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On the relationship between the turning and singular points in Sturm-Liouville equations

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Abstract. In this paper, we present some results about the Sturm-Liouville equation with turning points and singularities and transform them to each other. By applying a change of a variable, we can transform the differential equation with a turning point to the differential equation with a singularity. Also we will prove that a differential equation with a singularity will be transformed to a differential equation with a turning point in some cases.

Keywords: Sturm-Liouville operator; Turning point; Singularity.
2010 MSC: 34B24 34C05 34E20.

1 Introduction

Differential equations with turning points and singularities have emerged as an effective and powerful tool to study a wide class of related problems arising in various branches of mathematics, mechanics, physics, geophysics, computer sciences and other branches of natural sciences. Better knowledge and understanding from these equations provide us to study a wide class of problems arising in mathematical physics, radio electronics and other fields of sciences and technologies (see [1-6]). Differential equations with turning points and singularities have been studied in [8,10-12]. But the relationship between turning points and singularities has not yet been studied basically. In [11], differential equations with a turning point of the second type are transformed to a singularity. To the best of our knowledge, it does not exist an exact study on the relevance between a turning and singular point. For this reason, we would like to point out the transformation to convert two points to each other. This transformation is like the Liouville transformation, but with this difference that the Liouville transformation transforms the Sturm-Liouville equations with the positive weight function to the Sturm-Liouville equations with the weight function $r^2(x)=1$ and continuous potential function. Indeed we extend this transformation and would like to prove that the extended transformation can be applied for Sturm-Liouville equations with the turning point. We gave here only the main idea and refer the interested reader to [7] for further details. Therefore, we show that this transformation will omit the turning point and commute a turning point to a singularity. Moreover, we will prove that a singular point does not simply vanish. In other words, we will transform a singular point of second order to a turning point by the same transformation which indeed we use the reverse of this transformation to omit the singularity.

As mentioned above, the main goal of this paper is to study the relationship between a

turning point and a singularity in the Sturm-Liouville equations. We will apply a change of a variable to transform a turning point into a singularity. Taking the same transformation, we will convert merely a singularity of the second order to a turning point and give the interesting results. We mention that the approach considered in this paper can serve as a tool for various studies connected with the spectral theory of Sturm-Liouville equations and topics connected with these problems, like, for example, direct and inverse spectral problems. We note that direct and inverse spectral problems for differential equations having singularities and/or turning points were studied in [3,9,10] and other works.

2 Main results

We consider the differential equation

$$-y''(x) + q(x)y(x) = \lambda r^2(x)y(x), \quad x \in [0, T], \quad (2.1)$$

with the weight function $r^2(x)$. We assume that the weight function has the zeros in an interior point $x = a$ which is called a turning point. The functions $q(x)$ and $r^2(x)$ are real valued.

Lemma 1. Consider the Eq. (2.1). Let two real functions f and g be piecewise continuous. If the following change of the variable

$$u(z) = f(x)y(x), \quad z = \int_0^x g(v)dv, \quad (2.2)$$

omits the turning point, then $f^2(x) = g(x)$.

Proof. After obtaining y'' and substituting in the Eq. (2.1), we get

$$\frac{d^2y}{dx^2} = (2f^{-3}f'^2 - f^{-2}f'')u + (-2f^{-2}f'g' + f^{-1}g')\frac{du}{dz} + f^{-1}g^2\frac{d^2u}{dz^2},$$

where $f^n := (f(x))^n$ for $n \in \mathbb{Z}$. To eliminate $\frac{du}{dz}$, its coefficient must be deleted. So

$$2f^{-2}f'g' = f^{-1}g' \rightarrow 2f^{-1}f' = g^{-1}g' \rightarrow 2\frac{f'}{f} = \frac{g'}{g}.$$

By integrating from two sides of the above relation, we get

$$2\ln f = \ln g \rightarrow f^2 = g.$$

The Lemma 1 is proved. \square

Theorem 1. Consider the Eq. (2.1). Then under the following transformation

$$u(z) = r^{\frac{1}{2}}(x)y(x), \quad z = \int_0^x r(v)dv, \quad (2.3)$$

the Eq. (2.1) is transformed to

$$-u''(z) + Q(z)u(z) = \lambda u(z), \quad z \in [0, S], \quad (2.4)$$

where $S = \int_0^T r(v)dv$.

Proof. Using Lemma 1, we assume that the following transformation

$$u(z) = r^\alpha(x)y(x), \quad z = \int_0^x r^{2\alpha}(v)dv, \quad \alpha \neq 0, \quad (2.5)$$

omits the turning point in the Eq. (2.1). It is sufficient to find a unknown parameter α which is

a positive rational number. At first, we calculate the second derivation of the function $y(x)$.

