



Received: 13 March 2018
Accepted: 10 April 2018
First Published: 11 May 2018

*Corresponding author: Y. Khalili,
Department of Basic Sciences, Sari
Agricultural Sciences and Natural
Resources University, 578 Sari, Iran
E-mail: y.khalili@sanru.ac.ir

Reviewing editor:
Clemente Cesarano, International
Telematic University UNINETTUNO,
Italy

Additional information is available at
the end of the article

PURE MATHEMATICS | RESEARCH ARTICLE

On the relationship between the turning and singular points in Sturm–Liouville equations

Y. Khalili^{1*} and A. Neamaty²

Abstract: In this article, we present some results about the Sturm–Liouville equation with turning points and singularities and transform them to each other. By applying a change of a variable, we can transform the differential equation with a turning point to the differential equation with a singularity. Also we will prove that a differential equation with a singularity will be transformed to a differential equation with a turning point in some cases.

Subjects: Differential Equations; Mathematical Analysis; Pure Mathematics; Inverse Problems; Foundations and Theorems

Keywords: Sturm–Liouville operator; turning point; singularity

Maths Subject classifications: 34B24; 34C05; 34E20

1. Introduction

Differential equations with turning points and singularities have emerged as an effective and powerful tool to study a wide class of related problems arising in various branches of mathematics, mechanics, physics, geophysics, computer sciences and other branches of natural sciences. Better knowledge and understanding from these equations provide us to study a wide class of problems arising in mathematical physics, radio electronics, and other fields of sciences and technologies (see Constantin, 1998; Conway, 1995; Dehghan & Jodayree, 2013; Hryniv and Mykytyuk, 2004; Lapwood & Usami, 1981; Litvinenko & Soshnikov, 1964). Differential equations with turning points and singularities have been studied in Mosazadeh & Neamaty (2011); Neamaty & Khalili (2013); Neamaty & Mosazadeh (2011); and Yurko (1997). But the relationship between turning points and singularities has not yet been studied basically. In Neamaty & Mosazadeh (2011), differential equations with a turning point of the second type are transformed to a singularity. To the best of our knowledge, it does not exist an exact study on the relevance between a turning and singular point. For this reason, we would like to point out

ABOUT THE AUTHORS

Y. Khalili is an Assistant Professor of Mathematics, Department of Basic Sciences, Sari Agricultural Sciences and Natural Resources University, 578 Sari, Iran. His fields of research are direct and inverse spectral problems, asymptotic distribution of the eigenvalues, and applications of the inverse problems.

A. Neamaty is an Associate Professor of Mathematics, Department of Mathematics, University of Mazandaran, Babolsar, Iran. His favorite fields are direct and inverse spectral problems, asymptotic distribution of the eigenvalues, and applications of the inverse problems.

PUBLIC INTEREST STATEMENT

The theory of the inverse spectral problems is very helpful in the study of many areas of research, such as mathematics, mechanics, physics, geophysics, computer sciences, etc. In the recent decade, the investigation of the inverse problem is an important research topic and there is increasing interest in this field. The study of these problems helps to describe the behavior of nature by mathematical logic.

the transformation to convert two points to each other. This transformation is like the Liouville transformation, but with this difference that the Liouville transformation transforms the Sturm–Liouville equations with the positive weight function to the Sturm–Liouville equations with the weight function $r^2(x) = 1$ and continuous potential function. Indeed we extend this transformation and would like to prove that the extended transformation can be applied for Sturm–Liouville equations with the turning point. We gave here only the main idea and refer the interested reader to Miller & Michel (1982) for further details. Therefore, we show that this transformation will omit the turning point and commute a turning point to a singularity. Moreover, we will prove that a singular point does not simply vanish. In other words, we will transform a singular point of second order to a turning point by the same transformation which indeed we use the reverse of this transformation to omit the singularity.

