On \((\Lambda, \theta)\)-open sets in topological spaces

Chawalit Boonpok\textsuperscript{1*} and Chokchai Viriyapong\textsuperscript{1}

\textbf{ABSTRACT:} This article deals with the notions of \(s(\Lambda, \theta)\)-open, \(p(\Lambda, \theta)\)-open, \(\alpha(\Lambda, \theta)\)-open, \(\beta(\Lambda, \theta)\)-open, and \(b(\Lambda, \theta)\)-open sets. Several properties and the relationships between these concepts are discussed. Some characterizations of \(\Lambda_0\)-extremally disconnected spaces and \(\Lambda_0\)-hyperconnected spaces are investigated.

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1. Introduction

In 1966, the notions of \(\theta\)-open sets, \(\theta\)-closed sets, and \(\theta\)-closure operators were introduced by Veličko \cite{Velic} for the purpose of studying the important class of \(H\)-closed spaces in terms of arbitrary filterbases. Dickman and Porter \cite{DickmanPorter}, Joseph \cite{Joseph}, and Long and Herrington \cite{LongHerr} continued the work of Veličko. Noiri and Jafari \cite{NoiriJafari} have also obtained several new and interesting results related to these sets. Dickman and Porter \cite{DickmanPorter} proved that a compact subspace of a Hausdorff space is \(\theta\)-closed. Moreover, they showed that a \(\theta\)-closed subspace of a Hausdorff space is closed. Janković \cite{Jankovic} proved that a space \((X, \tau)\) is Hausdorff if and only if every compact set is \(\theta\)-closed. The family of all \(\theta\)-open sets forms a topology on \(X\) and is denoted by \(\tau_\theta\). This topology is coarser than \(\tau\) and a space \((X, \tau)\) is regular if and only if \(\tau = \tau_\theta\). It is also obvious that a set \(A\) is \(\theta\)-closed in \((X, \tau)\) if and only if it is closed in \((X, \tau_\theta)\). In \cite{Lee} the concepts of the \((\Lambda, \theta)\)-closure and \((\Lambda, \theta)\)-open sets were introduced by using \(\theta\)-open sets and \(\theta\)-closure operations due to Veličko \cite{Velic}. Al-Hawary introduced and studied the notions of \(C\)-open sets \cite{AlHawary}, generalized preopen sets \cite{AlHawary}, \(\omega\)-open sets \cite{Silwan}, \(\varphi\)-closed sets \cite{AlHawary}, and generalized \(b\)-closed sets \cite{Silwan}. In 1970, Willard \cite{Willard} introduced the concept of extremally disconnected spaces. It is well-known that each extremally disconnected compact and Hausdorff space is called a Stonean space. Extremally disconnected spaces play an important role in the theory of Boolean algebras and in some branches of functional analysis. There is a duality between Stonean spaces and the category of complete Boolean algebras. The importance of extremally disconnected spaces becomes clearer in a study of absolute relationships between these concepts. Moreover, several characterizations of \(\Lambda_0\)-extremally disconnected spaces and \(\Lambda_0\)-hyperconnected spaces are investigated.
topological spaces. Sivaraj [21] investigated some characterizations of extremally disconnected spaces by utilizing semi-open sets due to Levine [14]. Recently, Noiri [18] has obtained some characterizations of extremally disconnected spaces by utilizing preopen sets and semi-preopen sets. The notion of hyperconnected spaces was introduced by Steen and Seebach Jr. [22]. Several notions that are equivalent to hyperconnectedness were defined and investigated in the literature. Levine [15] called a topological space \((X, \tau)\) a \(D\)-space if every non-empty open set of \(X\) is dense in \(X\) and showed that \((X, \tau)\) is a \(D\)-space if and only if it is hyperconnected. Pipitone and Russo [19] defined a topological space \((X, \tau)\) to be semi-connected if \(X\) is not the union of two disjoint non-empty semi-open sets of \(X\) and showed that \((X, \tau)\) is semi-connected if and only if it is \(D\)-space. In 1992, Ajmal and Kohli [1] investigated the further properties of hyperconnected spaces. Shama [20] showed that \(D\)-spaces are equivalent to hyperconnected spaces due to Steen and Seebach Jr. [22] and among others established the following result: The semi-continuous image of a hyperconnected set is connected [20]. This article is divided as follows: In Section 3, the concepts of the following properties hold:

\[\begin{align*}
\text{Definition 2.1.} & \text{ Let } A \text{ be a subset of a topological space } (X, \tau) \text{ (or simply } X) \text{ always mean topological spaces and no separation axioms are assumed unless explicitly stated. For a subset } A \text{ of a topological space } (X, \tau), \text{ Cl}(A) \text{ and } \text{Int}(A) \text{ denote the closure and interior of } A \text{ in } (X, \tau), \text{ respectively. A point } x \in X \text{ is called a } \theta\text{-cluster point of } A \text{ if } A \cap \text{Cl}(U) \neq \emptyset \text{ for every open set } U \text{ of } X \text{ containing } x. \text{ The set of all } \theta\text{-cluster points of } A \text{ is called } \theta\text{-closure of } A \text{ and is denoted by } \text{Cl}_\theta(A). \text{ A subset } A \text{ of a topological space } (X, \tau) \text{ is called } \theta\text{-open if } A = \text{Cl}_\theta(A) [23]. \text{ The complement of a } \theta\text{-closed set is said to be } \theta\text{-open. The union of all } \theta\text{-open sets contained in } A \text{ is called the } \theta\text{-interior of } A \text{ and is denoted by } \text{Int}_\theta(A). \text{ It is shown in [23] that } \text{Cl}_\theta(V) = \text{Cl}(V) \text{ for every open set } V \text{ of } X \text{ and } \text{Cl}_\theta(B) \text{ is closed in } (X, \tau) \text{ for every subset } B \text{ of } X. \text{ The family of all } \theta\text{-open sets in a topological space } (X, \tau) \text{ is denoted by } \theta(X, \tau). \end{align*}\]

**Lemma 2.2.** [8] *For subsets \(A, B, \text{ and } A_i (i \in I) \text{ of a topological space } (X, \tau), \text{ the following properties hold:}*

\[\begin{align*}
(1) \quad & A \subseteq \text{Cl}_\theta(A). \\
(2) \quad & \text{If } A \subseteq B, \text{ then } \text{Cl}_\theta(A) \subseteq \text{Cl}_\theta(B). \\
(3) \quad & \text{Cl}_\theta(\text{Cl}_\theta(A)) = \text{Cl}_\theta(A). \\
(4) \quad & \text{Cl}_\theta(\bigcap\{A_i \mid i \in I\}) \subseteq \bigcap\{\text{Cl}_\theta(A_i) \mid i \in I\}. \\
(5) \quad & \text{Cl}_\theta(\bigcup\{A_i \mid i \in I\}) = \bigcup\{\text{Cl}_\theta(A_i) \mid i \in I\}. \\
\end{align*}\]

**Definition 2.3.** [8] *A subset \(A \subseteq \theta\text{-open set of a topological space } (X, \tau) \text{ is called a } \Lambda\theta\text{-set if } A = \text{Cl}_\theta(A). \)

**Lemma 2.4.** [8] *For subsets \(A \text{ and } A_i (i \in I) \text{ of a topological space } (X, \tau), \text{ the following properties hold:}*

\[\begin{align*}
(1) \quad & \text{Cl}_\theta(A) \text{ is a } \Lambda\theta\text{-set.} \\
(2) \quad & \text{If } A \text{ is a } \theta\text{-open, then } A \text{ is a } \Lambda\theta\text{-set.} \\
(3) \quad & \text{If } A_i \text{ is a } \Lambda\theta\text{-set for each } i \in I, \text{ then } \bigcap_{i \in I} A_i \text{ is a } \Lambda\theta\text{-set.} \\
(4) \quad & \text{If } A_i \text{ is a } \Lambda\theta\text{-set for each } i \in I, \text{ then } \bigcup_{i \in I} A_i \text{ is a } \Lambda\theta\text{-set.} \\
\end{align*}\]

**Definition 2.5.** Let \(A \subseteq \theta\text{-open set of a topological space } (X, \tau). \)

\[\begin{align*}
(1) \quad & \text{A is called a } (\Lambda, \theta)\text{-closed set [8] if } A = T \cap C, \text{ where } T \text{ is a } \Lambda\theta\text{-set and } C \text{ is a } \theta\text{-closed set. The complement of a } (\Lambda, \theta)\text{-closed set is called } (\Lambda, \theta)\text{-open. The collection of all } (\Lambda, \theta)\text{-open (resp. } (\Lambda, \theta)\text{-closed) sets in a topological space } (X, \tau) \text{ is denoted by } \Lambda\theta\text{Cl}(X, \tau) \text{ (resp. } \Lambda\theta\text{Int}(X, \tau)). \\
\end{align*}\]
(2) A point \( x \in X \) is called a \((\Lambda, \theta)\)-cluster point of \( A \) \cite{8} if for every \((\Lambda, \theta)\)-open set \( U \) of \( X \) containing \( x \), we have \( A \cap U \neq \emptyset \). The set of all \((\Lambda, \theta)\)-cluster points of \( A \) is called the \((\Lambda, \theta)\)-closure of \( A \) and is denoted by \( A^{(\Lambda, \theta)} \).

**Lemma 2.6.** \cite{8} Let \( A \) and \( B \) be subsets of a topological space \((X, \tau)\). For the \((\Lambda, \theta)\)-closure, the following properties hold:

1. \( A \subseteq A^{(\Lambda, \theta)} \).
2. \( A^{(\Lambda, \theta)} = \bigcap \{F | A \subseteq F \text{ and } F \text{ is } (\Lambda, \theta)\text{-closed} \} \).
3. If \( A \subseteq B \), then \( A^{(\Lambda, \theta)} \subseteq B^{(\Lambda, \theta)} \).
4. \( A^{(\Lambda, \theta)} \) is \((\Lambda, \theta)\)-closed.

**Lemma 2.7.** \cite{7} Let \( A \) be a subset of a topological space \((X, \tau)\). Then the following properties hold:

1. If \( A \) is \((\Lambda, \theta)\)-closed, then \( A = A_\theta(A) \cap Cl_\tau(A) \).
2. If \( A \) is \( \theta \)-closed, then \( A \) is \((\Lambda, \theta)\)-closed.
3. If \( A_i \) is \((\Lambda, \theta)\)-closed for each \( i \in I \), then \( \bigcap_{i \in I} A_i \) is \((\Lambda, \theta)\)-closed.

**Definition 2.8.** \cite{7} Let \((X, \tau)\) be a topological space, \( A \subseteq X \) and \( x \in X \). Then:

1. The \( \Lambda_\theta \)-kernel of \( A \), denoted by \( \Lambda_\theta Ker(A) \), is defined to be the set \( \Lambda_\theta Ker(A) = \cap \{ G \in \Lambda_\theta O(X, \tau) | A \subseteq G \} \);
2. \( \langle x \rangle = \{x \}^{(\Lambda, \theta)} \cap \Lambda_\theta Ker(\{x\}) \).

**Lemma 2.9.** \cite{7} Let \( A \) and \( B \) be subsets of a topological space \((X, \tau)\). Then the following properties hold:

1. \( A \subseteq B \) implies \( \Lambda_\theta Ker(A) \subseteq \Lambda_\theta Ker(B) \).
2. \( \Lambda_\theta Ker(\Lambda_\theta Ker(A)) = \Lambda_\theta Ker(A) \).

**Lemma 2.10.** \cite{7} Let \((X, \tau)\) be a topological space and \( x, y \in X \). Then, \( y \in \Lambda_\theta Ker(\{x\}) \) if and only if \( x \in \Lambda_\theta Ker(\{y\}) \).

**Lemma 2.11.** For a subset \( A \) of a topological space \((X, \tau)\), \( x \in A^{(\Lambda, \theta)} \) if and only if \( U \cap A \neq \emptyset \) for every \((\Lambda, \theta)\)-open set \( U \) containing \( x \).