Therefore

$$\frac{d^2 y}{dx^2} = \left(\alpha(\alpha+1)r^{-\alpha-2}(x)r'^2(x) - \alpha r^{-\alpha-1}(x)r''(x) \right) u + r^{3\alpha}(x) \frac{d^2 u}{dz^2}. \quad (2.6)$$

Substituting (2.5) and (2.6) in (2.1), we get

$$-\frac{d^2 u}{dz^2} + \left(-\alpha(\alpha+1)r^{-4\alpha-2}(x)r'^2(x) + \alpha r^{-4\alpha-1}(x)r''(x) + q(x)r^{-4\alpha}(x) \right) u(z) = \lambda r^{-4\alpha+2}(x)u(z). \quad (2.7)$$

Now, to commute the weight function with the turning point to the classical form, i.e., $r(x) = 1$, we have

$$-4\alpha + 2 = 0 \rightarrow \alpha = \frac{1}{2}. \quad (2.8)$$

Therefore, taking the parameter α , the Eq. (2.7) is converted to the following form

$$-\frac{d^2 u}{dz^2} + Q(z)u(z) = \lambda u(z), \quad (2.9)$$

where

$$Q(z) = \left(\frac{-3}{4} r^{-4}(x(z))r'^2(x(z)) + \frac{1}{2} r^{-3}(x(z))r''(x(z)) + q(x(z))r^{-2}(x(z)) \right). \quad (2.10)$$

The proof is completed. \square

Notation 1. Consider the Eq. (2.1). Let $\epsilon > 0$ be fixed, sufficiently small and $D_{0\epsilon} = [0, x_1 - \epsilon]$, $D_{\nu\epsilon} = [x_\nu + \epsilon, x_{\nu+1} - \epsilon]$ for $1 \leq \nu \leq m-1$, $D_{m\epsilon} = [x_m + \epsilon, T]$, $D_\epsilon = \bigcup_{\nu=0}^m D_{\nu\epsilon}$ and $I_{\nu\epsilon} = D_{\nu-1,\epsilon} \cup [x_\nu - \epsilon, x_\nu + \epsilon] \cup D_{\nu,\epsilon}$. We distinguish four different types of turning points $x_\nu \in (0, T)$. For $1 \leq \nu \leq m$

$$T_\nu = \begin{cases} I, & \text{if } l_\nu \text{ is even and } r^2(x)(x-x_\nu)^{-l_\nu} < 0 \text{ in } I_{\nu\epsilon}, \\ II, & \text{if } l_\nu \text{ is even and } r^2(x)(x-x_\nu)^{-l_\nu} > 0 \text{ in } I_{\nu\epsilon}, \\ III, & \text{if } l_\nu \text{ is odd and } r^2(x)(x-x_\nu)^{-l_\nu} < 0 \text{ in } I_{\nu\epsilon}, \\ IV, & \text{if } l_\nu \text{ is odd and } r^2(x)(x-x_\nu)^{-l_\nu} > 0 \text{ in } I_{\nu\epsilon}, \end{cases} \quad (2.11)$$

is called type of x_ν . Here $r^2(x) = \prod_{\nu=1}^m (x-x_\nu)^{l_\nu} r_0(x)$ where $r_0(x) > 0, l_\nu \in \mathbb{N}$.

To show the application of this transformation, we apply it for two different types of turning points.

Example 1. Consider the following differential equation

$$-y''(x) + q(x)y(x) = \lambda r^2(x)y(x), \quad x \in [0, T], \quad (2.12)$$

with the weight function $r^2(x) = (x-x_0)^{2l}$ for $l > 0, x_0 \in (0, T)$. Using the transformation (2.3), we have

$$u(z) = (x-x_0)^{\frac{l}{2}} y(x), \rightarrow y(x) = (x-x_0)^{\frac{-l}{2}} u(z), \quad (2.13)$$

$$z = \int_0^x (v - x_0)^l dv. \quad (2.14)$$

Taking (2.10) into account, we get

$$Q(x(z)) = \frac{-l}{4}(l+2)(x-x_0)^{-2l-2} + q(x(z))(x-x_0)^{-2l}. \quad (2.15)$$

Using (2.14), we obtain

$$x = x_0 + ((l+1)z + (-x_0)^{l+1})^{\frac{1}{l+1}}. \quad (2.16)$$

Also

$$x_0 = -(-l+1)z_0^{\frac{1}{l+1}}. \quad (2.17)$$

Now taking (2.15), (2.16) and (2.17), we have

$$Q(z) = \frac{\frac{-l(l+2)}{4(l+1)^2} + (l+1)^{\frac{-2l}{l+1}} q(x(z))(z-z_0)^{\frac{2}{l+1}}}{(z-z_0)^2}. \quad (2.18)$$

From Theorem 1, we arrive at the following differential equation

$$-\frac{d^2 u(z)}{dz^2} + \frac{Q_1(z)}{(z-z_0)^2} u(z) = \lambda u(z), \quad z \in [0, S], \quad (2.19)$$

where $Q_1(z) = \frac{-l(l+2)}{4(l+1)^2} + (l+1)^{\frac{-2l}{l+1}} q(x(z))(z-z_0)^{\frac{2}{l+1}}$.

