



Received: 31 January 2018
Accepted: 30 April 2018
First Published: 14 May 2018

*Corresponding author: Serbun Ufuk Değer, Department of Mathematics, Institute of Sciences, Kastamonu University, Kastamonu, Turkey; Department of Mathematics, Faculty of Art & Science, Kastamonu University, Kastamonu, Turkey
E-mail: sudeger@kastamonu.edu.tr

Reviewing editor:
Hari M. Srivastava, University of Victoria, Canada

Additional information is available at the end of the article

PURE MATHEMATICS | RESEARCH ARTICLE

On asymptotic stability of a class of time–delay systems

Serbun Ufuk Değer^{1,2*} and Yaşar Bolat^{1,2}

Abstract: In this paper, we give some new necessary and sufficient conditions for the asymptotic stability of a class of time–delay systems of the form $x'(t) + (1 - a)x(t) + A(x(t - k) + x(t - l)) = 0$, $t \geq 0$, where a is a real number, A is a 2×2 real constant matrix, and k, l are positive numbers such that $k > l$.

Subjects: Advanced Mathematics; Applied Mathematics; Dynamical Systems

Keywords: asymptotic stability; delay differential equations; characteristic equation

1. Introduction

In this paper, we study the asymptotic stability of the solutions of time–delay systems of the form

$$x'(t) + (1 - a)x(t) + A(x(t - k) + x(t - l)) = 0, \quad t \geq 0, \quad (1.1)$$

where a is a real number, A is a 2×2 real constant matrix, and k, l are positive numbers such that $k > l$. Time–delay systems are a type of differential equations in which the derivative of the unknown function at a certain time is given in terms of the values of the function at previous times. Also, they are called delay differential equations, retarded differential equations or differential-difference equations. On the other hand, since asymptotic stability is an interdisciplinary material, the asymptotic stability of these systems has a wide range of applications as biology,

ABOUT THE AUTHOR

My key research activities include Delay Differential Equations, Delay Difference Equations, Neutral Differential Equations, Neutral Difference Equations and stability of these equations. More generally, we can say Differential Equations, Difference Equations and stability of these equations.

The research reported in this paper relates to the stability of the systems that can be modeled by delay differential equations such as biology, physics and medicine. Thus, it is useful to use qualitative approaches to investigate the asymptotic stability of these systems. Our research represents a generalized method for describing the asymptotic stability of systems within of science branches such as biology and physics.

PUBLIC INTEREST STATEMENT

In this paper, asymptotic stability of a class of time–delay systems was investigated. This system is a generalized version of modeling compound optical resonators by using a matrix instead of a scalar. In addition, since asymptotic stability is an interdisciplinary material, the asymptotic stability of these systems has a wide range of applications as biology, physics and medicine.

We know that for the time–delay equations, an equation is asymptotically stable if and only if all roots of the associated characteristic equation have negative real parts. Stability analysis, however, does not require the exact calculation of the characteristic roots. This analysis can be performed by D-subdivision method, which gives a necessary and sufficient condition for stability based on the coefficients or delay parameters of the characteristic equation. Thus, this method reveals a qualitative approach in order to prove the asymptotic stability of the systems. Consequently, we have created new necessary and sufficient conditions for the systems, which are asymptotically stable.

physics, and medicine. For instance, Naresh, Tripathi, Tchenche, and Sharma (2009) formulated a nonlinear mathematical model to study the framework of disease Epidemiology. As another example Ikeda and Watanabe (2014) were investigated the stochastic differential equations and diffusion processes in their study about physic. Kruthika, Mahindrakar, and Pasumarthy (2017) were studied stability analysis of nonlinear time-delayed systems with application to biological models. They analyzed the local stability of a gene-regulatory network and immunotherapy for cancer modeled as nonlinear time-delay systems.

Many authors have also focused on the asymptotic stability of time-delay systems. Some important studies about related subject can be examined from the below authors: Bellman and Cooke (1963), Cooke and Grossman (1982), Cooke and van den Driessche (1986), Stepan (1989), Freedman and Kuang (1991), Kuang (1993), Ruan and Wei (2003), Elaydi (2005), Matsunaga (2008), Smith (2010), Gray, Greenhalgh, Hu, Mao, and Pan (2011), Khokhlova, Kipnis, and Malygina (2011), Xu (2012), Hrabalova (2013), Nakajima (2014), Liu, Jiang, Shi, Hayat, and Alsaedi (2016), and Li, Ma, Xiao, and Yang (2017). Also, an equation which is the special case of our system which is investigated by Kuang (1993) and he demonstrated that the zero solution of the delay differential equation with two delays of the form

$$x'(t) + q(x(t - k) + x(t - l)) = 0, \quad t \geq 0, \tag{1.2}$$

where $k \geq 0, l \geq 0$ and q is positive constant and is asymptotically stable if and only if

