cogent mathematics & statistics

Received: 12 February 2018
Accepted: 03 April 2018
First Published: 11 April 2018

*Corresponding author: Y. Golkhandy Pour, Faculty of Science, Department of Mathematics, Imam Khomeini International University, Qazvin, Iran
E-mail: y.golkhandypour@edu.ikiu.ac.ir

Reviewing editor:
Lishan Liu, Qufu Normal University, China

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PURE MATHEMATICS | RESEARCH ARTICLE

Maximum nullity of some Cayley graphs

E. Vatandoost[1] and Y. Golkhandy Pour[1*]

Abstract: Recently, the nullity, the algebraic multiplicity of the number zero in the spectrum of the adjacency matrix, of a molecular graph has received a lot of attention as it has a number of direct applications in organic chemistry. In this regard, many researchers have been trying to find an upper or lower bound for the maximum nullity (minimum rank), $M(\mathcal{G})$ $(mr(\mathcal{G}))$, for a graph $\mathcal{G}$. In this paper, using a well-known result which presents the spectrum of a Cayley graph in terms of irreducible characters of the underlying group, and using representation and character of groups, we give a lower bound for the maximum nullity of Cayley graph, $X_S(G)$, where $G = \langle a \rangle$ is a cyclic group, or $G = G_1 \times \cdots \times G_t$ such that $G_1 = \langle a \rangle$ is a cyclic group and G_i is an arbitrary finite group, for $2 \leq i \leq t$, with determine the spectrum of Cayley graphs.

Subjects: Algebra; Combinatorics; Pure Mathematics; Research Article

Keywords: maximum nullity; spectrum; Cayley graph

AMS subject classifications: 13C05; 13C99

1. Introduction

Motivation for founding the theory of graph spectra has come from applications in Chemistry and Physics. In theoretical chemistry, the π-electron energy of a conjugated carbon molecule computed using the *Hückel* theory. In the *Hückel* Molecular orbital method, for conjugated hydrocarbons, the energy of the j-th molecular orbital of the so-called π-electrons is related to the graph spectra. The nullity of a molecular graph, denoted by $N(\mathcal{G})$, is the algebraic multiplicity of the number zero in the spectrum of the adjacency matrix of the molecular graph. Recently, the nullity of a graph has received a lot of attention as it has a number of direct applications in organic chemistry.

For a positive integer n, let $S_n(\mathbb{R})$ be the set of all symmetric matrices of order n over the real number. Suppose that $A \in S_n(\mathbb{R})$. Then the graph of A which is denoted by $\mathcal{H}(A)$ is a graph with the vertex set $\{u_1, \ldots, u_n\}$ and the edge set $\{u_i \sim u_j : a_{ij} \neq 0, 0 \leq i < j \leq n\}$. It should be noted that the diagonal of A has no role in the determining of $\mathcal{H}(A)$.

ABOUT THE AUTHORS

E. Vatandoost is presently working as an assistant professor in the Department of Mathematics, Imam Khomeini International University, Qazvin, Iran. His field of specialities includes Group Theory and Graph Theory.

Y. Golkhandy Pour is a PhD candidate of pure algebra in the Department of Mathematics, Imam Khomeini International University, Qazvin, Iran. Her favorite fields are Group Theory and Graph Theory.

PUBLIC INTEREST STATEMENT

Maximum nullity and its related parameter zero forcing number are one of the interesting research areas in graph theory. The notion of maximum nullity was introduced by the well-known mathematicians *AIM Minimum Rank-Special Graphs Work Group*. Motivation for founding the theory of graph spectra has come from applications in Chemistry and Physics. Recently, the nullity of a graph has received a lot of attention as it has a number of direct applications in organic chemistry.

The *set of symmetric matrices* of graph $\mathcal{G}$ is the set $S(\mathcal{G}) = \{A \in S_n(\mathbb{R}) : \mathcal{H}(A) = \mathcal{G}\}$. The *minimum rank* of a graph was first introduced by AIM Minimum Rank-Special Graphs Work Group (2008) and is defined to be the minimum cardinality between the rank of symmetric matrices in $S(\mathcal{G})$ and denoted by $mr(\mathcal{G})$. Similarly, the *maximum nullity* of $\mathcal{G}$ is defined to be the maximum cardinality between the nullity of symmetric matrices in $S(\mathcal{G})$ and is denoted by $M(\mathcal{G})$. Clearly, $mr(\mathcal{G}) + M(\mathcal{G}) = n$.