**Proof.** Let \( x \in A^{(\Lambda, \theta)} \). Suppose that \( U \cap A = \emptyset \) for some \((\Lambda, \theta)\)-open set \( U \) containing \( x \). Then \( A \subseteq X - U \) and \( X - U \) is \((\Lambda, \theta)\)-closed. Since \( x \in A^{(\Lambda, \theta)} \), we have \( x \in X - U^{(\Lambda, \theta)} = X - U \); hence \( x \notin U \), which is a contradiction that \( x \notin U \). Conversely, we obtain \( U \cap A \neq \emptyset \) for every \((\Lambda, \theta)\)-open set \( U \) containing \( x \).

Conversely, assume that \( U \cap A \neq \emptyset \) for every \((\Lambda, \theta)\)-open set \( U \) containing \( x \). We shall show that \( x \in A^{(\Lambda, \theta)} \). Suppose that \( x \notin A^{(\Lambda, \theta)} \). Then, there exists a \((\Lambda, \theta)\)-closed set \( F \) such that \( A \subseteq F \) and \( x \notin F \). Therefore, we obtain \( X - F \) is a \((\Lambda, \theta)\)-open set containing \( x \) such that \( (X - F) \cap A = \emptyset \). This a contradiction to \( U \cap A \neq \emptyset \); hence \( x \in A^{(\Lambda, \theta)} \).

A subset \( B \) of a topological space \((X, \tau)\) is called a \((\Lambda, \theta)\)-neighborhood of a point \( x \in X \) \cite{7} if there exists a \((\Lambda, \theta)\)-open set \( U \) such that \( x \in U \subseteq B \).

**Lemma 2.12.** \cite{7} A subset \( A \) of a topological space \((X, \tau)\) is \((\Lambda, \theta)\)-open in \( X \) if and only if it is a \((\Lambda, \theta)\)-neighborhood of each point of \( A \).

**Definition 2.13.** Let \( A \) be a subset of a topological space \((X, \tau)\). The union of all \((\Lambda, \theta)\)-open sets contained in \( A \) is called the \((\Lambda, \theta)\)-interior of \( A \) and is denoted by \( A^{(\Lambda, \theta)} \).
Lemma 2.14. Let $A$ and $B$ be subsets of a topological space $(X, \tau)$. For the $(\Lambda, \theta)$-interior, the following properties hold:

1. $A_{(\Lambda, \theta)} \subseteq A$.
2. If $A \subseteq B$, then $A_{(\Lambda, \theta)} \subseteq B_{(\Lambda, \theta)}$.
3. $A$ is $(\Lambda, \theta)$-open if and only if $A_{(\Lambda, \theta)} = A$.
4. $A_{(\Lambda, \theta)}$ is $(\Lambda, \theta)$-open.

3. Generalized $(\Lambda, \theta)$-open sets

We begin this section by introducing the notions of $s(\Lambda, \theta)$-open, $p(\Lambda, \theta)$-open, $a(\Lambda, \theta)$-open and $\beta(\Lambda, \theta)$-open sets.

Definition 3.1. A subset $A$ of a topological space $(X, \tau)$ is said to be:

1. $s(\Lambda, \theta)$-open if $A \subseteq [A_{(\Lambda, \theta)}]^{(\Lambda, \theta)}$;
2. $p(\Lambda, \theta)$-open if $A \subseteq [A_{(\Lambda, \theta)}]^{(\Lambda, \theta)}$;
3. $a(\Lambda, \theta)$-open if $A \subseteq [A_{(\Lambda, \theta)}]^{\Lambda_{(\Lambda, \theta)}}$;
4. $\beta(\Lambda, \theta)$-open if $A \subseteq [A_{(\Lambda, \theta)}]^{\theta_{(\Lambda, \theta)}}$.

The family of all $s(\Lambda, \theta)$-open (resp. $p(\Lambda, \theta)$-open, $a(\Lambda, \theta)$-open, $\beta(\Lambda, \theta)$-open) sets in a topological space $(X, \tau)$ is denoted by $s_{\Lambda, \theta}O(X, \tau)$ (resp. $p_{\Lambda, \theta}O(X, \tau)$, $a_{\Lambda, \theta}O(X, \tau)$, $\beta_{\Lambda, \theta}O(X, \tau)$).

Definition 3.2. The complement of a $s(\Lambda, \theta)$-open (resp. $p(\Lambda, \theta)$-open, $a(\Lambda, \theta)$-open, $\beta(\Lambda, \theta)$-open) set is said to be $s(\Lambda, \theta)$-closed (resp. $p(\Lambda, \theta)$-closed, $a(\Lambda, \theta)$-closed, $\beta(\Lambda, \theta)$-closed).

The family of all $s(\Lambda, \theta)$-closed (resp. $p(\Lambda, \theta)$-closed, $a(\Lambda, \theta)$-closed, $\beta(\Lambda, \theta)$-closed) sets in a topological space $(X, \tau)$ is denoted by $s_{\Lambda, \theta}C(X, \tau)$ (resp. $p_{\Lambda, \theta}C(X, \tau)$, $a_{\Lambda, \theta}C(X, \tau)$, $\beta_{\Lambda, \theta}C(X, \tau)$).

Proposition 3.3. For a topological space $(X, \tau)$, the following properties hold:

1. $\Lambda_{\Lambda, \theta}O(X, \tau) \subseteq a_{\Lambda, \theta}O(X, \tau) \subseteq s_{\Lambda, \theta}O(X, \tau) \subseteq p_{\Lambda, \theta}O(X, \tau)$.
2. $a_{\Lambda, \theta}O(X, \tau) \subseteq p_{\Lambda, \theta}O(X, \tau)$.
3. $a_{\Lambda, \theta}O(X, \tau) = s_{\Lambda, \theta}O(X, \tau) \cap p_{\Lambda, \theta}O(X, \tau)$.

Proof:

1. Since $V = V_{(\Lambda, \theta)} \subseteq [V_{(\Lambda, \theta)}]^{(\Lambda, \theta)} \subseteq [V_{(\Lambda, \theta)}]^{\Lambda_{(\Lambda, \theta)}} \subseteq [V_{(\Lambda, \theta)}]^{\theta_{(\Lambda, \theta)}}$, we obtain $\Lambda_{\Lambda, \theta}O(X, \tau) \subseteq a_{\Lambda, \theta}O(X, \tau) \subseteq s_{\Lambda, \theta}O(X, \tau) \subseteq p_{\Lambda, \theta}O(X, \tau)$.

2. Since $V \subseteq [V_{(\Lambda, \theta)}]^{\Lambda_{(\Lambda, \theta)}} \subseteq [V_{(\Lambda, \theta)}]^{\theta_{(\Lambda, \theta)}}$, we have $a_{\Lambda, \theta}O(X, \tau) \subseteq p_{\Lambda, \theta}O(X, \tau)$.

3. By (1) and (2), we obtain $a_{\Lambda, \theta}O(X, \tau) \subseteq s_{\Lambda, \theta}O(X, \tau) \cap p_{\Lambda, \theta}O(X, \tau)$. On the other hand, let $V \in s_{\Lambda, \theta}O(X, \tau) \cap p_{\Lambda, \theta}O(X, \tau)$. Then, we have $V \in s_{\Lambda, \theta}O(X, \tau)$ and $V \in p_{\Lambda, \theta}O(X, \tau)$. Therefore, $V \subseteq [V_{(\Lambda, \theta)}]^{(\Lambda, \theta)}$ and $V \subseteq [V_{(\Lambda, \theta)}]^{\theta_{(\Lambda, \theta)}}$. This implies that $V \subseteq [V_{(\Lambda, \theta)}]^{\Lambda_{(\Lambda, \theta)}}$ and so $V \in a_{\Lambda, \theta}O(X, \tau)$. Hence,

$$s_{\Lambda, \theta}O(X, \tau) \cap p_{\Lambda, \theta}O(X, \tau) \subseteq a_{\Lambda, \theta}O(X, \tau).$$

Consequently, we obtain $a_{\Lambda, \theta}O(X, \tau) = s_{\Lambda, \theta}O(X, \tau) \cap p_{\Lambda, \theta}O(X, \tau)$.

Definition 3.4. A subset $A$ of a topological space $(X, \tau)$ is said to be $r(\Lambda, \theta)$-open (resp. $r(\Lambda, \theta)$-closed) if $A = [A_{(\Lambda, \theta)}]^{\Lambda_{(\Lambda, \theta)}}$ (resp. $A = [A_{(\Lambda, \theta)}]^{\theta_{(\Lambda, \theta)}}$).

The family of all $r(\Lambda, \theta)$-open (resp. $r(\Lambda, \theta)$-closed) sets in a topological space $(X, \tau)$ is denoted by $r_{\Lambda, \theta}O(X, \tau)$ (resp. $r_{\Lambda, \theta}C(X, \tau)$).
Proposition 3.5. For a subset $A$ of a topological space $(X, \tau)$, the following properties hold:

1. $A$ is $r(\Lambda, \theta)$-open if and only if $A = F_{(\Lambda, \theta)}$ for some $(\Lambda, \theta)$-closed set $F$.
2. $A$ is $r(\Lambda, \theta)$-closed if and only if $A = U_{(\Lambda, \theta)}$ for some $(\Lambda, \theta)$-open set $U$.

Lemma 3.6. For a subset $A$ of a topological space $(X, \tau)$, the following properties hold:

1. $|X - A|_{(\Lambda, \theta)} = X - A_{(\Lambda, \theta)}$.
2. $|X - A|_{(\Lambda, \theta)} = X - A_{(\Lambda, \theta)}$.

Proof. (1) Let $x \in X - A_{(\Lambda, \theta)}$. Then $x \in X - A$ and $x \notin A_{(\Lambda, \theta)}$. Consequently, we obtain $X - A_{(\Lambda, \theta)} = X - A_{(\Lambda, \theta)}$.

(2) This follows from (1).

Proposition 3.7. For a subset $A$ of a topological space $(X, \tau)$, the following properties hold:

1. $A$ is $s(\Lambda, \theta)$-closed if and only if $[A_{(\Lambda, \theta)}]_{(\Lambda, \theta)} \subseteq A$.
2. $A$ is $p(\Lambda, \theta)$-closed if and only if $[A_{(\Lambda, \theta)}]_{(\Lambda, \theta)} \subseteq A$.
3. $A$ is $a(\Lambda, \theta)$-closed if and only if $[A_{(\Lambda, \theta)}]_{(\Lambda, \theta)} \subseteq A$.
4. $A$ is $\beta(\Lambda, \theta)$-closed if and only if $[A_{(\Lambda, \theta)}]_{(\Lambda, \theta)} \subseteq A$.

Proof. (1) Suppose that $A$ is a $s(\Lambda, \theta)$-closed set. Then, we have $X - A$ is $s(\Lambda, \theta)$-open and so $X - A \subseteq |X - A|_{(\Lambda, \theta)}$. By Lemma 3.6,

$X - A \subseteq |X - A|_{(\Lambda, \theta)}, \Lambda, \theta = |X - A|_{(\Lambda, \theta)} = X - A_{(\Lambda, \theta)}$.

Consequently, we obtain $[A_{(\Lambda, \theta)}]_{(\Lambda, \theta)} \subseteq A$.

Conversely, suppose that $[A_{(\Lambda, \theta)}]_{(\Lambda, \theta)} \subseteq A$. Then, we have

$X - A \subseteq X - A_{(\Lambda, \theta)}$

and by Lemma 3.6, we obtain

$X - A \subseteq X - A_{(\Lambda, \theta)} = |X - A|_{(\Lambda, \theta)} = X - A_{(\Lambda, \theta)}$.

This implies that $X - A$ is $s(\Lambda, \theta)$-open and so $A$ is $s(\Lambda, \theta)$-closed.

The proofs of (2), (3) and (4) are similar to the proof of (1).