$$2q(k + l) \cos\left(\frac{k - l\pi}{k + l2}\right) < \pi. \tag{1.3}$$

2. Preliminaries

It is known that for the time-delay equations, an equation is asymptotically stable if and only if all roots of the associated characteristic equation have negative real parts. Stability analysis, however, does not require the exact calculation of the characteristic roots; only the sign of the real part of the critical root must be determined. This analysis can be performed by D-subdivision method (see, e.g., Insperger & Stépán, 2011; Stepan, 1998), which gives a necessary and sufficient condition for stability based on the coefficients of the characteristic equation. The aim of this paper is to obtain new results for the asymptotic stability of zero solution of system (1.1), while the characteristic equation of system (1.1) has roots on the imaginary axis when A is a constant matrix. If we obtain $x(t) = Py(t)$ for a regular matrix P in (1.1), then we have the following system:

$$y'(t) + (1 - a)y(t) + P^{-1}AP(y(t - k) + y(t - l)) = 0, \quad t \geq 0.$$

Thus, matrix A can be given one of the following two matrices in Jordan form [7]:

$$(I) A = \begin{pmatrix} q_1 & r \\ 0 & q_2 \end{pmatrix}, \quad q_1, q_2 \text{ and } r \text{ are real constants,}$$

$$(II) A = q \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad q, \theta \text{ are real constants and } |\theta| \leq \frac{\pi}{2}.$$

Here we consider the case (II), the other case should be considered similarly. The characteristic equation of system (1.1) is given as

$$F(\lambda) := \det\left(\lambda I_2 + (1 - a)I_2 + A(e^{-\lambda k} + e^{-\lambda l})\right) = 0, \tag{2.1}$$

where I_2 is the 2×2 identity matrix. Using (2.1), we obtain

$$F(\lambda) = \left(\lambda + (1 - a) + q \cos \theta (e^{-\lambda k} + e^{-\lambda l})\right)^2 + \left(q \sin \theta (e^{-\lambda k} + e^{-\lambda l})\right)^2$$

$$\begin{aligned}
 &= (\lambda + (1 - a) + q \cos|\theta|(e^{-\lambda k} + e^{-\lambda l}))^2 - (iq \sin|\theta|(e^{-\lambda k} + e^{-\lambda l}))^2 \\
 &= (\lambda + (1 - a) + qe^{i|\theta|}(e^{-\lambda k} + e^{-\lambda l}))(\lambda + (1 - a) + qe^{-i|\theta|}(e^{-\lambda k} + e^{-\lambda l})).
 \end{aligned}$$

If we let

$$f(\lambda) = (\lambda + (1 - a) + q(e^{-\lambda k + i|\theta|} + e^{-\lambda l + i|\theta|})), \tag{2.2}$$

then we have

$$F(\lambda) = f(\lambda)\overline{f(\bar{\lambda})},$$

where $\bar{\lambda}$ is the complex conjugate of any complex λ . Note that $\overline{f(\bar{\lambda})} = 0$ implies $f(\bar{\lambda}) = 0$.

3. Some auxiliary lemmas

In this section, we will investigate the distribution of the zeros of the characteristic equation of system (1.1). Thus, we state and prove some basic results on the roots of the characteristic equation of system (1.1).

Lemma 1 (Stepan, 1989) *The zero solution of (1.1) is asymptotically stable if and only if all the roots of equation*

$$f(\lambda, k, l) = \lambda + (1 - a) + q(e^{-\lambda k + i|\theta|} + e^{-\lambda l + i|\theta|}) = 0, \tag{3.1}$$

lie in the left half of the complex plane.

Since f is an analytic function of λ , k and l for the fixed numbers a, q and θ , one can regard the root $\lambda = \lambda(k, l)$ of (3.1) as a continuous function of k and l . The next lemma plays very important role for the main theorem.

Lemma 2 (Cooke & Grossman, 1982) *As k and l vary, the sum of the multiplicities of the roots of (3.1) in the open right half-plane can change only if a root appears on or crosses the imaginary axis.*

Consequently, we claim that (3.1) has only imaginary roots $\pm i\omega$. We will determine how the value of k and l change as Equation (3.1) has roots on the imaginary axis. Now, we can write the characteristic Equation (3.1) as follows:

$$\lambda + (1 - a) + q(e^{-\lambda k + i\theta} + e^{-\lambda l + i\theta}) = 0. \tag{3.2}$$

At the same time, we take $\lambda = i\omega$ such that $\omega \in \mathbb{R}$. Firstly, since $f(0) = (1 - a) + 2qe^{i\theta} \neq 0$, we see that $\omega \neq 0$. For $\omega \neq 0$, we obtain

$$f(i\omega) = i\omega + (1 - a) + q(e^{-i\omega k + i\theta} + e^{-i\omega l + i\theta}) = 0. \tag{3.3}$$

Using the real part and the imaginary part of (3.3)

$$\omega = q(\sin(\omega k - \theta) + \sin(\omega l - \theta)), \tag{3.4}$$

$$a - 1 = q(\cos(\omega k - \theta) + \cos(\omega l - \theta)), \tag{3.5}$$

which is equivalent to

$$\omega = 2q \sin\left(\frac{\omega(k+l)}{2} - \theta\right) \cos\left(\frac{\omega(k-l)}{2}\right), \tag{3.6}$$

$$a - 1 = 2q \cos\left(\frac{\omega(k+l)}{2} - \theta\right) \cos\left(\frac{\omega(k-l)}{2}\right), \tag{3.7}$$

is obtained.