One of the most interesting problems on minimum rank is to characterize $mr(\mathcal{G})$ for graphs. In this regard, many researchers have been trying to find an upper or lower bound for the minimum rank.

The adjacency matrix of a graph $\mathcal{G}$ is the matrix $A_{\mathcal{G}}$ whose the entry $a_{ij} = 1$ if and only if vertices u_i and u_j are adjacent, and $a_{ij} = 0$ otherwise. The eigenvalues of $\mathcal{G}$ are the eigenvalues of $A_{\mathcal{G}}$, and the spectrum of $\mathcal{G}$ is the collection of its eigenvalues together with multiplicities. If $\lambda_1 \dots, \lambda_t$ are distinct eigenvalues of a graph $\mathcal{G}$ with respective multiplicity $n_1, \dots n_k$, then we denote the spectrum of $\mathcal{G}$ by

$$spec(\mathcal{G}) = \left[\lambda_1^{n_1}, \dots, \lambda_t^{n_t}\right]. \tag{1}$$

Let G be a group, and let S be a subset of G that is closed under taking inverse and does not contain the identity, e. Then the *Cayley graph*, $X_S(G)$, is the graph with vertex set G and edge set

$$E = \{g_1 \sim g_2 : g_1 g_2^{-1} \in S\}. \tag{2}$$

Since S is inverse closed and does not contain the identity, it is a simple fact that $X_S(G)$ is undirected and has no loop.

In Babai (1979) presented the spectrum of a Cayley graph in terms of irreducible characters of the underlying group G. The following important theorem was the result of this paper.

THEOREM 1.1 *(Babai, 1979) Let G be a finite group of order n whose irreducible characters (over $\mathbb{C}$) are $\chi_1, \dots, \chi_h$ with respective degree $n_1, \dots, n_h$. Then the spectrum of the Cayley graph $X_S(G)$ can be arranged as $\Lambda = \{\lambda_{ijk} : i = 1, \dots, h; j, k = 1, \dots, n_i\}$ such that $\lambda_{ij1} = \dots = \lambda_{ijn_i}$ (this common value will be denoted by λ_{ij}), and*

$$\lambda_{i1}^t + \dots + \lambda_{in_i}^t = \sum_{s_1, \dots, s_t \in S} \chi_i\Big(\prod_{l=1}^{t} s_l\Big), \tag{3}$$

for any natural number t.

In this paper, using a well-known result of Babai (1979), we give a lower bound for the maximum nullity of Cayley graph, $X_S(G)$, where $G = \langle a \rangle$ is a cyclic group, or $G = G_1 \times \cdots \times G_t$ such that $G_1 = \langle a \rangle$ is a cyclic group and G_i is an arbitrary finite group, for some $2 \le i \le t$, with determine the spectrum of Cayley graphs.

2. Preliminaries

For any positive integer n, define *Möbius* number, $\mu(n)$, as the sum of the primitive n^{th} roots of unity. It has values in $\{-1, 0, 1\}$ depending on the factorization of n into prime factors.

(1) $\mu(1) = 1$,

(2) $\mu(n) = 0$, if n has a squared factor,

(3) $\mu(n) = (-1)^k$, if n is a square free with k number of prime factors.

Suppose that k is a positive integer. The number of solutions of $y_1 + \cdots + y_r \equiv t \pmod{k}$, where $y_1, \ldots, y_r$ and t are belonged to the least non-negative residue system modulo k, is obtained in terms of the von sterneck function, $\Phi(n, k)$. In particular, von Sterneck studied the case where the polynomial resulting from the expansion is reduced modulo a positive integer. This function is used in several equivalent forms and in the form used by *Hölder* (1936),

$$\Phi(k,n) = \frac{\phi(n)}{\phi(n/(n,k))}\mu(n/(n,k)), \tag{4}$$

where k and n are positive integers, (n, k) is the greatest common divisor of k and n, $\phi(n)$ is the *Eüler totient*, and $\mu(n)$ is the *Möbius* number. In the sequel, the following fundamental result is obtained by *Hölder*.

$$\Phi(r,n) = \sum_{(r,n)=1} exp(2\pi irk/n). \tag{5}$$

This properties was also studied by Daublebsky Von Sterneck , (1902), Nicol and Vandiver , (1954), and Apostol (1972).