Proposition 3.8. For a subset $A$ of a topological space $(X, \tau)$, the following properties hold:

1. $[[[A_{(\Lambda, \theta)}]_{(\Lambda, \theta)}]_{(\Lambda, \theta)}]_{(\Lambda, \theta)} = [A_{(\Lambda, \theta)}]_{(\Lambda, \theta)}$.
2. $[[A_{(\Lambda, \theta)}]_{(\Lambda, \theta)}]_{(\Lambda, \theta)} = [A_{(\Lambda, \theta)}]_{(\Lambda, \theta)}$.

Proof. (1) Since $[A_{(\Lambda, \theta)}]_{(\Lambda, \theta)} \subseteq [A_{(\Lambda, \theta)}]_{(\Lambda, \theta)}$, we have

$[A_{(\Lambda, \theta)}]_{(\Lambda, \theta)} = [A_{(\Lambda, \theta)}]_{(\Lambda, \theta)} \subseteq [[[A_{(\Lambda, \theta)}]_{(\Lambda, \theta)}]_{(\Lambda, \theta)}]_{(\Lambda, \theta)}$.

On the other hand, since $[[[A_{(\Lambda, \theta)}]_{(\Lambda, \theta)}]_{(\Lambda, \theta)}]_{(\Lambda, \theta)} \subseteq A_{(\Lambda, \theta)}$,

$[[[A_{(\Lambda, \theta)}]_{(\Lambda, \theta)}]_{(\Lambda, \theta)}]_{(\Lambda, \theta)} \subseteq [A_{(\Lambda, \theta)}]_{(\Lambda, \theta)}$.
Consequently, we obtain \( \|A^{(\Lambda, \theta)}\|_{(\Lambda, \theta)} = \|A^{(\Lambda, \theta)}\|_{(\Lambda, \theta)} \).

(2) The proof is similar to that of (1).

Proposition 3.9. For a subset \( A \) of a topological space \((X, \tau)\), the following properties are equivalent:

(1) \( A \) is \( r(\Lambda, \theta) \)-open.
(2) \( A \) is \((\Lambda, \theta)\)-open and \( s(\Lambda, \theta)\)-closed.
(3) \( A \) is \( a(\Lambda, \theta)\)-open and \( s(\Lambda, \theta)\)-closed.
(4) \( A \) is \( p(\Lambda, \theta)\)-open and \( s(\Lambda, \theta)\)-closed.
(5) \( A \) is \((\Lambda, \theta)\)-open and \( \beta(\Lambda, \theta)\)-closed.
(6) \( A \) is \( a(\Lambda, \theta)\)-open and \( \beta(\Lambda, \theta)\)-closed.

Proof. (1) \( \Rightarrow \) (2): Suppose that \( A \) is a \( r(\Lambda, \theta)\)-open set. Then, we have \( A = [A^{(\Lambda, \theta)}]_{(\Lambda, \theta)} \) and so \( [A^{(\Lambda, \theta)}]_{(\Lambda, \theta)} \subseteq A \). Therefore, we obtain

\[
A^{(\Lambda, \theta)} = \left[ [A^{(\Lambda, \theta)}]_{(\Lambda, \theta)} \right]_{(\Lambda, \theta)} = [A^{(\Lambda, \theta)}]_{(\Lambda, \theta)} = A.
\]

Consequently, \( A \) is \((\Lambda, \theta)\)-open and \( s(\Lambda, \theta)\)-closed.

(2) \( \Rightarrow \) (3): Follows from Proposition 3.3(1).

(3) \( \Rightarrow \) (4): Follows from Proposition 3.3(2).

(4) \( \Rightarrow \) (5): Suppose that \( A \) is \( p(\Lambda, \theta)\)-open and \( s(\Lambda, \theta)\)-closed. Then, we have \( A \subseteq [A^{(\Lambda, \theta)}]_{(\Lambda, \theta)} \) and \( [A^{(\Lambda, \theta)}]_{(\Lambda, \theta)} \subseteq A \). This implies that \( A = [A^{(\Lambda, \theta)}]_{(\Lambda, \theta)} \). Therefore, \( A \) is \( r(\Lambda, \theta)\)-open and hence, \( A \) is \((\Lambda, \theta)\)-open. Since every \( s(\Lambda, \theta)\)-closed set is \( \beta(\Lambda, \theta)\)-closed. Consequently, we obtain \( A \) is \((\Lambda, \theta)\)-open and \( \beta(\Lambda, \theta)\)-closed.

(5) \( \Rightarrow \) (6): Follows from Proposition 3.3(1).

(6) \( \Rightarrow \) (1): Suppose that \( A \) is \( a(\Lambda, \theta)\)-open and \( \beta(\Lambda, \theta)\)-closed. Then, we have

\[
A \subseteq [A^{(\Lambda, \theta)}]_{(\Lambda, \theta)}
\]

and \( [A^{(\Lambda, \theta)}]_{(\Lambda, \theta)} \subseteq A \). This implies that \( A = [A^{(\Lambda, \theta)}]_{(\Lambda, \theta)} \). Therefore, we obtain \( A^{(\Lambda, \theta)} = [A^{(\Lambda, \theta)}]_{(\Lambda, \theta)} \). Hence, \( A^{(\Lambda, \theta)} = [A^{(\Lambda, \theta)}]_{(\Lambda, \theta)} \) and hence, \( A^{(\Lambda, \theta)} = [A^{(\Lambda, \theta)}]_{(\Lambda, \theta)} = A \). Consequently, \( A \) is \( r(\Lambda, \theta)\)-open.

Corollary 3.10. For a subset \( A \) of a topological space \((X, \tau)\), the following properties are equivalent:

(1) \( A \) is \( r(\Lambda, \theta)\)-closed.
(2) \( A \) is \((\Lambda, \theta)\)-closed and \( s(\Lambda, \theta)\)-open.
(3) \( A \) is \( a(\Lambda, \theta)\)-closed and \( s(\Lambda, \theta)\)-open.
(4) \( A \) is \( p(\Lambda, \theta)\)-closed and \( s(\Lambda, \theta)\)-open.
(5) \( A \) is \((\Lambda, \theta)\)-closed and \( \beta(\Lambda, \theta)\)-open.
(6) \( A \) is \( a(\Lambda, \theta)\)-closed and \( \beta(\Lambda, \theta)\)-open.

Definition 3.11. A subset \( A \) of a topological space \((X, \tau)\) is called \((\Lambda, \theta)\)-clopen if \( A \) is both \((\Lambda, \theta)\)-open and \((\Lambda, \theta)\)-closed.

Proposition 3.12. For a subset \( A \) of a topological space \((X, \tau)\), the following properties are equivalent:

(1) \( A \) is \((\Lambda, \theta)\)-clopen.
(2) \( A \) is \( r(\Lambda, \theta)\)-open and \( r(\Lambda, \theta)\)-closed.
(3) \( A \) is \((\Lambda, \theta)\)-open and \( a(\Lambda, \theta)\)-closed.
(4) \( A \) is \((\Lambda, \theta)\)-open and \( p(\Lambda, \theta)\)-closed.
(5) \( A \) is \( a(\Lambda, \theta)\)-open and \( p(\Lambda, \theta)\)-closed.
(6) \( A \) is \( a(\Lambda, \theta)\)-open and \((\Lambda, \theta)\)-closed.
(7) A is \( p(\Lambda, \theta) \)-open and \( (\Lambda, \theta) \)-closed.

(8) A is \( p(\Lambda, \theta) \)-open and \( a(\Lambda, \theta) \)-closed.

Proof. (1) \( \Rightarrow \) (2): Suppose that A is \( (\Lambda, \theta) \)-closed set. Then, we have \( A = A_{(\Lambda, \theta)} = A^{(\Lambda, \theta)} \) and so 
\( A = [A_{(\Lambda, \theta)}]^{(\Lambda, \theta)} = [A^{(\Lambda, \theta)}]_{(\Lambda, \theta)} \). This shows that A is \( r(\Lambda, \theta) \)-open and \( r(\Lambda, \theta) \)-closed.

(2) \( \Rightarrow \) (3): Suppose that A is \( r(\Lambda, \theta) \)-open and \( r(\Lambda, \theta) \)-closed. Then, we have 
\( A = [A_{(\Lambda, \theta)}]^{(\Lambda, \theta)} = [A^{(\Lambda, \theta)}]_{(\Lambda, \theta)} \).

Therefore, 
\[ A_{(\Lambda, \theta)} = [A^{(\Lambda, \theta)}]_{(\Lambda, \theta)} = A \]
and 
\[ [A^{(\Lambda, \theta)}]_{(\Lambda, \theta)} = [A^{(\Lambda, \theta)}]_{(\Lambda, \theta)} = A. \]

Consequently, we obtain A is \( (\Lambda, \theta) \)-open and \( a(\Lambda, \theta) \)-closed.

(3) \( \Rightarrow \) (4): Suppose that A is \( (\Lambda, \theta) \)-open and \( a(\Lambda, \theta) \)-closed. Then, we have \( A = A_{(\Lambda, \theta)} \) and 
\[ [A^{(\Lambda, \theta)}]_{(\Lambda, \theta)} \subseteq A. \] By Proposition 3.8(2), 
\[ [A_{(\Lambda, \theta)}]^{(\Lambda, \theta)} = [[A^{(\Lambda, \theta)}]^{(\Lambda, \theta)}]_{(\Lambda, \theta)} = [A^{(\Lambda, \theta)}]^{(\Lambda, \theta)} \subseteq A. \]

Hence, A is \( p(\Lambda, \theta) \)-closed. Therefore, we obtain A is \( (\Lambda, \theta) \)-open and \( p(\Lambda, \theta) \)-closed.

(4) \( \Rightarrow \) (5): Suppose that A is \( (\Lambda, \theta) \)-open and \( p(\Lambda, \theta) \)-closed. Then, we have \( A = A^{(\Lambda, \theta)} \) and 
\[ [A^{(\Lambda, \theta)}]_{(\Lambda, \theta)} \subseteq A. \] Therefore, 
\[ A = A^{(\Lambda, \theta)} \subseteq [A^{(\Lambda, \theta)}]_{(\Lambda, \theta)} \subseteq A_{(\Lambda, \theta)}. \]

This implies that 
\[ [A^{(\Lambda, \theta)}]^{(\Lambda, \theta)} = A_{(\Lambda, \theta)} = A \text{ and so } A = a(\Lambda, \theta) \)-open. Consequently, we obtain A is \( a(\Lambda, \theta) \)-open and \( p(\Lambda, \theta) \)-closed.

(5) \( \Rightarrow \) (6): Suppose that A is \( a(\Lambda, \theta) \)-open and \( p(\Lambda, \theta) \)-closed. Then, we have 
\[ A \subseteq [A^{(\Lambda, \theta)}]_{(\Lambda, \theta)} \]
and 
\[ [A^{(\Lambda, \theta)}]_{(\Lambda, \theta)} \subseteq A. \] This implies that 
\[ A = [A^{(\Lambda, \theta)}]^{(\Lambda, \theta)} \text{ and hence, } A^{(\Lambda, \theta)} = [A^{(\Lambda, \theta)}]^{(\Lambda, \theta)} \text{ and } [A^{(\Lambda, \theta)}]_{(\Lambda, \theta)} \subseteq A. \]

By Proposition 3.8(2), we have \( A^{(\Lambda, \theta)} = [A^{(\Lambda, \theta)}]^{(\Lambda, \theta)} \). Since 
\[ [A^{(\Lambda, \theta)}]^{(\Lambda, \theta)} \subseteq A, \]
\[ A^{(\Lambda, \theta)} \subseteq A \text{ and so } A^{(\Lambda, \theta)} = A. \]

Therefore, we obtain A is \( (\Lambda, \theta) \)-closed and \( a(\Lambda, \theta) \)-open.