As mentioned earlier, the main goal of this article is to study the relationship between a turning point and a singularity in the Sturm–Liouville equations. We will apply a change of a variable to transform a turning point into a singularity. Taking the same transformation, we will convert merely a singularity of the second order to a turning point and give the interesting results. We mention that the approach considered in this article can serve as a tool for various studies connected with the spectral theory of Sturm–Liouville equations and topics connected with these problems, like, for example, direct and inverse spectral problems. We note that direct and inverse spectral problems for differential equations having singularities and/or turning points were studied in Dehghan & Jodayree (2013), Neamaty & Khalili (2013, 2014), and other works.

2. Main results

We consider the differential equation

$$-y''(x) + q(x)y(x) = \lambda r^2(x)y(x), \quad x \in [0, T], \tag{2.1}$$

with the weight function $r^2(x)$. We assume that the weight function has the zeros in an interior point $x = a$ which is called a turning point. The functions $q(x)$ and $r^2(x)$ are real valued.

Lemma 1. Consider the Equation (2.1). Let two real functions f and g be piecewise continuous. If the following change of the variable

$$u(z) = f(x)y(x), \quad z = \int_0^x g(v)dv, \tag{2.2}$$

omits the turning point, then $f^2(x) = g(x)$.

Proof. After obtaining y'' and substituting in the Equation (2.1), we get

$$\frac{d^2y}{dx^2} = (2f^{-3}f'^2 - f^{-2}f'')u + (-2f^{-2}f'g + f^{-1}g')\frac{du}{dz} + f^{-1}g^2\frac{d^2u}{dz^2},$$

where $f^n := (f(x))^n$ for $n \in \mathbb{Z}$. To eliminate $\frac{du}{dz}$, its coefficient must be deleted. So

$$2f^{-2}f'g = f^{-1}g' \rightarrow 2f^{-1}f' = g^{-1}g' \rightarrow 2\frac{f'}{f} = \frac{g'}{g}.$$

By integrating from two sides of the previous relation, we get

$$2\ln f = \ln g \rightarrow f^2 = g.$$

The Lemma 1 is proved. □

Theorem 1. Consider the Equation (2.1). Then under the following transformation

$$u(z) = r^{\frac{1}{2}}(x)y(x), \quad z = \int_0^x r(v)dv, \tag{2.3}$$

the Equation (2.1) is transformed to

$$-u''(z) + Q(z)u(z) = \lambda u(z), \quad z \in [0, S], \tag{2.4}$$

where $S = \int_0^T r(v)dv$.

Proof. Using Lemma 1, we assume that the following transformation

$$u(z) = r^\alpha(x)y(x), \quad z = \int_0^x r^{2\alpha}(v)dv, \quad \alpha \neq 0, \tag{2.5}$$

omits the turning point in the Equation (2.1). It is sufficient to find a unknown parameter α which is a positive rational number. At first, we calculate the second derivation of the function $y(x)$. Therefore

$$\frac{d^2y}{dx^2} = \left(\alpha(\alpha + 1)r^{-\alpha-2}(x)r'^2(x) - \alpha r^{-\alpha-1}(x)r''(x) \right) u + r^{3\alpha}(x) \frac{d^2u}{dz^2}. \tag{2.6}$$

Substituting (2.5) and (2.6) in (2.1), we get

$$\begin{aligned} & -\frac{d^2u}{dz^2} + \left(-\alpha(\alpha + 1)r^{-4\alpha-2}(x)r'^2(x) + \alpha r^{-4\alpha-1}(x)r''(x) + q(x)r^{-4\alpha}(x) \right) u(z) \\ & = \lambda r^{-4\alpha+2}(x)u(z). \end{aligned} \tag{2.7}$$

Now, to commute the weight function with the turning point to the classical form, i.e., $r(x) = 1$, we have

$$-4\alpha + 2 = 0 \rightarrow \alpha = \frac{1}{2}. \tag{2.8}$$

Therefore, taking the parameter α , the Equation (2.7) is converted to the following form

$$-\frac{d^2u}{dz^2} + Q(z)u(z) = \lambda u(z), \tag{2.9}$$

where

$$Q(z) = \left(\frac{-3}{4}r^{-4}(x(z))r'^2(x(z)) + \frac{1}{2}r^{-3}(x(z))r''(x(z)) + q(x(z))r^{-2}(x(z)) \right). \tag{2.10}$$