Suppose that $B(k,n) = \{t \in \mathbb{N} : t \leq n\ ,\ (t,n) = k\}$, and let $\omega = exp(2\pi i/n)$. Then the following function is called Ramanujan sum and is denoted by $C(r, n)$.

$$\sum_{k \in B(1,n)} \omega^{kr}, 0 \leq r \leq n-1, \tag{6}$$

In Ramanujan (2000), it was obtained that $C(r, n)$ have only integral values, for some positive integers r and n. Also, (5) and (6) state that $\Phi(r,n) = C(r,n)$.

LEMMA 2.1 *Suppose that $n > 1$ and $d > 1$ are two positive integers such that $d \mid n$. Also, let $B(d,n) = \{t \in \mathbb{N} : t \leq n\ ,\ (t,n) = d\}$. Then*

If $t \in B(d,n)$, then $C(t,n) = C(d,n)$,

$|B(d,n)| = \phi(n/d)$.

Proof. The proof is straightforward. □

THEOREM 2.1 *(Adams & Goldstein, 1976) For Eüler totient ϕ and positive integer n, we have $\sum_{d|n} \phi(d) = n$.*

LEMMA 2.2 *(James & Liebeck, 1993) The irreducible character of $G \times H$ is $\chi \times \psi$ such that χ and ψ are the irreducible characters of G and H, respectively. the value of $\chi \times \psi$ for any $g \in G$ and $h \in H$ is $(\chi \times \psi)(g,h) = \chi(g)\psi(h)$.*

LEMMA 2.3 *(James & Liebeck, 1993) Let $G = \langle a \rangle$ be a cyclic group of order n. Then irreducible characters of G are $\rho_j(a^k) = \omega^{jk}$, where $j, k = 0, 1, \ldots, n-1$.*

3. Main theorems

In the following theorem, we determine the spectrum of Cayley graph $X_S(G)$ whose G is a cyclic group of order n. Here, we define $F(n_i) = (-1)^{k_i}\phi(n)/\phi(n_i)$, where k_i is the number of prime factors in the decomposition of n_i.

Theorem 3.1 *Let n be a positive integer and D be its divisors set. Also, let $G = \langle a \rangle$ be a cyclic group of order n and $S = \{a^i : i \in B(1, n)\}$. Then*

$$spec(X_S(G)) = \left[\phi(n)^1, 0^{\sum_{x \in X} \phi(x)}, F(d_1)^{\phi(d_1)}, ..., F(d_t)^{\phi(d_t)}\right],$$

where $X = \{d \in D : p^2 \mid d\}$, for a prime p; and $d_i \in D \setminus X$, for some $1 \le i \le t$.

Proof. First, suppose that n is a prime number. Thus $X_S(G)$ is isomorphic to the complete graph K_n, and so

$$spec(X_S(G)) = \left[(p-1)^1, -1^{(p-1)}\right]. \tag{7}$$

Now, consider the case in that n is not prime. Let $\lambda_{\frac{n}{d_i}}$ be the eigenvalue of $X_S(G)$ corresponding to character of $\chi_{\frac{n}{d_i}}$, for some $d_i \in D$. By Lemma 1.1, $\lambda_{\frac{n}{d_i}} = C(\frac{n}{d_i}, n)$, and by the form used by *Hölder* in (4), we have

$$\lambda_{\frac{n}{d_i}} = \frac{\phi(n)}{\phi(d_i)} \mu(d_i). \tag{8}$$

On the other hand, lemma 2.1 implies that the multiplicity of $\lambda_{\frac{n}{d_i}}$ is equal to $\phi(d_i)$. If $d_i = 1$, then $\lambda_n = \phi(n)$ with multiplicity 1. Also, if $p^2 \mid d_i$, then definition of *Möbius* number implies that $\lambda_{\frac{n}{d_i}} = 0$. For other cases, $\lambda_{\frac{n}{d_i}} = F(d_i)$. □

The following theorem, which is proven by Akbari & Vatandoost (2017), help us to make a connection between the multiplicity of the eigenvalues of a graph $\mathcal{G}$ and its maximum nullity $M(\mathcal{G})$.