(6) \( \Rightarrow \) (7): Suppose that A is \( a(\Lambda, \theta) \)-open and \( (\Lambda, \theta) \)-closed. Then, we have 
\[ A \subseteq [A^{(\Lambda, \theta)}]_{(\Lambda, \theta)} \]
and 
\[ A^{(\Lambda, \theta)} = A. \] By Proposition 3.8(1), 
\[ A \subseteq [A^{(\Lambda, \theta)}]_{(\Lambda, \theta)} \subseteq [A^{(\Lambda, \theta)}]^{(\Lambda, \theta)} = [A^{(\Lambda, \theta)}]_{(\Lambda, \theta)} \]
\[ = [A^{(\Lambda, \theta)}]_{(\Lambda, \theta)} \]
\[ = [A^{(\Lambda, \theta)}]_{(\Lambda, \theta)} = [A^{(\Lambda, \theta)}]_{(\Lambda, \theta)}. \]

This shows that A is \( p(\Lambda, \theta) \)-open. Hence, A is \( p(\Lambda, \theta) \)-open and \( (\Lambda, \theta) \)-closed.

(7) \( \Rightarrow \) (8): Suppose that A is \( p(\Lambda, \theta) \)-open and \( (\Lambda, \theta) \)-closed. Then, we have 
\[ A \subseteq [A^{(\Lambda, \theta)}]_{(\Lambda, \theta)} \]
and 
\[ A^{(\Lambda, \theta)} = A. \] Thus, 
\[ [A^{(\Lambda, \theta)}]_{(\Lambda, \theta)} \subseteq A^{(\Lambda, \theta)} = A. \]

This shows that A is \( p(\Lambda, \theta) \)-open and \( a(\Lambda, \theta) \)-closed.

(8) \( \Rightarrow \) (1): Suppose that A is \( p(\Lambda, \theta) \)-open and \( a(\Lambda, \theta) \)-closed. Then, we have 
\[ A \subseteq [A^{(\Lambda, \theta)}]_{(\Lambda, \theta)} \]
and 
\[ A^{(\Lambda, \theta)} \subseteq [A^{(\Lambda, \theta)}]^{(\Lambda, \theta)} \subseteq A. \] Therefore, we have 
\[ A^{(\Lambda, \theta)} \subseteq [A^{(\Lambda, \theta)}]^{(\Lambda, \theta)} \subseteq A \]
and hence, \( A^{(\Lambda, \theta)} \subseteq A \). Consequently, \( A = A^{(\Lambda, \theta)} \) and so A is \( (\Lambda, \theta) \)-closed. Since 
\[ [A^{(\Lambda, \theta)}]_{(\Lambda, \theta)} \subseteq A, \]
\[ [A^{(\Lambda, \theta)}]^{(\Lambda, \theta)} \subseteq A \text{ and so } A \subseteq A^{(\Lambda, \theta)}. \] This implies that 
\[ A = A_{(\Lambda, \theta)}. \] Therefore, A is \( (\Lambda, \theta) \)-open. Consequently, we obtain A is \( (\Lambda, \theta) \)-clopen.
Definition 3.13. A subset $A$ of a topological space $(X, \tau)$ is called $\alpha(\Lambda, \theta)$-$\star$-open (resp. $\beta(\Lambda, \theta)$-$\star$-open) if $A = [A(\Lambda, \theta)]^{(\alpha, \theta)}_{(\alpha, \theta)}$ (resp. $A = [A(\Lambda, \theta)]^{(\beta, \theta)}_{(\beta, \theta)}$).

Proposition 3.14. A subset $A$ of a topological space $(X, \tau)$ is $r(\Lambda, \theta)$-open if and only if $A$ is $\alpha(\Lambda, \theta)$-$\star$-open.

Proof. Suppose that $A$ is a $r(\Lambda, \theta)$-open set. Then, we have $A = [A(\Lambda, \theta)]^{(\alpha, \theta)}_{(\alpha, \theta)}$. This implies that $A$ is $(\Lambda, \theta)$-open and so $A = [A(\Lambda, \theta)]^{(\alpha, \theta)}_{(\alpha, \theta)}$. Consequently, we obtain $A$ is $\alpha(\Lambda, \theta)$-$\star$-open.

Conversely, suppose that $A$ is a $\alpha(\Lambda, \theta)$-$\star$-open set. Then $A = [A(\Lambda, \theta)]^{(\alpha, \theta)}_{(\alpha, \theta)}$. By Proposition 3.8(1),

$$[A(\Lambda, \theta)]^{(\alpha, \theta)}_{(\alpha, \theta)} = [[A(\Lambda, \theta)]^{(\alpha, \theta)}_{(\alpha, \theta)}]^{(\alpha, \theta)}_{(\alpha, \theta)} = [A(\Lambda, \theta)]^{(\alpha, \theta)}_{(\alpha, \theta)} = A.$$ 

Consequently, we obtain $A$ is $r(\Lambda, \theta)$-open.

Proposition 3.15. A subset $A$ of a topological space $(X, \tau)$ is $r(\Lambda, \theta)$-closed if and only if $A$ is $\beta(\Lambda, \theta)$-$\star$-open.

Proof. Suppose that $A$ is a $r(\Lambda, \theta)$-closed set. Then, we have $A = [A(\Lambda, \theta)]^{(\beta, \theta)}_{(\beta, \theta)}$ and so $A$ is $(\Lambda, \theta)$-closed. Therefore, we obtain $A = [A(\Lambda, \theta)]^{(\beta, \theta)}_{(\beta, \theta)} = [A(\Lambda, \theta)]^{(\beta, \theta)}_{(\beta, \theta)}$. This shows that $A$ is $\beta(\Lambda, \theta)$-$\star$-open.

Conversely, suppose that $A$ is a $\beta(\Lambda, \theta)$-$\star$-open set. Then, we have

$$A = [A(\Lambda, \theta)]^{(\beta, \theta)}_{(\beta, \theta)}.$$ 

and by Proposition 3.8(1),

$$[A(\Lambda, \theta)]^{(\beta, \theta)}_{(\beta, \theta)} = [[A(\Lambda, \theta)]^{(\beta, \theta)}_{(\beta, \theta)}]^{(\beta, \theta)}_{(\beta, \theta)} = [A(\Lambda, \theta)]^{(\beta, \theta)}_{(\beta, \theta)} = A.$$ 

Therefore, we obtain $A$ is $r(\Lambda, \theta)$-closed.

Proposition 3.16. For a subset $A$ of a topological space $(X, \tau)$, the following properties are equivalent:

1. $A$ is $\beta(\Lambda, \theta)$-$\star$-open.
2. $A$ is $\beta(\Lambda, \theta)$-open and $(\Lambda, \theta)$-closed.
3. $A$ is $\beta(\Lambda, \theta)$-open and $\alpha(\Lambda, \theta)$-closed.

Proposition 3.17. For a subset $A$ of a topological space $(X, \tau)$, the following properties are equivalent:

1. $A$ is $\alpha(\Lambda, \theta)$-$\star$-open.
2. $A$ is $(\Lambda, \theta)$-open and $\beta(\Lambda, \theta)$-closed.
3. $A$ is $\alpha(\Lambda, \theta)$-open and $\beta(\Lambda, \theta)$-closed.

Definition 3.18. A subset $A$ of a topological space $(X, \tau)$ is said to be $b(\Lambda, \theta)$-open if $A \subseteq [A(\Lambda, \theta)]^{(\alpha, \theta)}_{(\alpha, \theta)}$. The complement of a $b(\Lambda, \theta)$-open set is said to be $b(\Lambda, \theta)$-closed.

The family of all $b(\Lambda, \theta)$-open (resp. $b(\Lambda, \theta)$-closed) sets in a topological space $(X, \tau)$ is denoted by $b_{\Lambda}O(X, \tau)$ (resp. $b_{\Lambda}C(X, \tau)$).

Remark 3.19. It is easy to see that for a topological space $(X, \tau)$, 

$$s_{\Lambda}O(X, \tau) \cup p_{\Lambda}O(X, \tau) \subseteq b_{\Lambda}O(X, \tau) \subseteq b_{\Lambda}O(X, \tau).$$ 

Proposition 3.20. For a topological space $(X, \tau)$, if $A = B \cup C$, where $B$ is $s(\Lambda, \theta)$-open and $C$ is $p(\Lambda, \theta)$-open, then $A$ is $b(\Lambda, \theta)$-open.

Lemma 3.21. A subset $A$ of a topological space $(X, \tau)$ is $b(\Lambda, \theta)$-closed if and only if $[A(\Lambda, \theta)]^{(\alpha, \theta)}_{(\alpha, \theta)} \cap [A(\Lambda, \theta)]^{(\beta, \theta)}_{(\beta, \theta)} \subseteq A$. 

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Proof. Suppose that $A$ is a $b(\Lambda, \theta)$-closed set. Then $X - A$ is $b(\Lambda, \theta)$-open and so

\[
X - A \subseteq \{X - A\}^{(\Lambda, \theta)}_{(\Lambda, \theta)} \cup \{X - A\}^{(\Lambda, \theta)}_{(\Lambda, \theta)} = \{X - A\}^{(\Lambda, \theta)}_{(\Lambda, \theta)} \cup \{X - A\}^{(\Lambda, \theta)}_{(\Lambda, \theta)}.
\]

Therefore, we obtain $[A^{(\Lambda, \theta)}]^{(\Lambda, \theta)}_{(\Lambda, \theta)} \cap [A^{(\Lambda, \theta)}]^{(\Lambda, \theta)}_{(\Lambda, \theta)} \subseteq A$. Conversely, suppose that $[A^{(\Lambda, \theta)}]^{(\Lambda, \theta)}_{(\Lambda, \theta)} \cap [A^{(\Lambda, \theta)}]^{(\Lambda, \theta)}_{(\Lambda, \theta)} \subseteq A$. Then, we have

\[
X - A \subseteq \{X - A\}^{(\Lambda, \theta)}_{(\Lambda, \theta)} \cup \{X - A\}^{(\Lambda, \theta)}_{(\Lambda, \theta)} = \{X - A\}^{(\Lambda, \theta)}_{(\Lambda, \theta)} \cup \{X - A\}^{(\Lambda, \theta)}_{(\Lambda, \theta)}.
\]

This shows that $X - A$ is $b(\Lambda, \theta)$-open and so $A$ is $b(\Lambda, \theta)$-closed.

**Corollary 3.22.** For a subset $A$ of a topological space $(X, \tau)$, the following properties are equivalent:

1. $A$ is $r(\Lambda, \theta)$-open.
2. $A$ is $(\Lambda, \theta)$-open and $b(\Lambda, \theta)$-closed.
3. $A$ is $a(\Lambda, \theta)$-open and $b(\Lambda, \theta)$-closed.

Proof. (1) $\Rightarrow$ (2): Suppose that $A$ is a $r(\Lambda, \theta)$-open set. Then, we have $A = [A^{(\Lambda, \theta)}]^{(\Lambda, \theta)}_{(\Lambda, \theta)}$. Therefore, $[A^{(\Lambda, \theta)}]^{(\Lambda, \theta)}_{(\Lambda, \theta)} \cap [A^{(\Lambda, \theta)}]^{(\Lambda, \theta)}_{(\Lambda, \theta)} = [A^{(\Lambda, \theta)}]^{(\Lambda, \theta)}_{(\Lambda, \theta)} \cap A \subseteq A$ and by Lemma 3.21, $A$ is $b(\Lambda, \theta)$-closed. Consequently, we obtain $A$ is $(\Lambda, \theta)$-open and $b(\Lambda, \theta)$-closed.

(2) $\Rightarrow$ (3): This is obvious since every $(\Lambda, \theta)$-open set is $a(\Lambda, \theta)$-open.