The proof is completed. □

Notation 1. Consider the Equation (2.1). Let $\epsilon > 0$ be fixed, sufficiently small and $D_{0\epsilon} = [0, x_1 - \epsilon]$, $D_{\nu\epsilon} = [x_\nu + \epsilon, x_{\nu+1} - \epsilon]$ for $1 \leq \nu \leq m - 1$, $D_{m\epsilon} = [x_m + \epsilon, T]$, $D_\epsilon = \bigcup_{\nu=0}^m D_{\nu\epsilon}$ and $I_{\nu\epsilon} = D_{\nu-1,\epsilon} \cup [x_\nu - \epsilon, x_\nu + \epsilon] \cup D_{\nu,\epsilon}$. We distinguish four different types of turning points $x_\nu \in (0, T)$. For $1 \leq \nu \leq m$

$$T_\nu = \begin{cases} I, & \text{if } l_\nu \text{ is even and } r^2(x)(x - x_\nu)^{-l_\nu} < 0 \text{ in } I_{\nu\epsilon}, \\ II, & \text{if } l_\nu \text{ is even and } r^2(x)(x - x_\nu)^{-l_\nu} > 0 \text{ in } I_{\nu\epsilon}, \\ III, & \text{if } l_\nu \text{ is odd and } r^2(x)(x - x_\nu)^{-l_\nu} < 0 \text{ in } I_{\nu\epsilon}, \\ IV, & \text{if } l_\nu \text{ is odd and } r^2(x)(x - x_\nu)^{-l_\nu} > 0 \text{ in } I_{\nu\epsilon}, \end{cases} \tag{2.11}$$

is called type of x_v . Here $r^2(x) = \prod_{v=1}^m (x - x_v)^{l_v} r_0(x)$ where $r_0(x) > 0, l_v \in \mathbb{N}$.

To show the application of this transformation, we apply it for two different types of turning points.

Example 1. Consider the following differential equation

$$-y''(x) + q(x)y(x) = \lambda r^2(x)y(x), \quad x \in [0, T], \quad (2.12)$$

with the weight function $r^2(x) = (x - x_0)^{2l}$ for $l > 0, x_0 \in (0, T)$. Using the transformation (2.3), we have

$$u(z) = (x - x_0)^{\frac{1}{2}} y(x), \rightarrow y(x) = (x - x_0)^{-\frac{1}{2}} u(z), \quad (2.13)$$

$$z = \int_0^x (v - x_0)^l dv. \quad (2.14)$$

Taking (2.10) into account, we get

$$Q(x(z)) = \frac{-l}{4}(l+2)(x-x_0)^{-2l-2} + q(x(z))(x-x_0)^{-2l}. \quad (2.15)$$

Using (2.14), we obtain

$$x = x_0 + ((l+1)z + (-x_0)^{l+1})^{\frac{1}{l+1}}. \quad (2.16)$$

Also

$$x_0 = -(-l+1)z_0)^{\frac{1}{l+1}}. \quad (2.17)$$

Now taking (2.15), (2.16) and (2.17), we have

$$Q(z) = \frac{\frac{-l(l+2)}{4(l+1)^2} + (l+1)^{\frac{-2l}{l+1}} q(x(z))(z-z_0)^{\frac{2l}{l+1}}}{(z-z_0)^2}. \quad (2.18)$$

From Theorem 1, we arrive at the following differential equation

$$-\frac{d^2 u(z)}{dz^2} + \frac{Q_1(z)}{(z-z_0)^2} u(z) = \lambda u(z), \quad z \in [0, S], \quad (2.19)$$

where $Q_1(z) = \frac{-l(l+2)}{4(l+1)^2} + (l+1)^{\frac{-2l}{l+1}} q(x(z))(z-z_0)^{\frac{2l}{l+1}}$.