Theorem 3.2 *(Akbari, Vatandoost & Golkhandy Pour, 2017) Let $\mathcal{G}$ be a graph of order n, and let λ_i be its eigenvalue with respective multiplicity n_i. Then $M(\mathcal{G}) \ge n_i$.*

As a result, Theorems 3.1 and 3.2, state the following corollary.

Corollary 3.2.1 *Let n be a positive integer and D be its divisors set. Also, let $G = \langle a \rangle$ be a cyclic group of order n, and let $S = \{a^i : i \in B(1, n)\}$. For some prime p and $d_i \in D$, the followings are established.*

(1) If n has a squared factor, then $M(X_S(G)) \ge max\left\{\sum_{p^2 \mid d_i} \phi(d_i), \phi(d_i)\right\}$.
(2) If n is a square free, then $M(X_S(G)) \ge \phi(d_i)$.

Definition 3.1 Let G be a group, and let S be a subset of G. Also, let $\Lambda = \{\chi_1, \dots, \chi_k\}$ be the set of irreducible characters with degree 1 of G. A character $\chi_i \in \Lambda$ is defined to be an ℓ-index character of G, if has the same value ℓ on all letters in S; in other word, $\chi_i \in \Lambda$ is an ℓ-index character of G if $\chi(s_i) = \ell$, for all $s_i \in S$. In the sequel, An ℓ-index number of G is defined to be the number of ℓ-index characters of G and is denoted by $N_G(\ell)$.

Theorem 3.3 *Let n be a positive integer whose divisors set is denoted by D. Also, let $G_1 = \langle a \rangle$ be a cyclic group of order n, and let $S' = \{a^i : i \in B(1, n)\}$. Suppose that $G_2, \dots, G_t$ are some arbitrary finite groups, and let S_k is a subset of G_k, for some $2 \le k \le t$. If $S = \{(a^i, \alpha_1, \dots, \alpha_t) : a^i \in S', \alpha_k \in S_k\}$, then for some prime p and $d_i \in D$, the followings are established.*

(1) If n has a square factor, then

$$M\left(X_S(G_1\times\cdots\times G_t)\right)\geq max$$
$$\left\{\left(\prod_{i=2}^{t}\left(N_{G_i}(\ell_i)|S_i|\right)\right)\left(\sum_{p^2|d_i}\phi(d_i)\right),\left(\prod_{i=2}^{t}\left(N_{G_i}(\ell_i)|S_i|\right)\right)\left(\phi(d_i)\right)\right\}.$$

(2) If n is a square free, then

$$M\left(X_S(G_1\times\cdots\times G_t)\right)\geq\left(\prod_{i=2}^{t}\left(N_{G_i}(\ell_i)|S_i|\right)\right)\left(\phi(d_i)\right).$$

Proof. For some $2\leq k\leq t$, suppose that ρ_{j_k} are the ℓ_k-index irreducible characters with degree 1 of G_k, and let $\chi_{\frac{n}{d_i}}$ be an irreducible character of G_1. Let $\lambda_{\frac{n}{d_i}j_1\cdots j_t}$ and $\lambda_{\frac{n}{d_i}}$ be the eigenvalues of $X_S(G_1\times\ldots\times G_t)$ and $X_{S'}(G_1)$ corresponding to irreducible characters of $\chi_{\frac{n}{d_i}}\times\rho_{j_1}\times\cdots\times\rho_{j_t}$ and $\chi_{\frac{n}{d_i}}$, respectively. Lemma 2.2 implies that