Lemma 3. Suppose that, $q > 0$ and $0 < \theta \leq \frac{\pi}{2}$. Let $\lambda = i\omega$ be a root (3.1) where $\omega \in \left(\frac{-\pi}{k-l}, \frac{\pi}{k-l}\right) - \left\{\frac{-n\pi+2\theta}{k+l}, \frac{n\pi+2\theta}{k+l}\right\}$ for $n = 0, 1, 2, \dots$, then the following conditions hold:

(i) If $\left(2q \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a - 1)^2 \leq 0$, then there exists no real number ω .

(ii) If $\left(2q \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a - 1)^2 > 0$, then the real numbers ω , q and the delays $(k + l)$ are as follows:

$$\omega = \pm\gamma = \pm\sqrt{\left(2q \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a - 1)^2}, \quad q = \frac{a-1}{2 \cos\left(\frac{\omega(k+l)}{2} - \theta\right) \cos\left(\frac{\omega(k-l)}{2}\right)} \quad \text{and} \quad k + l = (k_n + l_n)^\pm \quad \text{for } n = 0, 1, 2, \dots \text{ where } (k_n + l_n)^+ \text{ and } (k_n + l_n)^-;$$

$$(k_n + l_n)^+ = \frac{2}{\gamma} \left\{ 2n\pi + \arccos\left(\frac{a - 1}{2q \cos\left(\frac{\omega(k-l)}{2}\right)}\right) + \theta \right\}$$

$$(k_n + l_n)^- = \frac{2}{\gamma} \left\{ 2n\pi + \arccos\left(\frac{a - 1}{2q \cos\left(\frac{\omega(k-l)}{2}\right)}\right) - \theta \right\},$$

and $i\gamma$ or $-i\gamma$ is a root of (3.1) for the sum of delays $(k_n + l_n)^+$ or $(k_n + l_n)^-$.

Proof. By squaring both sides of (3.6) and (3.7), and adding them together, we obtain

$$\omega^2 + (a - 1)^2 = \left(2q \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2. \tag{3.8}$$

If $\left(2q \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a - 1)^2 \leq 0$, then statement (3.8) implies $\omega^2 < 0$, contradicts with $\omega^2 > 0$. Since $\omega \neq 0$, condition (i) is verified; that is, (3.1) has no root on the imaginary axis for all $k > l > 0$.

On the other hand, if $\left(2q \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a - 1)^2 > 0$, statement (3.8) implies that

$$\omega = \pm\sqrt{\left(2q \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a - 1)^2}. \tag{3.9}$$

If we let

$$\gamma = \sqrt{\left(2q \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a - 1)^2},$$

then we can write $\omega = \pm\gamma$. By (3.7), we have

$$q = \frac{a - 1}{\cos\left(\frac{\omega(k+l)}{2} - \theta\right) \cos\left(\frac{\omega(k-l)}{2}\right)}.$$

Now, we will show that $i\gamma$ is a root of (3.1).

In the case of $\omega = \gamma$, since $\cos\left(\frac{\omega(k-l)}{2}\right) > 0$ for $\omega \in \left(0, \frac{\pi}{k-l}\right) - \left\{\frac{n\pi+2\theta}{k+l}\right\}$ and using (3.6)

$$\sin\left(\frac{\omega(k+l)}{2} - \theta\right) > 0, \tag{3.10}$$

is obtained. Thus, we use (3.6) and (3.7), we have

$$\frac{\omega(k+l)}{2} - \theta = 2n\pi + \arccos\left(\frac{a-1}{2q \cos\left(\frac{\omega(k-l)}{2}\right)}\right) \quad n = 0, 1, 2, \dots,$$

which yields $(k_n + l_n)^+$. Also, from (3.6) and (3.7), we obtain

$$\sin\left(\arccos\left(\frac{a-1}{2q \cos\left(\frac{\omega(k-l)}{2}\right)}\right)\right) = \frac{\gamma}{2q \cos\left(\frac{\omega(k-l)}{2}\right)}$$

because of

$$\arccos\left(\frac{a-1}{2q \cos\left(\frac{\omega(k-l)}{2}\right)}\right) = \begin{cases} \arcsin\left(\frac{\gamma}{2q \cos\left(\frac{\omega(k-l)}{2}\right)}\right) & \text{if } a-1 \geq 0 \\ \pi - \arcsin\left(\frac{\gamma}{2q \cos\left(\frac{\omega(k-l)}{2}\right)}\right) & \text{if } a-1 < 0. \end{cases}$$