Example 2. Consider the following differential equation

$$-y''(x) + q(x)y(x) = \lambda r^2(x)y(x), \quad x \in [0, T], \quad (2.20)$$

with the weight function $r^2(x) = (x-x_0)^{2l+1}$ for $l > 0, x_0 \in (0, T)$. Using the transformation (2.3), we have

$$u(z) = (x-x_0)^{\frac{2l+1}{4}} y(x), \rightarrow y(x) = (x-x_0)^{-\frac{2l+1}{4}} u(z), \quad (2.21)$$

$$z = \int_0^x (v-x_0)^{\frac{2l+1}{2}} dv. \quad (2.22)$$

Taking (2.10), we have

$$Q(x(z)) = \frac{(2l+1)(2l+5)}{16}(x-x_0)^{-2l-3} + q(x(z))(x-x_0)^{-2l-1}. \quad (2.23)$$

Using (2.22), we obtain

$$x = x_0 + \left(\frac{2l+3}{2}z + (-x_0)^{\frac{2l+3}{2}}\right)^{\frac{2}{2l+3}}. \quad (2.24)$$

Also

$$x_0 = -\left(\frac{2l+3}{-2}z_0\right)^{\frac{2}{2l+3}}. \quad (2.25)$$

Now taking (2.23), (2.24) and (2.25), we have

$$Q(z) = \frac{\frac{(2l+1)(2l+5)}{4(2l+3)^2} + q(x(z))\left(\frac{2l+3}{2}\right)^{\frac{-4l-2}{2l+3}} (z-z_0)^{\frac{4}{2l+3}}}{(z-z_0)^2}. \quad (2.26)$$

From Theorem 1, we arrive at the following differential equation

$$-\frac{d^2u(z)}{dz^2} + \frac{Q_1(z)}{(z-z_0)^2}u(z) = \lambda u(z), \quad z \in [0, S], \quad (2.27)$$

where $Q_1(z) = \frac{(2l+1)(2l+5)}{4(2l+3)^2} + q(x(z))\left(\frac{2l+3}{2}\right)^{\frac{-4l-2}{2l+3}} (z-z_0)^{\frac{4}{2l+3}}$.

We can prove the problem for other types of turning points, analogously. Therefore, a turning point of any type is transformed to a singular point of the second order by this transformation.

Now we will show that singular points of the second order are only transformed to the turning point by this transformation.

Theorem 2. The transformation (2.3) only transmutes a singular point of the second order to a turning point in Sturm-Liouville equations.

Proof. Consider the following differential equation

$$-\frac{d^2u(z)}{dz^2} + Q_0(z)u(z) = \lambda u(z), \quad z \in [0, S], \quad (2.28)$$

where $Q_0(z) = \frac{\nu}{(z-z_0)^l}$ for $l > 0$, $z_0 \in (0, S)$ and $\nu \neq 0$. It is sufficient to show that a singular point only vanishes for $l = 2$ by this transformation. Using the transformation (2.3), we have

$$\frac{d^2u(z)}{dz^2} = \left(\frac{-3}{4} r^{\frac{-7}{2}}(x)r'^2(x) + \frac{1}{2} r^{\frac{-5}{2}}(x)r''(x) \right) y + r^{\frac{-3}{2}}(x) \frac{d^2y}{dx^2}.$$

Substituting the above relation in (2.28), we get

$$-\frac{d^2y}{dx^2} + q_0(x)y = \lambda r^2(x)y, \quad x \in [0, T], \quad (2.29)$$

where

$$q_0(x) = \left(\frac{3}{4} r^2(x)r'^2(x) - \frac{1}{2} r^{-1}(x)r''(x) + Q_0(z)r^2(x) \right). \quad (2.30)$$

Let $r(x) = (x-x_0)^n$ for $n \in \mathbb{N}$, $x_0 \in (0, T)$. Therefore we have

$$q_0(x) = \left(\frac{3n^2}{4(x-x_0)^2} + \frac{-n(n-1)}{2(x-x_0)^2} + \frac{\nu(x-x_0)^{2n}}{(z-z_0)^l} \right).$$