Example 2. Consider the following differential equation

$$-y''(x) + q(x)y(x) = \lambda r^2(x)y(x), \quad x \in [0, T], \quad (2.20)$$

with the weight function $r^2(x) = (x - x_0)^{2l+1}$ for $l > 0, x_0 \in (0, T)$. Using the transformation (2.3), we have

$$u(z) = (x - x_0)^{\frac{2l+1}{4}} y(x), \rightarrow y(x) = (x - x_0)^{-\frac{2l+1}{4}} u(z), \quad (2.21)$$

$$z = \int_0^x (v - x_0)^{\frac{2l+1}{2}} dv. \quad (2.22)$$

Taking (2.10), we have

$$Q(x(z)) = \frac{(2l+1)(2l+5)}{16}(x-x_0)^{-2l-3} + q(x(z))(x-x_0)^{-2l-1}. \quad (2.23)$$

Using (2.22), we obtain

$$x = x_0 + \left(\frac{2l+3}{2} z + (-x_0)^{\frac{2l+3}{2}} \right)^{\frac{2}{2l+3}}. \tag{2.24}$$

Also

$$x_0 = - \left(\frac{2l+3}{-2} z_0 \right)^{\frac{2}{2l+3}}. \tag{2.25}$$

Now taking (2.23), (2.24) and (2.25), we have

$$Q(z) = \frac{\frac{(2l+1)(2l+5)}{4(2l+3)^2} + q(x(z)) \left(\frac{2l+3}{2} \right)^{\frac{-4l-2}{2l+3}} (z - z_0)^{\frac{4}{2l+3}}}{(z - z_0)^2}. \tag{2.26}$$

From Theorem 1, we arrive at the following differential equation

$$- \frac{d^2 u(z)}{dz^2} + \frac{Q_1(z)}{(z - z_0)^2} u(z) = \lambda u(z), \quad z \in [0, S], \tag{2.27}$$

where $Q_1(z) = \frac{(2l+1)(2l+5)}{4(2l+3)^2} + q(x(z)) \left(\frac{2l+3}{2} \right)^{\frac{-4l-2}{2l+3}} (z - z_0)^{\frac{4}{2l+3}}$. We can prove the problem for other types of turning points, analogously. Therefore, a turning point of any type is transformed to a singular point of the second order by this transformation. Now we will show that singular points of the second order are only transformed to the turning point by this transformation.

Theorem 2. The transformation (2.3) only transmutes a singular point of the second order to a turning point in Sturm–Liouville equations.

Proof. Consider the following differential equation

$$- \frac{d^2 u(z)}{dz^2} + Q_0(z) u(z) = \lambda u(z), \quad z \in [0, S], \tag{2.28}$$

where $Q_0(z) = \frac{\nu}{(z - z_0)^l}$ for $l > 0$, $z_0 \in (0, S)$ and $\nu \neq 0$. It is sufficient to show that a singular point only vanishes for $l = 2$ by this transformation. Using the transformation (2.3), we have

$$\frac{d^2 u(z)}{dz^2} = \left(\frac{-3}{4} r^{-7}(x) r'^2(x) + \frac{1}{2} r^{-5}(x) r''(x) \right) y + r^{-3}(x) \frac{d^2 y}{dx^2}.$$

Substituting the aforementioned relation in (2.28), we get

$$- \frac{d^2 y}{dx^2} + q_0(x) y = \lambda r^2(x) y, \quad x \in [0, T], \tag{2.29}$$

where

$$q_0(x) = \left(\frac{3}{4} r^2(x) r'^2(x) - \frac{1}{2} r^{-1}(x) r''(x) + Q_0(z) r^2(x) \right). \tag{2.30}$$

Let $r(x) = (x - x_0)^n$ for $n \in \mathbb{N}$, $x_0 \in (0, T)$. Therefore we have

$$q_0(x) = \left(\frac{3n^2}{4(x - x_0)^2} + \frac{-n(n - 1)}{2(x - x_0)^2} + \frac{\nu(x - x_0)^{2n}}{(z - z_0)^l} \right).$$

Also $z - z_0 = \frac{(x - x_0)^{n+1}}{n+1}$. So

$$q_0(x) = \left(\frac{3n^2}{4(x-x_0)^2} + \frac{-n(n-1)}{2(x-x_0)^2} + \frac{\nu(n+1)^l}{(x-x_0)^{-2n+nl+l}} \right). \tag{2.31}$$

Now we have to eliminate the potential function $q_0(x)$. We consider two cases:

If $l = 2$ then,

$$q_0(x) = \frac{n^2 + 2n + 4\nu(n+1)^2}{4(x-x_0)^2}. \tag{2.32}$$

This is possible for $\nu = \frac{n(n+2)}{-4(n+1)^2}$.