$$\begin{aligned}\lambda_{\frac{n}{d_i}j_2\cdots j_t}&=\sum_{(g_1,\ldots,g_t)\in S}\left(\chi_{\frac{n}{d_i}}\times\rho_{j_2}\times\ldots\times\rho_{j_t}\right)(g_1,\ldots,g_t)\\&=\sum_{(g_1,\ldots,g_t)\in S}\left(\chi_{\frac{n}{d_i}}(g_1)\times\rho_{j_2}(g_2)\times\ldots\times\rho_{j_t}(g_t)\right)\\&=\left(\prod_{k=2}^{t}\left(N_{G_k}(\ell_k)|S_k|\right)\right)\sum_{s'\in S'}\left(\chi_{\frac{n}{d_i}}(s')\right).\end{aligned}\tag{9}$$

We have,

$$\lambda_{\frac{n}{d_i}j_2\cdots j_t}=\left(\prod_{k=2}^{t}\left(N_{G_k}(\ell_k)|S_k|\right)\right)(\lambda_{\frac{n}{d_i}}).\tag{10}$$

Hence, by Theorem 3.1, if n is a square free, then $\lambda_{\frac{n}{d_i}j_2\cdots j_t}$ is an eigenvalue of $X_S(G_1\times\cdots\times G_t)$ with multiplicity

$$\left(\prod_{k=2}^{t}\left(N_{G_k}(\ell_k)|S_k|\right)\right)(\phi(d_i)),\tag{11}$$

and if n is divided by a prime number, then $\lambda_{\frac{n}{d_i}j_2\cdots j_t}$ is an eigenvalue of $X_S(G_1\times\cdots\times G_t)$ with multiplicity

$$\left(\prod_{k=2}^{t}\left(N_{G_k}(\ell_k)|S_k|\right)\right)(\phi(d_i)),\tag{12}$$

where $\lambda_{\frac{n}{d_i}}\neq 0$, or with multiplicity

$$\left(\prod_{k=2}^{t}\left(N_{G_k}(\ell_k)|S_k|\right)\right)\left(\sum_{p^2|d_i}\phi(d_i)\right),\tag{13}$$

where $\lambda_{\frac{n}{d_i}}\neq 0$. Therefore, theorem 3.2, completed the proof. □

We use theorem 3.3 and consider the maximum nullity of dihedral groups D_n, with presentation

$$\langle a,b:a^n=b^2=e,(ab)^2=e\rangle,\tag{14}$$

Table 1. Character table of dihedral groups D_n, where $n = 2m + 1$

	a^i	a^{ib}
χ_j	$\omega^{jk} + \omega^{-jk}$	$0 \quad (j = 1, \ldots, m)$
χ_{m+1}	1	−1
χ_{m+2}	1	1

where $n = 2m + 1$ is odd. In this case, D_n has m irreducible character of degree 2 and 2 characters of degree 1. See Table 1, for more details.

Example 1 Let $G = \langle g \rangle$ be a group of odd order n, and let $S = \{(g^i, a^j) : g^i \in G, a^j \in D_n, i,j \in B(1,n)\}$. Obviously, χ_{m+1} and χ_{m+2} are two 1-index irreducible characters of D_n, and so $N_{D_n}(1) = 2$. Hence, by Theorem 3.3, we have

(1) If n has a square factor, then

$$M(X_S(G \times D_n)) \geq max\left\{ 2\phi(n)\Big(\sum_{p^2|d_i} \phi(d_i)\Big), 2\phi(n)\big(\phi(d_i)\big) \right\}.$$

(2) If n is square free, then $M(X_S(G \times D_n)) \geq 2\phi(n)(\phi(d_i))$.

Funding
The authors are very grateful to the referee for his/her useful comments. This research was partially supported by the Imam Khomeini International University.

Author details
E. Vatandoost[1]
E-mail: vatandoost@sci.ikiu.ac.ir
Y. Golkhandy Pour[1]
E-mail: y.golkhandypour@edu.ikiu.ac.ir
[1] Faculty of Science, Department of Mathematics, Imam Khomeini International University, Qazvin, Iran.

Citation information
Cite this article as: Maximum nullity of some Cayley graphs, Y. Golkhandy Pour & E. Vatandoost, *Cogent Mathematics & Statistics* (2018), 5: 1462658.

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