(3) $\Rightarrow$ (1): Suppose that $A$ is $a(\Lambda, \theta)$-open and $b(\Lambda, \theta)$-closed. Then, we have $[A^{(\Lambda, \theta)}]^{(\Lambda, \theta)}_{(\Lambda, \theta)} \cap [A^{(\Lambda, \theta)}]^{(\Lambda, \theta)}_{(\Lambda, \theta)} \subseteq A$ and $A \subseteq [A^{(\Lambda, \theta)}]^{(\Lambda, \theta)}_{(\Lambda, \theta)}$. Therefore, we obtain $[A^{(\Lambda, \theta)}]^{(\Lambda, \theta)}_{(\Lambda, \theta)} \subseteq [A^{(\Lambda, \theta)}]^{(\Lambda, \theta)}_{(\Lambda, \theta)}$ and hence,

\[
[A^{(\Lambda, \theta)}]^{(\Lambda, \theta)}_{(\Lambda, \theta)} \cap [A^{(\Lambda, \theta)}]^{(\Lambda, \theta)}_{(\Lambda, \theta)} = [A^{(\Lambda, \theta)}]^{(\Lambda, \theta)}_{(\Lambda, \theta)} \cap [A^{(\Lambda, \theta)}]^{(\Lambda, \theta)}_{(\Lambda, \theta)} = [A^{(\Lambda, \theta)}]^{(\Lambda, \theta)}_{(\Lambda, \theta)} \cap [A^{(\Lambda, \theta)}]^{(\Lambda, \theta)}_{(\Lambda, \theta)}
\]

This implies that $A = [A^{(\Lambda, \theta)}]^{(\Lambda, \theta)}_{(\Lambda, \theta)} = [A^{(\Lambda, \theta)}]^{(\Lambda, \theta)}_{(\Lambda, \theta)} \cap [A^{(\Lambda, \theta)}]^{(\Lambda, \theta)}$. By Proposition 3.8(2), we have $A^{(\Lambda, \theta)} = [A^{(\Lambda, \theta)}]^{(\Lambda, \theta)}_{(\Lambda, \theta)}$. Hence, $A = [A^{(\Lambda, \theta)}]^{(\Lambda, \theta)}_{(\Lambda, \theta)} \subseteq A^{(\Lambda, \theta)}$. Consequently, $A$ is $r(\Lambda, \theta)$-open.

**Lemma 3.23.** Let $A$ be a subset of a topological space $(X, \tau)$. If $A$ is both $s(\Lambda, \theta)$-closed and $\beta(\Lambda, \theta)$-open, then $A$ is $s(\Lambda, \theta)$-open.

Proof. Suppose that $A$ is both $s(\Lambda, \theta)$-closed and $\beta(\Lambda, \theta)$-open. Since $A$ is $s(\Lambda, \theta)$-closed, it follows from Proposition 3.7(1) that $[A^{(\Lambda, \theta)}]^{(\Lambda, \theta)}_{(\Lambda, \theta)} \subseteq A$. Since $A$ is $\beta(\Lambda, \theta)$-open, $[A^{(\Lambda, \theta)}]^{(\Lambda, \theta)}_{(\Lambda, \theta)} \subseteq A \subseteq [A^{(\Lambda, \theta)}]^{(\Lambda, \theta)}_{(\Lambda, \theta)}$ and so $[A^{(\Lambda, \theta)}]^{(\Lambda, \theta)}_{(\Lambda, \theta)} \subseteq A$. Therefore, we obtain $[A^{(\Lambda, \theta)}]^{(\Lambda, \theta)}_{(\Lambda, \theta)} \subseteq A$. This shows that $A$ is $s(\Lambda, \theta)$-open.

**Proposition 3.24.** Let $A$ be a subset of a topological space $(X, \tau)$. If $A$ is $b(\Lambda, \theta)$-open, then $A^{(\Lambda, \theta)}$ is $r(\Lambda, \theta)$-closed.

Proof. Suppose that $A$ is a $b(\Lambda, \theta)$-open set. Since $A$ is $b(\Lambda, \theta)$-open, we have

\[
A \subseteq [A^{(\Lambda, \theta)}]^{(\Lambda, \theta)}_{(\Lambda, \theta)} \cup [A^{(\Lambda, \theta)}]^{(\Lambda, \theta)}_{(\Lambda, \theta)} \subseteq [A^{(\Lambda, \theta)}]^{(\Lambda, \theta)}_{(\Lambda, \theta)}
\]

Therefore, we obtain $A^{(\Lambda, \theta)} \subseteq [A^{(\Lambda, \theta)}]^{(\Lambda, \theta)}_{(\Lambda, \theta)} \subseteq A^{(\Lambda, \theta)}$ and so

\[
A^{(\Lambda, \theta)} = [A^{(\Lambda, \theta)}]^{(\Lambda, \theta)}_{(\Lambda, \theta)}
\]

Hence, $A^{(\Lambda, \theta)}$ is $r(\Lambda, \theta)$-closed.
Corollary 3.25. For a subset $A$ of a topological space $(X, r)$, the following properties hold:

1. If $A$ is $s(\Lambda, \theta)$-open, then $A^{(\Lambda, \theta)}$ is $r(\Lambda, \theta)$-closed.
2. If $A$ is $p(\Lambda, \theta)$-open, then $A^{(\Lambda, \theta)}$ is $r(\Lambda, \theta)$-closed.
3. If $A$ is $a(\Lambda, \theta)$-open, then $A^{(\Lambda, \theta)}$ is $r(\Lambda, \theta)$-closed.

Proposition 3.26. For a subset $A$ of a topological space $(X, r)$, the following properties are equivalent:

1. $A \in \beta \Lambda \cup O(X, r)$.
2. $A^{(\Lambda, \theta)} \in r \Lambda \cup C(X, r)$.
3. $A^{(\Lambda, \theta)} \in \beta \Lambda \cup O(X, r)$.
4. $A^{(\Lambda, \theta)} \in s \Lambda \cup O(X, r)$.
5. $A^{(\Lambda, \theta)} \in b \Lambda \cup O(X, r)$.

**Proof.** (1) $\Rightarrow$ (2): Suppose that $A \in \beta \Lambda \cup O(X, r)$. Then, we have

$A \subseteq \{A^{(\Lambda, \theta)}\}_{\theta}^{(\Lambda, \theta)}$

and hence, $A^{(\Lambda, \theta)} \subseteq \{A^{(\Lambda, \theta)}\}_{\theta}^{(\Lambda, \theta)} \subseteq A^{(\Lambda, \theta)}$. This implies that $A^{(\Lambda, \theta)} = \{A^{(\Lambda, \theta)}\}_{\theta}^{(\Lambda, \theta)}$.

Therefore, we obtain $A^{(\Lambda, \theta)} \in r \Lambda \cup C(X, r)$.

(2) $\Rightarrow$ (3): Suppose that $A^{(\Lambda, \theta)} \in r \Lambda \cup C(X, r)$. Then, we have

$A^{(\Lambda, \theta)} = \{A^{(\Lambda, \theta)}\}_{\theta}^{(\Lambda, \theta)}$

and so $A^{(\Lambda, \theta)} = \{A^{(\Lambda, \theta)}\}_{\theta}^{(\Lambda, \theta)} = \{A^{(\Lambda, \theta)}\}_{\theta}^{(\Lambda, \theta)}$. Therefore, we obtain $A^{(\Lambda, \theta)} \in \beta \Lambda \cup O(X, r)$.

(3) $\Rightarrow$ (4): Suppose that $A^{(\Lambda, \theta)} \in \beta \Lambda \cup O(X, r)$. Then, we have

$A^{(\Lambda, \theta)} \subseteq \{A^{(\Lambda, \theta)}\}_{\theta}^{(\Lambda, \theta)}$

and so $A^{(\Lambda, \theta)} \subseteq \{A^{(\Lambda, \theta)}\}_{\theta}^{(\Lambda, \theta)}$. Consequently, we obtain $A^{(\Lambda, \theta)} \in s \Lambda \cup O(X, r)$.

(4) $\Rightarrow$ (5): Follows from Remark 3.19.

(5) $\Rightarrow$ (1): Suppose that $A^{(\Lambda, \theta)} \in b \Lambda \cup O(X, r)$. Then, we have

$A \subseteq A^{(\Lambda, \theta)} \subseteq \{A^{(\Lambda, \theta)}\}_{\theta}^{(\Lambda, \theta)} \cup \{A^{(\Lambda, \theta)}\}_{\theta}^{(\Lambda, \theta)} = \{A^{(\Lambda, \theta)}\}_{\theta}^{(\Lambda, \theta)} \cup \{A^{(\Lambda, \theta)}\}_{\theta}^{(\Lambda, \theta)} = \{A^{(\Lambda, \theta)}\}_{\theta}^{(\Lambda, \theta)}$. Consequently, we obtain $A \in \beta \Lambda \cup O(X, r)$.

Corollary 3.27. For a subset $A$ of a topological space $(X, r)$, the following properties are equivalent:

1. $A \in \beta \Lambda \cup C(X, r)$.
2. $A_{\theta}^{(\Lambda, \theta)} \in r \Lambda \cup O(X, r)$.
3. $A_{\theta}^{(\Lambda, \theta)} \in \beta \Lambda \cup C(X, r)$.
4. $A_{\theta}^{(\Lambda, \theta)} \in s \Lambda \cup C(X, r)$.
5. $A_{\theta}^{(\Lambda, \theta)} \in b \Lambda \cup C(X, r)$.

Definition 3.28. A subset $A$ of a topological space $(X, r)$ is called $r(\Lambda, \theta)$-open if there exists a $r(\Lambda, \theta)$-open set $U$ such that $U \subseteq A \subseteq U^{(\Lambda, \theta)}$. The complement of a $r(\Lambda, \theta)$-open set is called $r(\Lambda, \theta)$-closed.

The family of all $r(\Lambda, \theta)$-open (resp. $r(\Lambda, \theta)$-closed) sets in a topological space $(X, r)$ is denoted by $r\Lambda \cup O(X, r)$ (resp. $r\Lambda \cup C(X, r)$).
Proposition 3.29. Let $A$ be a subset of a topological space $(X, \tau)$. If $A$ is $rs(\Lambda, \theta)$-open, then $A \subseteq [A(\Lambda, \theta)|^{(\Lambda, \theta)}].$

Proof. Suppose that $A$ is a $rs(\Lambda, \theta)$-open set. Then, there exists a $r(\Lambda, \theta)$-open set $U$ such that $U \subseteq A \subseteq U^{(\Lambda, \theta)}$. Therefore, we obtain $U \subseteq A^{(\Lambda, \theta)}$ and so

$$A \subseteq [A^{(\Lambda, \theta)}]^{(\Lambda, \theta)}.$$ 

Proposition 3.30. For a subset $A$ of a topological space $(X, \tau)$, the following properties are equivalent:

1. $A$ is $rs(\Lambda, \theta)$-open.
2. $A$ is $s(\Lambda, \theta)$-open and $s(\Lambda, \theta)$-closed.
3. $A$ is $b(\Lambda, \theta)$-open and $s(\Lambda, \theta)$-closed.
4. $A$ is $\beta(\Lambda, \theta)$-open and $s(\Lambda, \theta)$-closed.
5. $A$ is $s(\Lambda, \theta)$-open and $b(\Lambda, \theta)$-closed.
6. $A$ is $s(\Lambda, \theta)$-open and $\beta(\Lambda, \theta)$-closed.

Proof. (1) $\Rightarrow$ (2): Suppose that $A$ is a $rs(\Lambda, \theta)$-open set. Then, there exist a $r(\Lambda, \theta)$-open set $V$ such that $V \subseteq A \subseteq V^{(\Lambda, \theta)}$. Therefore, we obtain $V \subseteq A^{(\Lambda, \theta)}$ and hence, $A \subseteq (\Lambda, \theta)]^{(\Lambda, \theta)}$. This shows that $A$ is $s(\Lambda, \theta)$-open. Since $V^{(\Lambda, \theta)} = A^{(\Lambda, \theta)}$ and $V$ is $r(\Lambda, \theta)$-open, $[A^{(\Lambda, \theta)}]^{(\Lambda, \theta)} = V^{(\Lambda, \theta)} = V \subseteq A$. Thus, by Proposition 3.7(1), $A$ is $s(\Lambda, \theta)$-closed. Consequently, we obtain $A$ is $s(\Lambda, \theta)$-open and $s(\Lambda, \theta)$-closed.

(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4): This is obvious since $s_A \subseteq O(X, \tau) \subseteq b_A \subseteq \beta_A(X, \tau)$.

(4) $\Rightarrow$ (5): Follows from Lemma 3.23 and hence, $s_A \subseteq O(X, \tau) \subseteq b_A \subseteq \beta_A(X, \tau)$.