Also $z-z_0 = \frac{(x-x_0)^{n+1}}{n+1}$. So

$$q_0(x) = \left(\frac{3n^2}{4(x-x_0)^2} + \frac{-n(n-1)}{2(x-x_0)^2} + \frac{\nu(n+1)^l}{(x-x_0)^{-2n+nl+l}} \right). \quad (2.31)$$

Now we have to eliminate the potential function $q_0(x)$. We consider two cases:

If $l = 2$ then,

$$q_0(x) = \frac{n^2 + 2n + 4\nu(n+1)^2}{4(x-x_0)^2}. \quad (2.32)$$

This is possible for $\nu = \frac{n(n+2)}{-4(n+1)^2}$.

For $l \neq 2$, two cases will occur. At the case $-2n + nl + l < 2$, we get for $x \neq x_0$,

$$q_0(x) = \frac{n^2 + 2n + 4\nu(n+1)^2(x-x_0)^{2-(-2n+nl+l)}}{4(x-x_0)^2}. \quad (2.33)$$

Otherwise, if $-2n + nl + l > 2$, then for $x \neq x_0$,

$$q_0(x) = \frac{(n^2 + 2n)(x-x_0)^{-2n+nl+l-2} + 4\nu(n+1)^2}{4(x-x_0)^{-2n+nl+l}}. \quad (2.34)$$

The potential functions (2.33) and (2.34) do not always vanish and the singular point x_0 remains after taking this transformation in the case $l \neq 2$. Therefore, the transformation (2.3) apply for $l = 2$, i.e., using this transformation, we can only omit the singular point of the second order. Theorem 2 is proved. \square

It seems that singular points of other orders do not vanish. In this case, we only mention the following theorem.

Theorem 3. The following transformation made on the discontinuous function

$$u(z) = q_1^\alpha(x)y(x), \quad z = \int_0^x q_1^{2\alpha}(v)dv; \quad \alpha < \frac{1}{2l}, \quad (2.35)$$

does not eliminate the singular point in Sturm-Liouville equations.

Proof. We consider the following differential equation

$$-y'' + q_1(x)y = \lambda y, \quad x \in [0, T], \quad (2.36)$$

where $q_1(x) = \frac{1}{(x-x_0)^l}$ is a real function for $l > 0, x_0 \in (0, T)$. Using (2.35), we have for $x \neq x_0$

$$\frac{d^2y}{dx^2} = (\alpha(\alpha+1)q_1^{-\alpha-2}(x)q_1^{l^2}(x) - \alpha q_1^{-\alpha-1}(x)q_1^{l'}(x))u + q_1^{3\alpha}(x)\frac{d^2u}{dz^2}. \quad (2.37)$$

Substituting (2.35) and (2.37) in (2.36), we give

$$-\frac{d^2u}{dz^2} + (-\alpha(\alpha+1)q_1^{-4\alpha-2}(x)q_1^{l^2}(x) + \alpha q_1^{-4\alpha-1}(x)q_1^{l'}(x) + q_1^{-4\alpha+1}(x))u(z) = \lambda q_1^{-4\alpha}(x)u(z). \quad (2.38)$$

Taking $q_1(x) = \frac{1}{(x-x_0)^l}$, we have

$$-\frac{d^2u}{dz^2} + q_2(x)u(z) = \lambda(x-x_0)^{4\alpha l}u(z), \quad (2.39)$$

where

$$q_2(x) = \frac{-\alpha(\alpha+1)l^2}{(x-x_0)^{2-4\alpha l}} + \frac{\alpha l(l+1)}{(x-x_0)^{2-4\alpha l}} + \frac{1}{(x-x_0)^{1-4\alpha}}. \quad (2.40)$$

We have to assume $\alpha \geq \frac{1}{2l}$ to omit the singular point. This is inconsistent to the integrable condition (2.35). Therefore, the singular point does not vanish by taking the transformation (2.35). Theorem 3 is proved. \square

3 Conclusions

In this paper, we verify a transformation to omit the turning points in Sturm-Liouville equations. In other words, we proposed a transformation and proved that this transformation can eliminate a turning point (Theorem 1). Also we showed that differential equations with a singular point of the second order are transformed to a differential equation with a turning point by using this transformation.

Moreover, we proved that any transformation of the form (2.35) can not omit the singularity.

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PUBLIC INTEREST STATEMENT

The theory of the inverse spectral problems is very helpful in the study of many areas of research, such as mathematics, mechanics, physics, geophysics, computer sciences, etc. In the recent decade, the investigation of the inverse problem is an important research topic and there is increasing interest in this field. The study of these problems helps to describe the behavior of nature by mathematical logic.

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