions of $(\alpha_n + \beta_n)^\pm$ as follows;

$$\min\{(\alpha_n + \beta_n)^\pm : n = 0, 1, \dots\} = \begin{cases} (\alpha_0 + \beta_0)^+ & \text{if } (a-1) - 2q \cos \theta \cos\left(\frac{\omega(k-l)}{2}\right) < 0 \\ (\alpha_0 + \beta_0)^- & \text{if } (a-1) - 2q \cos \theta \cos\left(\frac{\omega(k-l)}{2}\right) \geq 0. \end{cases}$$

Lemma 5. Suppose that $0 < \theta \leq \frac{\pi}{2}$, $\omega \in \left(\frac{-\pi}{k-l}, \frac{\pi}{k-l}\right) - \{0\}$ and

$$\omega^2 + q^2 \omega(k+l) \sin(\omega(l-k)) > 0.$$

Also, the following conditions

$$(3.11) \quad \begin{cases} \frac{\sin(\omega k - \theta)}{l \sin \omega(k-l)} > q \text{ if } \frac{q}{\omega} > 0 \\ \frac{\sin(\omega k - \theta)}{l \sin \omega(k-l)} < q \text{ if } \frac{q}{\omega} < 0 \end{cases} \text{ and } \begin{cases} \frac{\sin(\omega l - \theta)}{k \sin \omega(k-l)} > q \text{ if } \frac{q}{\omega} > 0 \\ \frac{\sin(\omega l - \theta)}{k \sin \omega(k-l)} < q \text{ if } \frac{q}{\omega} < 0, \end{cases}$$

are provided. Then all the roots of equation (3.1) on the imaginary axis move in the right half-plane as k and l increase.

Proof. Let $\lambda = i\omega$ be a root of (3.1) where $\omega \in \left(\frac{-\pi}{k-l}, \frac{\pi}{k-l} \right) - \{0\}$ is a real number. It will be enough to show

$$\operatorname{Re} \frac{\partial \lambda}{\partial k} \Big|_{\lambda=i\omega} > 0 \text{ and } \operatorname{Re} \frac{\partial \lambda}{\partial l} \Big|_{\lambda=i\omega} > 0.$$

Firstly, we take the derivative of λ with respect to k on equation (3.1), we have

$$\frac{\partial \lambda}{\partial k} - q \lambda e^{-\lambda k + i\theta} - q k \lambda e^{-\lambda k + i\theta} \frac{\partial \lambda}{\partial k} - q l e^{-\lambda l + i\theta} \frac{\partial \lambda}{\partial k} = 0,$$

$$\frac{\partial \lambda}{\partial k} = \frac{q \lambda e^{-\lambda k + i\theta}}{1 - q(k e^{-\lambda k + i\theta} + l e^{-\lambda l + i\theta})}.$$

Substituting $\lambda = i\omega$ into the above equation, we get

$$\begin{aligned} \frac{\partial \lambda}{\partial k} \Big|_{\lambda=i\omega} &= \frac{i\omega + 1 - a}{1 - q(k e^{-i(\omega k - \theta)} + l e^{-i(\omega l - \theta)})} \\ &= \frac{i\omega + 1 - a}{1 - q(k \cos(\omega k - \theta) + l \cos(\omega l - \theta)) + iq(k \sin(\omega k - \theta) + l \sin(\omega l - \theta))} \end{aligned}$$

If we multiply with the complex conjugate of the denominator in the above equation, then we can write

$$(3.12) \quad \operatorname{Re} \frac{\partial \lambda}{\partial k} \Big|_{\lambda=i\omega} = \frac{q\omega \sin(\omega k - \theta) - q^2 \omega l \sin(\omega(k-l))}{M},$$

where

$$M = (1 - q(k \cos(\omega k - \theta) + l \cos(\omega l - \theta)))^2 + q^2 (k \sin(\omega k - \theta) + l \sin(\omega l - \theta))^2.$$

Since (3.11), we can write $\operatorname{Re} \frac{\partial \lambda}{\partial k} \Big|_{\lambda=i\omega} > 0$. On the other hand, we take the derivative of λ with respect to l on equation (3.1), similar to (3.12)

$$(3.13) \quad \operatorname{Re} \frac{\partial \lambda}{\partial l} \Big|_{\lambda=i\omega} = \frac{q\omega \sin(\omega l - \theta) - q^2 \omega k \sin(\omega(k-l))}{M},$$

is obtained. From (3.11), we get $\operatorname{Re} \frac{\partial \lambda}{\partial l} \Big|_{\lambda=i\omega} > 0$. Moreover, by adding both (3.12) and (3.13) together, we have

$$(3.14) \quad \operatorname{Re} \frac{\partial \lambda}{\partial k} \Big|_{\lambda=i\omega} + \operatorname{Re} \frac{\partial \lambda}{\partial l} \Big|_{\lambda=i\omega} = \frac{\omega^2 + q^2 \omega(k+l) \sin(\omega(l-k))}{M} > 0.$$

Hence, the proof is completed.

Now we can state and prove main theorems.