(5) $\Rightarrow$ (6): This is obvious since $b_A \subseteq \beta_A(X, \tau)$.

(6) $\Rightarrow$ (1): Suppose that $A$ is $s(\Lambda, \theta)$-open and $\beta(\Lambda, \theta)$-closed. Since $A$ is $s(\Lambda, \theta)$-open and $\beta(\Lambda, \theta)$-closed, it follows from Lemma 3.23 that $A$ is $s(\Lambda, \theta)$-closed. Thus, by Proposition 3.7(1), $A^{(\Lambda, \theta)} = A \subseteq A^{(\Lambda, \theta)} \subseteq [A^{(\Lambda, \theta)}]^{(\Lambda, \theta)}$. Put $V = [A^{(\Lambda, \theta)}]^{(\Lambda, \theta)}$, then $V$ is a $r(\Lambda, \theta)$-open set such that $V \subseteq A \subseteq V^{(\Lambda, \theta)}$. Therefore, we obtain $A$ is $rs(\Lambda, \theta)$-open.

Proposition 3.31. Let $(X, \tau)$ be a topological space and $x \in X$. Then $\{x\}$ is ($\Lambda, \theta$)-open if and only if $\{x\}$ is $s(\Lambda, \theta)$-open.

Proof. The necessity is clear. Suppose that $\{x\}$ is $s(\Lambda, \theta)$-open. Then, we have $\{x\} \subseteq [\{x\}]^{(\Lambda, \theta)}$. Now, $\{x\}^{(\Lambda, \theta)}$ is either $\{x\}$ or $\emptyset$. Since $\emptyset^{(\Lambda, \theta)} = \emptyset$ and $\{x\} \subseteq [\{x\}]^{(\Lambda, \theta)}$, $\{x\}^{(\Lambda, \theta)} \neq \emptyset$. Therefore, we obtain $\{x\}^{(\Lambda, \theta)} = \{x\}$. This shows that $\{x\}$ is ($\Lambda, \theta$)-open.

Lemma 3.32. Let $A$ be a subset of a topological space $(X, \tau)$ and $U \in \Lambda \cup O(X, \tau)$. If $U \cap A = \emptyset$, then $A \cap U^{(\Lambda, \theta)} = \emptyset$.

Proposition 3.33. Let $(X, \tau)$ be a topological space and $x \in X$. Then, the following properties are equivalent:

1. $\{x\}$ is $p(\Lambda, \theta)$-open.
2. $\{x\}$ is $b(\Lambda, \theta)$-open.
3. $\{x\}$ is $\beta(\Lambda, \theta)$-open.

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) follows from Remark 3.19.

(3) $\Rightarrow$ (1): Let $\{x\}$ be $\beta(\Lambda, \theta)$-open. Suppose that $\{x\}$ is not $p(\Lambda, \theta)$-open. Then, we have $\{x\} \not\subseteq [\{x\}]^{(\Lambda, \theta)}$ and so $\{x\} \cap [\{x\}]^{(\Lambda, \theta)} = \emptyset$. Since $[\{x\}]^{(\Lambda, \theta)}$ is ($\Lambda, \theta$)-open, it follows from
Lemma 3.32 that \( \{x\}^{(\Lambda^\theta)} \cap \{\{x\}^{(\Lambda^\theta)}\}_{x} = \emptyset \) and hence, \( \{\{x\}^{(\Lambda^\theta)}\}_{x} = \emptyset \). Therefore, we obtain \( \{\{x\}^{(\Lambda^\theta)}\}_{x} = 0 \). This is a contradiction.

**Proposition 3.34.** Let \( (X, \tau) \) be a topological space and \( x \in X \). Then \( \{x\} \) is \( p(\Lambda, \theta) \)-open or \( a(\Lambda, \theta) \)-closed.

**Proof.** Suppose that \( \{x\} \) is not \( p(\Lambda, \theta) \)-open. Then, we have \( \{x\} \not\subseteq \{\{x\}^{(\Lambda^\theta)}\}_{x} \) and so \( \{x\} \cap \{\{x\}^{(\Lambda^\theta)}\}_{x} = \emptyset \). Since \( \{\{x\}^{(\Lambda^\theta)}\}_{x} = (\Lambda, \theta) \)-open, it follows from Lemma 3.32 that \( \{x\}^{(\Lambda^\theta)} \cap \{\{x\}^{(\Lambda^\theta)}\}_{x} = \emptyset \) and hence, \( \{\{x\}^{(\Lambda^\theta)}\}_{x} = \emptyset \). Therefore, we obtain, \( \{\{x\}^{(\Lambda^\theta)}\}_{x} = 0 \). By Proposition 3.7(3), \( \{x\} \) is \( a(\Lambda, \theta) \)-closed.

**Proposition 3.35.** A subset \( A \) of a topological space \( (X, \tau) \) is \( s(\Lambda, \theta) \)-open if and only if there exists a \( (\Lambda, \theta) \)-open set \( U \) such that \( U \subseteq A \subseteq U^{(\Lambda^\theta)} \).

**Proof.** Suppose that \( A \) is \( s(\Lambda, \theta) \)-open. Then, we have \( A \subseteq [A]^{(\Lambda^\theta)} \). Put \( U = [A]^{(\Lambda^\theta)} \), then \( U \) is a \( (\Lambda, \theta) \)-open set such that \( U \subseteq A \subseteq U^{(\Lambda^\theta)} \).

Conversely, suppose that there exists \( a(\Lambda, \theta) \)-open set \( U \) such that \( U \subseteq A \subseteq U^{(\Lambda^\theta)} \). Then \( U \subseteq A^{(\Lambda^\theta)} \) and thus \( U^{(\Lambda^\theta)} \subseteq [A]^{(\Lambda^\theta)} \), but \( A \subseteq U^{(\Lambda^\theta)} \), so \( A \subseteq [A]^{(\Lambda^\theta)} \). Consequently, we obtain \( A \) is \( s(\Lambda, \theta) \)-open.

**Proposition 3.36.** Let \( A \) be a subset of a topological space \( (X, \tau) \). If there exists a \( p(\Lambda, \theta) \)-open set \( U \) such that \( U \subseteq A \subseteq U^{(\Lambda^\theta)} \), then \( A \) is \( \beta(\Lambda, \theta) \)-open.

**Proof.** Since \( U \subseteq A \subseteq U^{(\Lambda^\theta)} \), it follows that \( A^{(\Lambda^\theta)} = U^{(\Lambda^\theta)} \) and \( [A]^{(\Lambda^\theta)} = [U^{(\Lambda^\theta)}] \), but \( U \) is \( p(\Lambda, \theta) \)-open, so \( U \subseteq [A]^{(\Lambda^\theta)} \) and \( A \subseteq U^{(\Lambda^\theta)} \). Therefore, we obtain \( A \subseteq [A]^{(\Lambda^\theta)} \). This shows that \( A \) is \( \beta(\Lambda, \theta) \)-open.

**Definition 3.37.** A subset \( D \) of a topological space \( (X, \tau) \) is called \( \Lambda^\theta \)-dense if \( D^{(\Lambda^\theta)} = X \). \( D \) is called \( \Lambda^\theta \)-codense if \( X - D \) is \( \Lambda^\theta \)-dense.

**Proposition 3.38.** For a subset \( D \) of a topological space \( (X, \tau) \), the following properties are equivalent:

1. \( D \) is \( \Lambda^\theta \)-dense.
2. If \( F \) is any \( (\Lambda, \theta) \)-closed set and \( D \subseteq F \), then \( F = X \).
3. Each non-empty \( (\Lambda, \theta) \)-open set contains an element of \( D \).
4. The complement of \( D \) has empty \( (\Lambda, \theta) \)-interior.

**Proof.** (1) \( \Rightarrow \) (2): Let \( F \) be a \( (\Lambda, \theta) \)-closed set such that \( D \subseteq F \). Then \( X = D^{(\Lambda^\theta)} \subseteq F^{(\Lambda^\theta)} \).

(2) \( \Rightarrow \) (3): Let \( U \) be non-empty \( (\Lambda, \theta) \)-open set such that \( U \cap D = \emptyset \); then \( D \subseteq X - U \neq \emptyset \), which contradicts (2), since \( X - U \) is \( (\Lambda, \theta) \)-closed.

(3) \( \Rightarrow \) (4): Suppose that \( |X - D|^{(\Lambda^\theta)} \neq \emptyset \); since \( |X - D|^{(\Lambda^\theta)} \) is a \( (\Lambda, \theta) \)-open set such that \( |X - D|^{(\Lambda^\theta)} \subseteq X - D \), we have \( |X - D|^{(\Lambda^\theta)} \) contains no point of \( D \).

(4) \( \Rightarrow \) (1): \( |X - D|^{(\Lambda^\theta)} = X - D^{(\Lambda^\theta)} = \emptyset \) so that \( D^{(\Lambda^\theta)} = X \).

**Remark 3.39.** Let \( A \) be a subset of a topological space \( (X, \tau) \). If \( A \) is \( \Lambda^\theta \)-dense, then \( A \) is \( p(\Lambda, \theta) \)-open.

**Proposition 3.40.** Let \( A \) be a subset of a topological space \( (X, \tau) \). If \( A \) is \( p(\Lambda, \theta) \)-open, then \( A \) is the intersection of a \( r(\Lambda, \theta) \)-open set and a \( \Lambda^\theta \)-dense set.
Proof. Suppose that $A$ is $p(\Lambda, \theta)$-open. Then, we have $A \subseteq A^{(\Lambda, \theta)}_{(\Lambda, \theta)}$ and so $A = [A \cup X - A^{(\Lambda, \theta)}_{(\Lambda, \theta)}] \cap A^{(\Lambda, \theta)}_{(\Lambda, \theta)}$. Let $C = A^{(\Lambda, \theta)}_{(\Lambda, \theta)}$ and $D = A \cup X - A^{(\Lambda, \theta)}_{(\Lambda, \theta)}$. Then $C$ is $r(\Lambda, \theta)$-open by Proposition 3.8(1), also $A^{(\Lambda, \theta)} \subseteq D^{(\Lambda, \theta)}$ since $A \subseteq D$ and $X - A^{(\Lambda, \theta)} \subseteq D^{(\Lambda, \theta)}$. Thus, $D^{(\Lambda, \theta)} = X$. 

Corollary 3.41. Let $A$ be a subset of a topological space $(X, r)$. If $A$ is $p(\Lambda, \theta)$-closed, then $A$ is the union of a $r(\Lambda, \theta)$-closed set and a set has empty $(\Lambda, \theta)$-interior.

Proposition 3.42. Let $A$ be a subset of a topological space $(X, r)$. If $A$ is $s(\Lambda, \theta)$-open, then $A$ is the intersection of a $r(\Lambda, \theta)$-closed set and a set whose $(\Lambda, \theta)$-interior is $\Lambda^{\theta}$-dense.

Proof. Suppose that $A$ is $s(\Lambda, \theta)$-open, that is $A \subseteq A^{(\Lambda, \theta)}_{(\Lambda, \theta)}$. Then, we have $A = [A \cup X - A^{(\Lambda, \theta)}_{(\Lambda, \theta)}] \cap A^{(\Lambda, \theta)}_{(\Lambda, \theta)}$. Let $F = A^{(\Lambda, \theta)}_{(\Lambda, \theta)}$ and $C = A \cup X - A^{(\Lambda, \theta)}_{(\Lambda, \theta)}$.

Then $F$ is $r(\Lambda, \theta)$-closed by Proposition 3.8(2), also $A^{(\Lambda, \theta)}_{(\Lambda, \theta)} \subseteq C^{(\Lambda, \theta)}_{(\Lambda, \theta)}$ since $A \subseteq C$ but $X - A^{(\Lambda, \theta)}_{(\Lambda, \theta)} \subseteq C$ and $X - A^{(\Lambda, \theta)}_{(\Lambda, \theta)}$ is $(\Lambda, \theta)$-open, so $X - A^{(\Lambda, \theta)}_{(\Lambda, \theta)} \subseteq C_{(\Lambda, \theta)} \subseteq [C^{(\Lambda, \theta)}_{(\Lambda, \theta)}]^{(\Lambda, \theta)}$. Consequently, $C^{(\Lambda, \theta)}_{(\Lambda, \theta)} = X$.