For $l \neq 2$, two cases will occur. At the case $-2n + nl + l < 2$, we get for $x \neq x_0$,

$$q_0(x) = \frac{n^2 + 2n + 4\nu(n+1)^2(x-x_0)^{2-(-2n+nl+l)}}{4(x-x_0)^2}. \tag{2.33}$$

Otherwise, if $-2n + nl + l > 2$, then for $x \neq x_0$,

$$q_0(x) = \frac{(n^2 + 2n)(x-x_0)^{-2n+nl+l-2} + 4\nu(n+1)^2}{4(x-x_0)^{-2n+nl+l}}. \tag{2.34}$$

The potential functions (2.33) and (2.34) do not always vanish and the singular point x_0 remains after taking this transformation in the case $l \neq 2$. Therefore, the transformation (2.3) apply for $l = 2$, i.e., using this transformation, we can only omit the singular point of the second order.

Theorem 2 is proved. □

It seems that singular points of other orders do not vanish. In this case, we only mention the following theorem.

Theorem 3. The following transformation made on the discontinuous function

$$u(z) = q_1^\alpha(x)y(x), \quad z = \int_0^x q_1^{2\alpha}(v)dv; \quad \alpha < \frac{1}{2l}, \tag{2.35}$$

does not eliminate the singular point in Sturm–Liouville equations.

Proof. We consider the following differential equation

$$-y'' + q_1(x)y = \lambda y, \quad x \in [0, T], \tag{2.36}$$

where $q_1(x) = \frac{1}{(x-x_0)^l}$ is a real function for $l > 0, x_0 \in (0, T)$. Using (2.35), we have for $x \neq x_0$

$$\frac{d^2y}{dx^2} = (\alpha(\alpha+1)q_1^{-\alpha-2}(x)q_1'^2(x) - \alpha q_1^{-\alpha-1}(x)q_1''(x))u + q_1^{3\alpha}(x)\frac{d^2u}{dz^2}. \tag{2.37}$$

Substituting (2.35) and (2.37) in (2.36), we give

$$\begin{aligned} & -\frac{d^2u}{dz^2} + (-\alpha(\alpha+1)q_1^{-4\alpha-2}(x)q_1'^2(x) + \alpha q_1^{-4\alpha-1}(x)q_1''(x) + q_1^{-4\alpha+1}(x))u(z) \\ & = \lambda q_1^{-4\alpha}(x)u(z). \end{aligned} \tag{2.38}$$

Taking $q_1(x) = \frac{1}{(x-x_0)^l}$, we have

$$-\frac{d^2 u}{dz^2} + q_2(x)u(z) = \lambda(x - x_0)^{4\alpha}u(z), \quad (2.39)$$

where

$$q_2(x) = \frac{-\alpha(\alpha + 1)l^2}{(x - x_0)^{2-4\alpha}} + \frac{\alpha l(l + 1)}{(x - x_0)^{2-4\alpha}} + \frac{1}{(x - x_0)^{1-4\alpha}}. \quad (2.40)$$

We have to assume $\alpha \geq \frac{1}{2l}$ to omit the singular point. This is inconsistent to the integrable condition (2.35). Therefore, the singular point does not vanish by taking the transformation (2.35). Theorem 3 is proved. \square

3. Conclusions

In this article, we verify a transformation to omit the turning points in Sturm–Liouville equations. In other words, we proposed a transformation and proved that this transformation can eliminate a turning point (Theorem 1). Also we showed that differential equations with a singular point of the second order are transformed to a differential equation with a turning point by using this transformation.

Moreover, we proved that any transformation of the form (2.35) cannot omit the singularity.

Funding

The authors received no direct funding for this research.