4. MAIN RESULTS

We will show that the stability analysis with a qualitative approach, as we have already mentioned in section 1.

Theorem 1. *Suppose that $0 < \theta \leq \frac{\pi}{2}$ and the conditions of lemma 5 are satisfied. Let the matrix A of the system (1.1) be in the form (II). Then system (1.1) is asymptotically stable if and only if either*

$$(4.1) \quad \begin{cases} (a-1) - 2q \cos \theta \cos\left(\frac{\omega(k-l)}{2}\right) < 0 \\ \left(2q \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a-1)^2 \leq 0, \end{cases}$$

or

$$(4.2) \quad \begin{cases} (a-1) - 2q \cos \theta \cos\left(\frac{\omega(k-l)}{2}\right) < 0 \\ \left(2q \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a-1)^2 > 0 \\ k+l < \frac{2 \operatorname{sgn}(q)}{\sqrt{\left(2q \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a-1)^2}} \left\{ \arccos\left(\frac{a-1}{2q \cos\left(\frac{\omega(k-l)}{2}\right)}\right) - \theta \right\}. \end{cases}$$

Proof. In case of $k=0$ and $l=0$, the root of (3.1) is only $\lambda(0,0) = a-1-2b \cos \theta - i2b \sin \theta$. Thus, the root of the equation (3.1) has a negative real part. By the continuity of the roots with respect to k and l , we can say that all the roots of (3.1) lie in the left half plane for $k > 0$ and

$l > 0$ sufficiently small.

For the sufficiency, here our claim is: If either condition (4.1) or (4.2) holds, then (3.1) does not have a root on the imaginary axis. By condition (4.1) and lemma 3, our claim is true for $k > 0$ and $l > 0$. Now, suppose that condition (4.2) holds: Since

$$\frac{2\text{sgn}(q)}{\sqrt{\left(2q\cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a-1)^2}} \left\{ \arccos\left(\frac{a-1}{2q\cos\left(\frac{\omega(k-l)}{2}\right)}\right) - \theta \right\} = \begin{cases} (k_0 + l_0)^- & \text{if } q > 0 \\ (\alpha_0 + \beta_0)^+ & \text{if } q < 0, \end{cases}$$

and remark 1, we get $(k+l) \neq (k_n + l_n)^\pm$, $(k+l) \neq (\alpha_n + \beta_n)^\pm$ for $n = 0, 1, 2, \dots$. Thus, we obtain the contraposition with lemma 1, our other claim is also true. By the above argument and lemma 2, we can say that if either condition (4.1) or (4.2) holds, then all the roots of (3.1) lie in the left half plane.

For the necessity, we will show the following contraposition; either

$$(4.3) \quad (a-1) - 2q \cos \theta \cos\left(\frac{\omega(k-l)}{2}\right) \geq 0,$$

or

$$(4.4) \quad \left\{ \begin{array}{l} \left(2q\cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a-1)^2 > 0 \\ k+l \geq \frac{2\text{sgn}(q)}{\sqrt{\left(2q\cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a-1)^2}} \left\{ \arccos\left(\frac{a-1}{2q\cos\left(\frac{\omega(k-l)}{2}\right)}\right) - \theta \right\} \end{array} \right\}.$$

Thus, if (4.3) and (4.4) hold, then there exists roots λ_j of (3.1) such that $Re(\lambda_j) > 0$ for $j=1, 2$. Assume that (4.3) holds and let $\lambda_1(k, l)$ be the branch of the root of satisfying $\lambda_1(0, 0) = a-1-2b\cos\theta-i2b\sin\theta$. Then, lemma 5 or the continuity of $\lambda_1(k, l)$ implies that $Re(\lambda_1) > 0$ for $k > 0$ and $l > 0$ sufficiently small. From here, we can say that $\lambda_1(k, l)$ cannot move in the left half-plane crossing on the imaginary axis as k and l increase. Hence, we have $Re(\lambda_1) > 0$ for all $k > 0$ and $l > 0$. Assume that (4.4) holds and let $\lambda_2(k, l)$ be the branch of the

root of satisfying $\lambda_2(k_0, l_0) = -i\text{sgn}(q)\sqrt{\left(2q\cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a-1)^2}$. Then, lemma 5 or the continuity of $\lambda_2(k, l)$ implies that $Re(\lambda_2) > 0$ for $k-k_0 > 0$ and $l-l_0 > 0$ sufficiently small. From here, we can say that $\lambda_2(k, l)$ cannot move in the left half-plane crossing on the imaginary axis as k and l increase. Hence, we have $Re(\lambda_2) > 0$ for all $k > k_0$ and $l > l_0$.

The proof is completed. \square

Remark 2. We consider the delay differential system (1.1) where matrix A is given as in case (I), that is,

$$(4.5) \quad x'(t) + (1-a)x(t) + \begin{pmatrix} q_1 & r \\ 0 & q_2 \end{pmatrix} (x(t-k) + x(t-l)) = 0, \quad t \geq 0.$$

Then, characteristic equation of (4.5) is as follows

$$(4.6) \quad (\lambda + (1-a) + q_1(e^{-\lambda k} + e^{-\lambda l}))(\lambda + (1-a) + q_2(e^{-\lambda k} + e^{-\lambda l})) = 0.$$

It is obvious that for $a=1$ and $i=1,2$, the equation $\lambda + q_i(e^{-\lambda k} + e^{-\lambda l}) = 0$ is the characteristic equation of (4.6) with $q = q_i$, and so one can immediately get the following corollary from the previous result given by Kuang (1993).