Corollary 3.43. Let $A$ be a subset of a topological space $(X, r)$. If $A$ is $s(\Lambda, \theta)$-closed, then $A$ is the union of a $r(\Lambda, \theta)$-open set and a set whose $(\Lambda, \theta)$-closure has empty $(\Lambda, \theta)$-interior.

Proposition 3.44. Let $A$ be a subset of a topological space $(X, r)$. If $A$ is $b(\Lambda, \theta)$-open, then $A$ is the intersection of a $r(\Lambda, \theta)$-closed set and a $\Lambda^{\theta}$-dense set.

Proof. Suppose that $A$ is $b(\Lambda, \theta)$-open. Then, we have $A \subseteq [A^{(\Lambda, \theta)}_{(\Lambda, \theta)}]^{(\Lambda, \theta)}$ and so $A = [A \cup X - A^{(\Lambda, \theta)}_{(\Lambda, \theta)}] \cap [A^{(\Lambda, \theta)}_{(\Lambda, \theta)}]^{(\Lambda, \theta)}$. Let $F = [A^{(\Lambda, \theta)}_{(\Lambda, \theta)}]^{(\Lambda, \theta)}$ and $D = A \cup X - A^{(\Lambda, \theta)}_{(\Lambda, \theta)}$. Then $F$ is $r(\Lambda, \theta)$-closed by Proposition 3.8(2), also $A^{(\Lambda, \theta)} \subseteq D^{(\Lambda, \theta)}$. Since $X - A^{(\Lambda, \theta)} \subseteq D \subseteq D^{(\Lambda, \theta)}$, we have $D^{(\Lambda, \theta)} = X$.

Corollary 3.45. Let $A$ be a subset of a topological space $(X, r)$. If $A$ is $b(\Lambda, \theta)$-closed, then $A$ is the union of a $r(\Lambda, \theta)$-open set and a set has empty $(\Lambda, \theta)$-interior.

Theorem 3.46. For a topological space $(X, r)$, the following properties are equivalent:

1. Every $s(\Lambda, \theta)$-open set is $a(\Lambda, \theta)$-open.
2. Every $s(\Lambda, \theta)$-open set is $p(\Lambda, \theta)$-open.
3. Every $b(\Lambda, \theta)$-open set is $p(\Lambda, \theta)$-open.
4. Every $b(\Lambda, \theta)$-open set is $p(\Lambda, \theta)$-open.
5. Every $r(\Lambda, \theta)$-open set is $p(\Lambda, \theta)$-open.
6. Every $r(\Lambda, \theta)$-open set is $p(\Lambda, \theta)$-open.
7. Every $r(\Lambda, \theta)$-closed set is $p(\Lambda, \theta)$-open.
8. Every $r(\Lambda, \theta)$-closed set is $(\Lambda, \theta)$-open.

Proof. (1) $\Rightarrow$ (2): Let $A$ be a $s(\Lambda, \theta)$-open set. By (1), we have $A$ is $a(\Lambda, \theta)$-open and so $A \subseteq [A^{(\Lambda, \theta)}_{(\Lambda, \theta)}]^{(\Lambda, \theta)} \subseteq A^{(\Lambda, \theta)}_{(\Lambda, \theta)}$. Consequently, we obtain $A$ is $p(\Lambda, \theta)$-open.

(2) $\Rightarrow$ (3): Let $A$ be a $b(\Lambda, \theta)$-open set. Then, we have $A \subseteq [A^{(\Lambda, \theta)}_{(\Lambda, \theta)}]^{(\Lambda, \theta)}$. It follows from Proposition 3.5(2) that $B = [A^{(\Lambda, \theta)}_{(\Lambda, \theta)}]^{(\Lambda, \theta)}$ is $r(\Lambda, \theta)$-closed and so $B$ is $s(\Lambda, \theta)$-open. By (2), $B$ is $p(\Lambda, \theta)$-open and hence, $A \subseteq B \subseteq B^{(\Lambda, \theta)}_{(\Lambda, \theta)} = B_{(\Lambda, \theta)}$. Also, it is clear that $B \subseteq A^{(\Lambda, \theta)}$ and thus, $B_{(\Lambda, \theta)} \subseteq A^{(\Lambda, \theta)}_{(\Lambda, \theta)}$. Therefore, $A \subseteq A^{(\Lambda, \theta)}_{(\Lambda, \theta)}$. This shows that $A$ is $p(\Lambda, \theta)$-open.

(3) $\Rightarrow$ (4): Let $A$ be a $b(\Lambda, \theta)$-open set. By (3) and Remark 3.19, we obtain $A$ is $p(\Lambda, \theta)$-open.
(4) \Rightarrow (5): It follows from Proposition 3.30 that \( r_s\Lambda_\theta O(X, \tau) \subseteq s\Lambda_\theta O(X, \tau) \), but \( s\Lambda_\theta O(X, \tau) \subseteq b\Lambda_\theta O(X, \tau) \), thus \( r_s\Lambda_\theta O(X, \tau) \subseteq b\Lambda_\theta O(X, \tau) \). Therefore, the result follows from (4).

(5) \Rightarrow (6): Since every \( r_s(\Lambda, \theta) \)-open set is \( s(\Lambda, \theta) \)-closed, it follows from (5) that a \( rs(\Lambda, \theta) \)-open set is both \( s(\Lambda, \theta) \)-closed and \( p(\Lambda, \theta) \)-open. Thus by Proposition 3.9(4), (6) follows.

(6) \Rightarrow (7): Suppose that every \( rs(\Lambda, \theta) \)-open set is \( r(\Lambda, \theta) \)-open. Let \( A \) be a \( r(\Lambda, \theta) \)-closed set. Then, we have \( A = [A(\Lambda, \theta)](\Lambda, \theta) \). Put \( V = [A(\Lambda, \theta)](\Lambda, \theta) \), then \( V \) is a \( r(\Lambda, \theta) \)-open set such that \( V \subseteq A \subseteq V(\Lambda, \theta) \). This shows that \( A \) is \( rs(\Lambda, \theta) \)-open. By (6), we obtain \( A \) is \( r(\Lambda, \theta) \)-open and so \( A \) is \( p(\Lambda, \theta) \)-open.

(7) \Rightarrow (8): Suppose that every \( r(\Lambda, \theta) \)-closed set is \( p(\Lambda, \theta) \)-open. Let \( A \) be a \( r(\Lambda, \theta) \)-closed set. By (7), we have \( A \) is \( p(\Lambda, \theta) \)-open. Therefore,

\[ A \subseteq [A(\Lambda, \theta)](\Lambda, \theta) = [A(\Lambda, \theta)](\Lambda, \theta) = A(\Lambda, \theta). \]

Consequently, we obtain \( A \) is \( (\Lambda, \theta) \)-open.

(8) \Rightarrow (1): Let \( A \) be a \( s(\Lambda, \theta) \)-open set. Then by Corollary 3.25, \( A(\Lambda, \theta) \) is \( r(\Lambda, \theta) \)-closed. Thus by (8), \( A(\Lambda, \theta) \) is \( (\Lambda, \theta) \)-open and so \( A \subseteq A(\Lambda, \theta) = [A(\Lambda, \theta)](\Lambda, \theta) \). Consequently, \( A \) is \( p(\Lambda, \theta) \)-open. Since \( A \in s\Lambda_\theta O(X, \tau) \cap p\Lambda_\theta O(X, \tau) = a\Lambda_\theta O(X, \tau) \), (1) follows.

**Remark 3.47.** It is clear from Proposition 3.30 that if \( A \) is a \( rs(\Lambda, \theta) \)-open set of a topological space \((X, \tau)\), then \( X - A \) is \( rs(\Lambda, \theta) \)-open.

**Corollary 3.48.** For a topological space \((X, \tau)\), the following properties are equivalent:

1. \( \alpha\Lambda_\theta O(X, \tau) = s\Lambda_\theta O(X, \tau) \).
2. Every \( rs(\Lambda, \theta) \)-open set of \( X \) is \( p(\Lambda, \theta) \)-closed.
3. Every \( rs(\Lambda, \theta) \)-open set of \( X \) is \( r(\Lambda, \theta) \)-closed.

**Proof.** Follows from Theorem 3.46 and Remark 3.47.

**Definition 3.49.** Let \((X, \tau)\) be a topological space. A subset \( A \) of \( X \) is called \( p(\Lambda, \theta) \)-clopen if \( A \) is both \( p(\Lambda, \theta) \)-open and \( p(\Lambda, \theta) \)-closed.

**Corollary 3.50.** For a topological space \((X, \tau)\), the following properties are equivalent:

1. \( \alpha\Lambda_\theta O(X, \tau) = s\Lambda_\theta O(X, \tau) \).
2. Every \( rs(\Lambda, \theta) \)-open set of \( X \) is \( p(\Lambda, \theta) \)-clopen.
3. Every \( rs(\Lambda, \theta) \)-open set of \( X \) is \( (\Lambda, \theta) \)-clopen.

**Proof.** Follows from Theorem 3.46 and Corollary 3.48.

**Proposition 3.51.** For a topological space \((x, \tau)\), the following properties are equivalent:

1. Every \( \alpha(\Lambda, \theta) \)-open subset of \( X \) is \( p(\Lambda, \theta) \)-open.
2. Every \( \alpha(\Lambda, \theta) \)-open subset of \( X \) is \( s(\Lambda, \theta) \)-open.

**Proof.** Follows from Proposition 3.3(3).

**Definition 3.52.** A topological space \((X, \tau)\) is said to be \( \Lambda_\theta \)-submaximal if each \( \Lambda_\theta \)-dense subset of \( X \) is \( (\Lambda, \theta) \)-open.

**Proposition 3.53.** Let \((X, \tau)\) be a topological space. If each \( p(\Lambda, \theta) \)-open set is \( s(\Lambda, \theta) \)-open and each \( \alpha(\Lambda, \theta) \)-open set is \( (\Lambda, \theta) \)-open, then \((X, \tau)\) is \( \Lambda_\theta \)-submaximal.

**Proof.** Let \( D \) be a \( \Lambda_\theta \)-dense subset of \( X \). Since \( D(\Lambda, \theta) = X \), then \( D \) is a \( p(\Lambda, \theta) \)-open set. This implies that \( D \) is a \( s(\Lambda, \theta) \)-open set. Since any set is \( \alpha(\Lambda, \theta) \)-open if and only if it is \( s(\Lambda, \theta) \)-open and
$p(\Lambda, \theta)$-open, then $D$ is a $\alpha(\Lambda, \theta)$-open set. Hence, since each $\alpha(\Lambda, \theta)$-open set is $(\Lambda, \theta)$-open, we have $D$ is $(\Lambda, \theta)$-open. Thus, $(X, \tau)$ is $\Lambda_\theta$-submaximal.

**Proposition 3.54.** Let $(X, \tau)$ be a topological space. If each $p(\Lambda, \theta)$-open set is $(\Lambda, \theta)$-open, then $(X, \tau)$ is $\Lambda_\theta$-submaximal.

**Proof.** Suppose that each $p(\Lambda, \theta)$-open set is $(\Lambda, \theta)$-open. It follows that every $p(\Lambda, \theta)$-open set is $s(\Lambda, \theta)$-open. Since each $\alpha(\Lambda, \theta)$-open set is $p(\Lambda, \theta)$-open, then each $\alpha(\Lambda, \theta)$-open set is $(\Lambda, \theta)$-open. Thus, by Proposition 3.53, $(X, \tau)$ is $\Lambda_\theta$-submaximal.

**Proposition 3.55.** For a topological space $(X, \tau)$, the following properties are equivalent:

1. $(X, \tau)$ is $\Lambda_\theta$-submaximal.
2. Each $\Lambda_\theta$-codense subset $C$ of $X$ is $(\Lambda, \theta)$-closed.