Hence, for the case $k+l = (k_n + l_n)^+$, $i\gamma$ is a root of (3.1). Indeed,

$$\begin{aligned} f(i\omega) &= i\omega + (1-a) + q\left(e^{i(\omega k - \theta)} + e^{i(\omega l - \theta)}\right) \\ &= i\sqrt{\left(2q \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a-1)^2} + (1-a) + 2q \cos\left(\frac{\omega(k-l)}{2}\right) e^{-i\left(\frac{\omega(k+l)}{2} - \theta\right)}, \\ &= i\sqrt{\left(2q \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a-1)^2} + (1-a) + \\ &\quad + 2q \cos\left(\frac{\omega(k-l)}{2}\right) e^{-i\left(2n\pi + \arccos\left(\frac{a-1}{2q \cos\left(\frac{\omega(k-l)}{2}\right)}\right)\right)}, \\ &= i\sqrt{\left(2q \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a-1)^2} + (1-a) + 2q \cos\left(\frac{\omega(k-l)}{2}\right) \times \\ &\quad \times \left\{ \cos\left(\arccos\left(\frac{a-1}{2q \cos\left(\frac{\omega(k-l)}{2}\right)}\right)\right) - i \sin\left(\arccos\left(\frac{a-1}{2q \cos\left(\frac{\omega(k-l)}{2}\right)}\right)\right) \right\}, \\ &= i\sqrt{\left(2q \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a-1)^2} - i\sqrt{\left(2q \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a-1)^2}, \\ &= 0. \end{aligned}$$

Thus, this implies that $i\gamma$ is a root of (3.1). Similarly, in the case $\omega < 0$, it can be shown that $-i\gamma$ is a root of (3.1) for the sum of delays $(k_n + l_n)^-$. The proof is completed.

When $q > 0$, we have the following analogous result.

Lemma 4. Suppose that, $q < 0$ and $0 < \theta \leq \frac{\pi}{2}$. Let $\lambda = i\omega$ be a root of (3.1) where $\omega \in \left(\frac{-\pi}{k-1}, \frac{\pi}{k-1}\right) - \left\{\frac{-n\pi+2\theta}{k+l}, \frac{n\pi+2\theta}{k+l}\right\}$ for $n = 0, 1, 2, \dots$, then the following conditions hold:

(i) If $\left(2q \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a-1)^2 \leq 0$, then there exists no real number ω .

(ii) If $\left(2q \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a-1)^2 > 0$, then the real numbers ω , q and the delays $(k+l)$ are as follows:

$$\omega = \pm\gamma = \pm\sqrt{\left(2q \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a-1)^2}, \quad q = \frac{a-1}{2 \cos\left(\frac{\omega(k+l)}{2}-\theta\right) \cos\left(\frac{\omega(k-l)}{2}\right)} \text{ and } k+l = (\alpha_n + \beta_n)^\pm \text{ for}$$

$n = 0, 1, 2, \dots$, where $(\alpha_n + \beta_n)^+$ and $(\alpha_n + \beta_n)^-$;

$$(\alpha_n + \beta_n)^+ = \frac{2}{\gamma} \left\{ 2n\pi - \arccos\left(\frac{a-1}{2q \cos\left(\frac{\omega(k-l)}{2}\right)}\right) + \theta \right\}$$

$$(\alpha_n + \beta_n)^- = \frac{2}{\gamma} \left\{ (2n+2)\pi - \arccos\left(\frac{a-1}{2q \cos\left(\frac{\omega(k-l)}{2}\right)}\right) - \theta \right\},$$

and $i\gamma$ or $-i\gamma$ is a root of (3.1) for the sum of delays $(\alpha_n + \beta_n)^+$ or $(\alpha_n + \beta_n)^-$.

Proof. The proof is similar to Lemma 3.

Remark 1. For $q > 0$, from the definitions of $(k_n + l_n)^\pm$;

$$\min\{(k_n + l_n)^\pm : n = 0, 1, \dots\} = \begin{cases} (k_0 + l_0)^- \text{ if } (a-1) - 2q \cos \theta \cos\left(\frac{\omega(k-l)}{2}\right) < 0 \\ (k_0 + l_0)^+ \text{ if } (a-1) - 2q \cos \theta \cos\left(\frac{\omega(k-l)}{2}\right) \geq 0, \end{cases}$$

is obtained. Similarly, for $q < 0$, we obtain from the definitions of $(\alpha_n + \beta_n)^\pm$ as follows:

$$\min\{(\alpha_n + \beta_n)^\pm : n = 0, 1, \dots\} = \begin{cases} (\alpha_0 + \beta_0)^+ \text{ if } (a-1) - 2q \cos \theta \cos\left(\frac{\omega(k-l)}{2}\right) < 0 \\ (\alpha_0 + \beta_0)^- \text{ if } (a-1) - 2q \cos \theta \cos\left(\frac{\omega(k-l)}{2}\right) \geq 0. \end{cases}$$