Author details

Y. Khalili¹

E-mail: y.khalili@sanru.ac.ir

A. Neamaty²

E-mail: namaty@umz.ac.ir

¹ Department of Basic Sciences, Sari Agricultural Sciences and Natural Resources University, 578 Sari, Iran.

² Department of Mathematics, University of Mazandaran, Babolsar, Iran.

Citation information

Cite this article as: On the relationship between the turning and singular points in Sturm–Liouville equations, Y. Khalili & A. Neamaty, *Cogent Mathematics & Statistics* (2018), 5: 1464880.

References

- Constantin, A. (1998). On the inverse spectral problem for the Camassa–Holm equation. *Journal of Functional Analysis*, 155, 352–363. doi:10.1006/jfan.1997.3231
- Conway, J. B. (1995). *Functions of one complex variable* (Vol. I, 2nd ed.). New York, NY: Springer-Verlag.
- Dehghan, M., & Jodayree, A. A. (2013). Reconstructing the potential function for indefinite Sturm–Liouville problems using infinite product forms. *Electronic Journal of Differential Equations*, 182, 1–16.
- Hryniv, R. O., & Mykytyuk, Y. V. (2004). Half inverse spectral problems for Sturm–Liouville operators with

singular potentials. *Inverse Problems*, 20, 1423–1444. doi:10.1088/0266-5611/20/5/006

Lapwood, F. R., & Usami, T. (1981). *Free oscillations of the earth*. Cambridge (UK): Cambridge University Press.

Litvinenko, O. N., & Soshnikov, V. I. (1964). *The theory of heterogenous lines and their applications in radio engineering*. Moscow: Radio.

Miller, R. K., & Michel, A. N. (1982). *Ordinary differential equations*. New York, NY: Academic Press.

Mosazadeh, S., & Neamaty, A. (2011). On the uniqueness of the solution of dual equation of a singular Sturm–Liouville problem. *Proceedings – Mathematical Sciences*, 121, 469–480. doi:10.1007/s12044-011-0044-5

Neamaty, A., & Khalili, Y. (2013). The differential pencils with turning point on the half line. *Arab Journal of Mathematical Sciences*, 19(1), 95–104.

Neamaty, A., & Khalili, Y. (2014). Determination of a differential operator with discontinuity from interior spectral data. *Inverse Problems in Science and Engineering*, 22(6), 1002–1008. doi:10.1080/17415977.2013.848436

Neamaty, A., & Mosazadeh, S. (2011). On the canonical solution of the Sturm–Liouville problem with singularity and turning point of even order. *Canadian Mathematical Bulletin*, 54(3), 506–518. doi:10.4153/CMB-2011-069-7

Yurko, V. A. (1997). Integral transforms connected with differential operators having singularities inside the interval. *Integral Transforms and Special Functions*, 5, 309–322. doi:10.1080/10652469708819143



© 2018 The Author(s). This open access article is distributed under a Creative Commons Attribution (CC-BY) 4.0 license.

You are free to: Share — copy and redistribute the material in any medium or format.

Adapt — remix, transform, and build upon the material for any purpose, even commercially.

The licensor cannot revoke these freedoms as long as you follow the license terms. Under the following terms:

Attribution — You must give appropriate credit, provide a link to the license, and indicate if changes were made.

You may do so in any reasonable manner, but not in any way that suggests the licensor endorses you or your use. No additional restrictions

You may not apply legal terms or technological measures that legally restrict others from doing anything the license permits.



***Cogent Mathematics & Statistics* (ISSN: 2574-2558) is published by Cogent OA, part of Taylor & Francis Group.**

Publishing with Cogent OA ensures:

- Immediate, universal access to your article on publication
- High visibility and discoverability via the Cogent OA website as well as Taylor & Francis Online
- Download and citation statistics for your article
- Rapid online publication
- Input from, and dialog with, expert editors and editorial boards
- Retention of full copyright of your article
- Guaranteed legacy preservation of your article
- Discounts and waivers for authors in developing regions

Submit your manuscript to a Cogent OA journal at www.CogentOA.com