Corollary 1. Suppose that $a=1$ for the system (4.6). Let the matrix A of the system (4.6) is written as the form (I). Then system (4.6) is asymptotically stable if and only if for $i=1,2$

$$(4.7) \quad 2q_i(k+l) \cos\left(\frac{k-l}{k+l} \frac{\pi}{2}\right) < \pi.$$

Theorem 2. Suppose that conditions of lemma 5 are satisfied. Let the matrix A of the system (1.1) be in the form (I). Then system (1.1) is asymptotically stable if and only if for $i=1,2$ either

$$(4.8) \quad \begin{cases} (a-1) - 2q_i \cos\left(\frac{\omega(k-l)}{2}\right) < 0 \\ \left(2q_i \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a-1)^2 \leq 0, \end{cases}$$

or

$$(4.9) \quad \begin{cases} (a-1) - 2q_i \cos\left(\frac{\omega(k-l)}{2}\right) < 0 \\ \left(2q_i \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a-1)^2 > 0 \\ k+l < \frac{2 \operatorname{sgn}(q)}{\sqrt{\left(2q_i \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a-1)^2}} \left\{ \arccos\left(\frac{a-1}{2q_i \cos\left(\frac{\omega(k-l)}{2}\right)}\right) \right\}. \end{cases}$$

Proof. The proof is similar to the theorem 1.

5. AN EXTENSION TO A SYSTEM OF HIGHER DIMENSION

Finally, a higher dimensional linear delay differential system with two delays is considered

$$(5.1) \quad x'(t) + (1-a)x(t) + A(x(t-k) + x(t-l)) = 0 \quad t \geq 0,$$

where a is a real number, A is a $d \times d$ real constant matrix, and k, l are positive numbers such that $k > l$.

Theorem 3. Let $q_j e^{i\theta_j}$ ($j=1,2,\dots,d$) be the eigenvalues of matrix A . Then the system (5.1) is asymptotically stable iff

$$(5.2) \quad \begin{cases} (a-1) - 2q_j \cos \theta_j \cos\left(\frac{\omega(k-l)}{2}\right) < 0 \\ \left(2q_j \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a-1)^2 \leq 0, \end{cases}$$

or

$$(5.3) \quad \left. \begin{aligned} & (a-1) - 2q_j \cos \theta_j \cos\left(\frac{\omega(k-l)}{2}\right) < 0 \\ & \left(2q_j \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a-1)^2 > 0 \end{aligned} \right\} k+l < \frac{2 \operatorname{sgn}(q)}{\sqrt{\left(2q_j \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a-1)^2}} \left\{ \arccos\left(\frac{a-1}{2q_j \cos\left(\frac{\omega(k-l)}{2}\right)}\right) - \theta_j \right\}.$$

where q_j, θ_j are real numbers and $|\theta_j| \leq \frac{\pi}{2}$.

Proof. Since $q_j e^{i\theta_j}$ ($j=1,2,\dots,d$) be the eigenvalues of matrix A , the characteristic equation of the system (5.1) is given by

$$f(\lambda) = \prod_{j=1}^d \left(\lambda + (1-a) + q_j \left(e^{-\lambda k + i|\theta_j|} + e^{-\lambda l + i|\theta_j|} \right) \right) = 0.$$

Thus, theorem 3 can be seen as a result of theorem 1 and theorem 3. □

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PUBLIC INTEREST STATEMENT

In this paper, asymptotic stability of a class of time-Delay Systems was investigated. This system is a generalized version of modeling compound optical resonators by using a matrix instead of a scalar. In addition, since asymptotic stability is an interdisciplinary material, the asymptotic stability of these systems has a wide range of applications as biology, physics and medicine.

We know that for the time-delay equations, an equation is asymptotically stable if and only if all roots of the associated characteristic equation have negative real parts. Stability analysis, however, does not require the exact calculation of the characteristic roots. This analysis can be performed by D-subdivision method, which gives a necessary and sufficient condition for stability based on the coefficients or delay parameters of the characteristic equation. Thus, this method reveals a qualitative approach in order to prove the asymptotic stability of the systems. Consequently, we have created new necessary and sufficient conditions for the systems, which are asymptotically stable.

AUTHOR STATEMENT

My key research activities include; Delay Differential Equations, Delay Difference Equations, Neutral Differential Equations, Neutral Difference Equations and stability of these equations. More generally, we can say Differential Equations, Difference Equations and stability of these equations.

The research reported in this paper relates to the stability of the systems that can be modeled by delay-differential equations such as biology, physics and medicine. Thus, it is useful to use qualitative approaches to investigate the asymptotic stability of these systems. Our research represents a generalized method for describing the asymptotic stability of systems within of science branches such as biology and physics.

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