Lemma 5. Suppose that $0 < \theta \leq \frac{\pi}{2}$, $\omega \in \left(\frac{-\pi}{k-1}, \frac{\pi}{k-1}\right) - \{0\}$ and

$$\omega^2 + q^2 \omega(k+l) \sin(\omega(l-k)) > 0.$$

Also, the following conditions

$$\begin{cases} \frac{\sin(\omega k - \theta)}{l \sin \omega(k-l)} > q \text{ if } > \frac{q}{\omega} > 0 \\ \frac{\sin(\omega k - \theta)}{l \sin \omega(k-l)} < q \text{ if } < \frac{q}{\omega} < 0 \end{cases} \text{ and } \begin{cases} \frac{\sin(\omega l - \theta)}{k \sin \omega(k-l)} > q \text{ if } > \frac{q}{\omega} > 0 \\ \frac{\sin(\omega l - \theta)}{k \sin \omega(k-l)} < q \text{ if } < \frac{q}{\omega} < 0, \end{cases} \quad (3.11)$$

are provided. Then all the roots of Equation (3.1) on the imaginary axis move in the right half-plane as k and l increase.

Proof. Let $\lambda = i\omega$ be a root of (3.1) where $\omega \in \left(\frac{-\pi}{k-1}, \frac{\pi}{k-1}\right) - \{0\}$ is a real number. It will be enough to show

$$\operatorname{Re} \frac{\partial \lambda}{\partial k} \Big|_{\lambda=i\omega} > 0 \text{ and } \operatorname{Re} \frac{\partial \lambda}{\partial l} \Big|_{\lambda=i\omega} > 0.$$

Firstly, we take the derivative of λ with respect to k on Equation (3.1), we have

$$\frac{\partial \lambda}{\partial k} - q\lambda e^{-\lambda k + i\theta} - qk\lambda e^{-\lambda k + i\theta} \frac{\partial \lambda}{\partial k} - qle^{-\lambda l + i\theta} \frac{\partial \lambda}{\partial k} = 0,$$

$$\frac{\partial \lambda}{\partial k} = \frac{q\lambda e^{-\lambda k + i\theta}}{1 - q(ke^{-\lambda k + i\theta} + le^{-\lambda l + i\theta})}$$

Substituting $\lambda = i\omega$ into the above equation, we obtain

$$\begin{aligned} \left. \frac{\partial \lambda}{\partial k} \right|_{\lambda=i\omega} &= \frac{i\omega + 1 - a}{1 - q(ke^{-i(\omega k - \theta)} + le^{-i(\omega l - \theta)})} \\ &= \frac{i\omega + 1 - a}{1 - q(k \cos(\omega k - \theta) + l \cos(\omega l - \theta)) + iq(k \sin(\omega k - \theta) + l \sin(\omega l - \theta))} \end{aligned}$$

If we multiply with the complex conjugate of the denominator in the above equation, then we can write

$$\operatorname{Re} \left. \frac{\partial \lambda}{\partial k} \right|_{\lambda=i\omega} = \frac{q\omega \sin(\omega k - \theta) - q^2 \omega l \sin(\omega(k-l))}{M}, \tag{3.12}$$

where

$$M = (1 - q(k \cos(\omega k - \theta) + l \cos(\omega l - \theta)))^2 + q^2(k \sin(\omega k - \theta) + l \sin(\omega l - \theta))^2.$$

Since (3.11), we can write $\operatorname{Re} \left. \frac{\partial \lambda}{\partial k} \right|_{\lambda=i\omega} > 0$. On the other hand, we take the derivative of λ with respect to l on Equation (3.1), similar to (3.12)

$$\operatorname{Re} \left. \frac{\partial \lambda}{\partial l} \right|_{\lambda=i\omega} = \frac{q\omega \sin(\omega l - \theta) - q^2 \omega k \sin(\omega(k-l))}{M}, \tag{3.13}$$

is obtained. From (3.11), we obtain $\operatorname{Re} \left. \frac{\partial \lambda}{\partial l} \right|_{\lambda=i\omega} > 0$. Moreover, by adding both (3.12) and (3.13) together, we have

$$\operatorname{Re} \left. \frac{\partial \lambda}{\partial k} \right|_{\lambda=i\omega} + \operatorname{Re} \left. \frac{\partial \lambda}{\partial l} \right|_{\lambda=i\omega} = \frac{\omega^2 + q^2 \omega(k+l) \sin(\omega(k-l))}{M} > 0. \tag{3.14}$$

Hence, the proof is completed.

Now we can state and prove main theorems.

4. Main results

We will show that the stability analysis with a qualitative approach, as we have already mentioned in section 1.

Theorem 1. Suppose that $0 < \theta \leq \frac{\pi}{2}$ and the conditions of Lemma 5 are satisfied. Let the matrix A of system (1.1) be in the form (II). Then system (1.1) is asymptotically stable if and only if either

$$\begin{cases} (a - 1) - 2q \cos \theta \cos\left(\frac{\omega(k-l)}{2}\right) < 0 \\ \left(2q \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a - 1)^2 \leq 0, \end{cases} \tag{4.1}$$

or

$$\begin{cases} (a - 1) - 2q \cos \theta \cos\left(\frac{\omega(k-l)}{2}\right) < 0 \\ \left(2q \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a - 1)^2 > 0 \\ k + l < \frac{2 \operatorname{sgn}(q)}{\sqrt{\left(2q \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a - 1)^2}} \left\{ \arccos\left(\frac{a - 1}{2q \cos\left(\frac{\omega(k-l)}{2}\right)}\right) - \theta \right\} \end{cases} \tag{4.2}$$

Proof. In the case of $k = 0$ and $l = 0$, the root of (3.1) is only $\lambda(0, 0) = a - 1 - 2b \cos \theta - i2b \sin \theta$. Thus, the root of the Equation (3.1) has a negative real part. By the continuity of the roots with respect to k and l , we can say that all the roots of (3.1) lie in the left half plane for $k > 0$ and $l > 0$ sufficiently small.

For the sufficiency, here our claim is: If either condition (4.1) or (4.2) holds, then (3.1) does not have a root on the imaginary axis. By condition (4.1) and Lemma 3, our claim is true for $k > 0$ and $l > 0$. Now, suppose that condition (4.2) holds: Since

$$\frac{2\text{sgn}(q)}{\sqrt{\left(2q \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a-1)^2}} \left\{ \arccos\left(\frac{a-1}{2q \cos\left(\frac{\omega(k-l)}{2}\right)}\right) - \theta \right\} = \begin{cases} (k_0 + l_0)^- & \text{if } q > 0 \\ (\alpha_0 + \beta_0)^+ & \text{if } q < 0, \end{cases}$$

and Remark 1, we obtain $(k+l) \neq (k_n + l_n)^\pm$, $(k+l) \neq (\alpha_n + \beta_n)^\pm$ for $n = 0, 1, 2, \dots$. Thus, we obtain the contraposition with Lemma 1, our other claim is also true. By the above argument and Lemma 2, we can say that if either condition (4.1) or (4.2) holds, then all the roots of (3.1) lie in the left half plane.

For the necessity, we will show the following contraposition: either

$$(a-1) - 2q \cos \theta \cos\left(\frac{\omega(k-l)}{2}\right) \geq 0, \tag{4.3}$$

or

$$\left\{ \begin{array}{l} \left(2q \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a-1)^2 > 0 \\ k+l \geq \frac{2\text{sgn}(q)}{\sqrt{\left(2q \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a-1)^2}} \left\{ \arccos\left(\frac{a-1}{2q \cos\left(\frac{\omega(k-l)}{2}\right)}\right) - \theta \right\}. \end{array} \right. \tag{4.4}$$

Thus, if (4.3) and (4.4) hold, then there exists roots λ_j of (3.1) such that $\text{Re}(\lambda_j) > 0$ for $j = 1, 2$. Assume that (4.3) holds and let $\lambda_1(k, l)$ be the branch of the root of satisfying $\lambda_1(0, 0) = a - 1 - 2b \cos \theta - i2b \sin \theta$. Then, Lemma 5 or the continuity of $\lambda_1(k, l)$ implies that $\text{Re}(\lambda_1) > 0$ for $k > 0$ and $l > 0$ sufficiently small. From here, we can say that $\lambda_1(k, l)$ cannot move in the left half-plane crossing on the imaginary axis as k and l increase. Hence, we have $\text{Re}(\lambda_1) > 0$ for all $k > 0$ and $l > 0$. Assume that (4.4) holds and let $\lambda_2(k, l)$ be the branch of the root of satisfying $\lambda_2(k_0, l_0) = -i\text{sgn}(q) \sqrt{\left(2q \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a-1)^2}$. Then, Lemma 5 or the continuity of $\lambda_2(k, l)$ implies that $\text{Re}(\lambda_2) > 0$ for $k - k_0 > 0$ and $l - l_0 > 0$ sufficiently small. From here, we can say that $\lambda_2(k, l)$ cannot move in the left half-plane crossing on the imaginary axis as k and l increase. Hence, we have $\text{Re}(\lambda_2) > 0$ for all $k > k_0$ and $l > l_0$.

The proof is completed.

Remark 2. We consider the delay differential system (1.1) where matrix A is given as in case (I), i.e.,

$$x'(t) + (1-a)x(t) + \begin{pmatrix} q_1 & r \\ 0 & q_2 \end{pmatrix} (x(t-k) + x(t-l)) = 0, \quad t \geq 0. \tag{4.5}$$

Then, characteristic equation of (4.5) is as follows:

$$\left(\lambda + (1-a) + q_1(e^{-\lambda k} + e^{-\lambda l})\right) \left(\lambda + (1-a) + q_2(e^{-\lambda k} + e^{-\lambda l})\right) = 0. \tag{4.6}$$

It is obvious that for $a = 1$ and $i = 1, 2$, the equation $\lambda + q_i(e^{-\lambda k} + e^{-\lambda l}) = 0$ is the characteristic equation of (4.6) with $q = q_i$, and so one can immediately obtain the following corollary from the previous result given by Kuang (1993).

Corollary 1. Suppose that $a = 1$ for system (4.6) Let the matrix A of system (4.6) is written as the form (I). Then system (4.6) is asymptotically stable if and only if for $i = 1, 2$

$$2q_i(k+l) \cos\left(\frac{k-l\pi}{k+l}\right) < \pi. \tag{4.7}$$

Theorem 2. Suppose that conditions of Lemma 5 are satisfied. Let the matrix A of system (1.1) be in the form (I). Then system (1.1) is asymptotically stable if and only if for $i = 1, 2$ either

$$\begin{cases} (a-1) - 2q_i \cos\left(\frac{\omega(k-l)}{2}\right) < 0 \\ \left(2q_i \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a-1)^2 \leq 0, \end{cases} \tag{4.8}$$

or

$$\begin{cases} (a-1) - 2q_i \cos\left(\frac{\omega(k-l)}{2}\right) < 0 \\ \left(2q_i \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a-1)^2 > 0 \\ k+l < \frac{2\text{sgn}(q)}{\sqrt{\left(2q_i \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a-1)^2}} \left\{ \arccos\left(\frac{a-1}{2q_i \cos\left(\frac{\omega(k-l)}{2}\right)}\right) \right\}. \end{cases} \tag{4.9}$$

Proof. The proof is similar to Theorem 1.

5. An extension to a system of higher dimension

Finally, a higher dimensional linear delay differential system with two delays is considered

$$x'(t) + (1-a)x(t) + A(x(t-k) + x(t-l)) = 0 \quad t \geq 0, \tag{5.1}$$

where a is a real number, A is a $d \times d$ real constant matrix, and k, l are positive numbers such that $k > l$.

Theorem 3. Let $q_j e^{i\theta_j}$ ($j = 1, 2, \dots, d$) be the eigenvalues of matrix A . Then system (5.1) is asymptotically stable iff

$$\begin{cases} (a-1) - 2q_j \cos \theta_j \cos\left(\frac{\omega(k-l)}{2}\right) < 0 \\ \left(2q_j \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a-1)^2 \leq 0, \end{cases} \tag{5.2}$$

or

$$\begin{cases} (a-1) - 2q_j \cos \theta_j \cos\left(\frac{\omega(k-l)}{2}\right) < 0 \\ \left(2q_j \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a-1)^2 > 0 \\ k+l < \frac{2\text{sgn}(q)}{\sqrt{\left(2q_j \cos\left(\frac{\omega(k-l)}{2}\right)\right)^2 - (a-1)^2}} \left\{ \arccos\left(\frac{a-1}{2q_j \cos\left(\frac{\omega(k-l)}{2}\right)}\right) - \theta_j \right\}. \end{cases} \tag{5.3}$$

where q_j, θ_j are real numbers and $|\theta_j| \leq \frac{\pi}{2}$.

Proof. Since $q_j e^{i\theta_j}$ ($j = 1, 2, \dots, d$) be the eigenvalues of matrix A' the characteristic equation of system (5.1) is given by

$$f(\lambda) = \prod_{j=1}^d \left(\lambda + (1-a) + q_j \left(e^{-\lambda k + i|\theta_j|} + e^{-\lambda l + i|\theta_j|} \right) \right) = 0.$$

Thus, Theorem 3 can be seen as a result of Theorems 1 and 3.

Funding

This work was supported by the Serbun Ufuk DEĞER.

Author details

Serbun Ufuk Değer^{1,2}

E-mail: sudeger@kastamonu.edu.tr

Yaşar Bolat^{1,2}

E-mail: ybolat@kastamonu.edu.tr

¹ Department of Mathematics, Institute of Sciences, Kastamonu University, Kastamonu, Turkey.

² Department of Mathematics, Faculty of Art & Science, Kastamonu University, Kastamonu, Turkey.

Citation information

Cite this article as: On asymptotic stability of a class of time–delay systems, Serbun Ufuk Değer & Yaşar Bolat, *Cogent Mathematics & Statistics* (2018), 5: 1473709.

References

- Bellman, R., & Cooke, K. L. (1963). *Differential-difference equations*. New York, NY: Academic Press.
- Cooke, K. L., & Grossman, Z. (1982). Discrete delay, distributed delay and stability switches. *Journal of Mathematical Analysis Applications*, 86, 592–627. doi:10.1016/0022-247X(82)90243-8
- Cooke, K. L., & van den Driessche, P. (1986). On zeroes of some transcendental equations. *Funkcialaj Ekvacioj*, 29, 77–90. (MR865215 (87m:34098))
- Elaydi, S. (2005). *An introduction to difference equations* (3rd ed.). New York, NY: Springer-Verlag.
- Freedman, H. I., & Kuang, Y. (1991). Stability switches in linear scalar neutral delay equation. *Funkcialaj Ekvacioj*, 34, 187–209.
- Gray, A., Greenhalgh, D., Hu, L., Mao, X., & Pan, J. A. (2011). Stochastic differential equation SIS epidemic model. *SIAM Journal on Applied Mathematics*, 71(3), 876–902. doi:10.1137/10081856X
- Hrabalova, J. (2013). Stability properties of a discretized neutral delay differential equation. *Tatra Mountains Mathematical Publications*, 54, 83–92.
- Ikeda, N., & Watanabe, S. (2014). *Stochastic differential equations and diffusion processes* (Vol. 24). North Holland: Elsevier.
- Inspurger, T., & Stépán, G. (2011). *Semi-discretization for time-delay systems: Stability and engineering applications* (Vol. 178). New York, Dordrecht, Heidelberg, London: Springer Science & Business Media.
- Khokhlova, T., Kipnis, M., & Malygina, V. (2011). The stability cone for a delay differential matrix equation. *Applied Mathematics Letters*, 24, 742–745. doi:10.1016/j.aml.2010.12.020
- Kruthika, H. A., Mahindrakar, A. D., & Pasumarthy, R. (2017). Stability analysis of nonlinear time–delayed systems with application to biological models. *International Journal of Applied Mathematics and Computer Science*, 27(1), 91–103. doi:10.1515/amcs-2017-0007
- Kuang, Y. (1993). *Delay differential equations with applications in population dynamics*. MR1218880 (94f:34001) Boston: Academic Press.
- Li, Y., Ma, W., Xiao, L., & Yang, W. (2017). Global stability analysis of the equilibrium of an improved time–delayed dynamic model to describe the development of T Cells in the thymus. *Filomat*, 31(2), 347–361. doi:10.2298/FIL1702347L
- Liu, Q., Jiang, D., Shi, N., Hayat, T., & Alsaedi, A. (2016). Asymptotic behaviors of a stochastic delayed SIR epidemic model with nonlinear incidence. *Communications in Nonlinear Science and Numerical Simulation*, 40, 89–99. doi:10.1016/j.cnsns.2016.04.023
- Matsunaga, H. (2008). Delay dependent and delay independent stability criteria for a delay differential system. *American Mathematical Society*, 42, 4305–4312. (136 Fields Inst. Commun.)
- Nakajima, H. (2014). On the stability of a linear retarded differential-difference equation. *Funkcialaj Ekvacioj*, 57, 43–56. doi:10.1619/fesi.57.43
- Naresh, R., Tripathi, A., Tchuente, J. M., & Sharma, D. (2009). Stability analysis of a time delayed SIR epidemic model with nonlinear incidence rate. *Computers & Mathematics with Applications*, 58(2), 348–359. doi:10.1016/j.camwa.2009.03.110
- Ruan, S., & Wei, J. (2003). On the zeros of transcendental functions with applications to stability of delay differential equations with two delays. *Dynamic of Continuous, Discrete and Impulsive Systems*, 10, 863–874.
- Smith, H. (2010). *An introduction to delay differential equations with applications to the life science*. New York, NY: Springer.
- Stepan, G. (1989). *Retarded dynamical systems: Stability and characteristic functions, pitman research notes in mathematics series* (Vol. 210). New York, NY: Academic Press.
- Stepan, G. (1998). Delay-differential equation models for machine tool chatter. *Dynamics and Chaos in Manufacturing Processes*, 471152935, 165–192.
- Xu, R. (2012). Global stability of a delayed epidemic model with latent period and vaccination strategy. *Applied Mathematical Modelling*, 36(11), 5293–5300. doi:10.1016/j.apm.2011.12.037



© 2018 The Author(s). This open access article is distributed under a Creative Commons Attribution (CC-BY) 4.0 license.

You are free to:

Share — copy and redistribute the material in any medium or format.

Adapt — remix, transform, and build upon the material for any purpose, even commercially.

The licensor cannot revoke these freedoms as long as you follow the license terms.

Under the following terms:

Attribution — You must give appropriate credit, provide a link to the license, and indicate if changes were made.

You may do so in any reasonable manner, but not in any way that suggests the licensor endorses you or your use.

No additional restrictions

You may not apply legal terms or technological measures that legally restrict others from doing anything the license permits.



Cogent Mathematics & Statistics (ISSN: 2574-2558) is published by Cogent OA, part of Taylor & Francis Group.

Publishing with Cogent OA ensures:

- Immediate, universal access to your article on publication
- High visibility and discoverability via the Cogent OA website as well as Taylor & Francis Online
- Download and citation statistics for your article
- Rapid online publication
- Input from, and dialog with, expert editors and editorial boards
- Retention of full copyright of your article
- Guaranteed legacy preservation of your article
- Discounts and waivers for authors in developing regions

Submit your manuscript to a Cogent OA journal at www.CogentOA.com