**Definition 3.56.** Let $(X, \tau)$ be a topological space. A subset $A$ of $X$ is said to be locally $(\Lambda, \theta)$-closed if $A = U \cap F$, where $U \in \Lambda_\theta O(X, \tau)$ and $F$ is $(\Lambda, \theta)$-closed.

**Theorem 3.57.** For a subset $A$ of a topological space $(X, \tau)$, the following properties are equivalent:

1. $A$ is locally $(\Lambda, \theta)$-closed.
2. $A = U \cap A^{(\Lambda, \theta)}$ for some $U \in \Lambda_\theta O(X, \tau)$.
3. $A^{(\Lambda, \theta)} - A$ is $(\Lambda, \theta)$-closed.
4. $A \cup [X - A^{(\Lambda, \theta)}] \in \Lambda_\theta O(X, \tau)$.
5. $A \subseteq [A \cup [X - A^{(\Lambda, \theta)}]]^{(\Lambda, \theta)}$.

**Proof.** (1) $\Rightarrow$ (2): Suppose that $A = U \cap F$, where $U \in \Lambda_\theta O(X, \tau)$ and $F$ is $(\Lambda, \theta)$-closed. Since $A \subseteq F$, we have $A^{(\Lambda, \theta)} \subseteq F^{(\Lambda, \theta)} = F$. Since $A \subseteq U$,

$$A \subseteq U \cap A^{(\Lambda, \theta)} \subseteq U \cap F = A.$$  

Therefore, we obtain $A = U \cap A^{(\Lambda, \theta)}$ for some $U \in \Lambda_\theta O(X, \tau)$.

(2) $\Rightarrow$ (3): Suppose that $A = U \cap A^{(\Lambda, \theta)}$ for some $U \in \Lambda_\theta O(X, \tau)$. Then, $A^{(\Lambda, \theta)} - A = [X - (U \cap A^{(\Lambda, \theta)})] \cap A^{(\Lambda, \theta)} = (X - U) \cap A^{(\Lambda, \theta)}$. Since $(X - U) \cap A^{(\Lambda, \theta)}$ is $(\Lambda, \theta)$-closed and hence, $A^{(\Lambda, \theta)} - A$ is $(\Lambda, \theta)$-closed.

(3) $\Rightarrow$ (4): We have $X - [A^{(\Lambda, \theta)} - A] = [X - A^{(\Lambda, \theta)}] \cup A$ and hence, by (3) we obtain $A \cup [X - A^{(\Lambda, \theta)}] \in \Lambda_\theta O(X, \tau)$.

(4) $\Rightarrow$ (5): By (4), $A \subseteq A \cup [X - A^{(\Lambda, \theta)}] = [A \cup [X - A^{(\Lambda, \theta)}]]^{(\Lambda, \theta)}$.

(5) $\Rightarrow$ (1): We put $U = [A \cup [X - A^{(\Lambda, \theta)}]]^{(\Lambda, \theta)}$.

Then $U \in \Lambda_\theta O(X, \tau)$ and

$$A = A \cap U \subseteq U \cap A^{(\Lambda, \theta)} \subseteq [A \cup [X - A^{(\Lambda, \theta)}]]^{(\Lambda, \theta)} \cap A^{(\Lambda, \theta)} = A \cap A^{(\Lambda, \theta)} = A.$$  

Therefore, we obtain $A = U \cap A^{(\Lambda, \theta)}$, where $U \in \Lambda_\theta O(X, \tau)$ and $A^{(\Lambda, \theta)}$ is $(\Lambda, \theta)$-closed. Consequently, $A$ is locally $(\Lambda, \theta)$-closed.

**Theorem 3.58.** For a topological space $(X, \tau)$, the following properties are equivalent:

1. $(X, \tau)$ is $\Lambda_\theta$-submaximal.
2. Each subset of $X$ is a locally $(\Lambda, \theta)$-closed set.
3. Each $\Lambda_\theta$-dense subset of $X$ is an intersection of a $(\Lambda, \theta)$-closed set and a $(\Lambda, \theta)$-open set.

**Proof.** (1) $\Rightarrow$ (2): Suppose that $(X, \tau)$ is $\Lambda_\theta$-submaximal and $A \subseteq X$. This implies that $[X - A^{(\Lambda, \theta)} - A]^{(\Lambda, \theta)} = [A \cup [X - A^{(\Lambda, \theta)}]]^{(\Lambda, \theta)} = X$. Since $X - [A^{(\Lambda, \theta)} - A]$ is $\Lambda_\theta$-dense set, then $X - [A^{(\Lambda, \theta)} - A]$ is a $(\Lambda, \theta)$-open set. It follows that $A^{(\Lambda, \theta)} - A$ is $(\Lambda, \theta)$-closed. Consequently, $X - [A^{(\Lambda, \theta)} - A] = A \cup [X - A^{(\Lambda, \theta)}]$ is $(\Lambda, \theta)$-open. Thus, $A = [A \cup [X - A^{(\Lambda, \theta)}]] \cap A^{(\Lambda, \theta)}$ is a locally $(\Lambda, \theta)$-closed set.
(2) $\Rightarrow$ (3): Suppose that every subset of $X$ is a locally $(\Lambda, \theta)$-closed set. Let $D$ be a $\Lambda_\theta$-dense set. By (2), we have $D$ is locally $(\Lambda, \theta)$-closed. Consequently, we obtain $D$ is an intersection of a $(\Lambda, \theta)$-closed set and a $(\Lambda, \theta)$-open set.

(3) $\Rightarrow$ (1): Let $D$ be a $\Lambda_\theta$-dense set. There exist a $(\Lambda, \theta)$-open set $U$ and a $(\Lambda, \theta)$-closed set $F$ such that $D = U \cap F$. Since $D \subseteq F$ and $D$ is a $\Lambda_\theta$-dense set, then $F_{(\Lambda, \theta)} \supseteq |D_{(\Lambda, \theta)}|_{(\Lambda, \theta)} = X_{(\Lambda, \theta)} = X$. This implies that $F = X$ and $D = U$ is $(\Lambda, \theta)$-open. Hence, $(X, \tau)$ is $\Lambda_\theta$-submaximal.

**Theorem 3.59.** For a topological space $(X, \tau)$, the following properties are equivalent:

(1) $(X, \tau)$ is $\Lambda_\theta$-submaximal.

(2) Each $\Lambda_\theta$-dense subset of $X$ is an union of a $(\Lambda, \theta)$-open set and a $(\Lambda, \theta)$-closed set.

**Proof.** This is an immediate consequence of Theorem 3.58.

**Definition 3.60.** [7] A topological space $(X, \tau)$ is said to be:

(1) $\Lambda_\theta$-$T_0$ if for any distinct pair of points $x$ and $y$ in $X$, there exists a $(\Lambda, \theta)$-open set containing one of the points but not the other;

(2) $\Lambda_\theta$-$T_1$ if for any distinct pair of points $x$ and $y$ in $X$, there exist a $(\Lambda, \theta)$-open set $U$ containing $x$ but not $y$ and a $(\Lambda, \theta)$-open set $V$ containing $y$ but not $x$.

First, we investigate characterizations of $\Lambda_\theta$-$T_0$ and $\Lambda_\theta$-$T_1$ spaces.

**Theorem 3.61.** A topological space $(X, \tau)$ is $\Lambda_\theta$-$T_0$ if and only if for each pair of distinct points $x, y$ in $X$, $\{x\}^{(\Lambda, \theta)} \neq \{y\}^{(\Lambda, \theta)}$.

**Proof.** Suppose that $x, y \in X$, $x \neq y$ and $\{x\}^{(\Lambda, \theta)} \neq \{y\}^{(\Lambda, \theta)}$. Let $z$ be a point of $X$ such that $z \in \{x\}^{(\Lambda, \theta)}$ but $z \notin \{y\}^{(\Lambda, \theta)}$. We claim that $x \notin \{y\}^{(\Lambda, \theta)}$. For, if $x \in \{y\}^{(\Lambda, \theta)}$, then $\{x\}^{(\Lambda, \theta)} \subseteq \{y\}^{(\Lambda, \theta)}$. This contradicts the fact that $z \notin \{y\}^{(\Lambda, \theta)}$. Consequently, $x$ belongs to the $(\Lambda, \theta)$-open set $X - \{y\}^{(\Lambda, \theta)}$ to which $y$ does not belong.

Conversely, suppose that $(X, \tau)$ is a $\Lambda_\theta$-$T_0$ space and let $x, y$ be any two distinct points of $X$. There exists a $(\Lambda, \theta)$-open set $G$ containing $x$ or $y$, say $x$ but not $y$. Then $X - G$ is a $(\Lambda, \theta)$-closed set which does not contain $x$ but contains $y$. Since $\{y\}^{(\Lambda, \theta)}$ is the smallest $(\Lambda, \theta)$-closed set containing $y$, $\{y\}^{(\Lambda, \theta)} \subseteq X - G$, and so $x \notin \{y\}^{(\Lambda, \theta)}$. Consequently, we obtain $\{x\}^{(\Lambda, \theta)} \neq \{y\}^{(\Lambda, \theta)}$.

**Theorem 3.62.** A topological space $(X, \tau)$ is $\Lambda_\theta$-$T_1$ if and only if the singletons are $(\Lambda, \theta)$-closed sets.

**Proof.** Suppose that $(X, \tau)$ is $\Lambda_\theta$-$T_1$ and $x$ any point of $X$. Let $y \in X - \{x\}$. Then $x \neq y$ and so there exists a $(\Lambda, \theta)$-open set $U_y$ such that $y \in U_y$ but $x \notin U_y$. Consequently, $y \in U_y \subseteq X - \{x\}$, i.e., $X - \{x\} = \bigcup\{U_y \mid y \in X - \{x\}\}$ which is $(\Lambda, \theta)$-open.

Conversely, suppose that $\{x\}$ is $(\Lambda, \theta)$-closed for every $z \in X$. Let $x, y \in X$ such that $x \neq y$. Now $x \neq y$ implies $y \notin X - \{x\}$. Hence, $X - \{y\}$ is a $(\Lambda, \theta)$-open set containing $y$ but not containing $x$. Similarly, $X - \{y\}$ is a $(\Lambda, \theta)$-open set containing $x$ but not containing $y$. This means that $(X, \tau)$ is a $\Lambda_\theta$-$T_1$ space.

**Definition 3.63.** [7] A topological space $(X, \tau)$ is said to be:

(1) $\Lambda_\theta$-$R_0$ if every $(\Lambda, \theta)$-open set contains the $(\Lambda, \theta)$-closure of each of its singletons;

(2) $\Lambda_\theta$-$R_1$ if for any points $x$ and $y$ in $X$, with $\{x\}^{(\Lambda, \theta)} \neq \{y\}^{(\Lambda, \theta)}$, there exist disjoint $(\Lambda, \theta)$-open sets $U$ and $V$ such that $\{x\}^{(\Lambda, \theta)}$ is a subset of $U$ and $\{y\}^{(\Lambda, \theta)}$ is a subset of $V$.

**Corollary 3.64.** [7] A topological space $(X, \tau)$ is $\Lambda_\theta$-$R_0$ space if and only if for any $x$ and $y$ in $X$, $\{x\}^{(\Lambda, \theta)} \neq \{y\}^{(\Lambda, \theta)}$ implies $\{x\}^{(\Lambda, \theta)} \cap \{y\}^{(\Lambda, \theta)} = \emptyset$.

**Proposition 3.65.** [7] For a topological space $(X, \tau)$, the following properties are equivalent:

(1) $(X, \tau)$ is a $\Lambda_\theta$-$R_0$ space.
(2) \( \{x\}^{(\Lambda, \theta)} \) for each \( x \in X \).
(3) \( \{x\} \) is \( (\Lambda, \theta) \)-closed for each \( x \in X \).

**Proposition 3.66.** [7] If \((X, \tau)\) is a \( \Lambda_{\theta}-R_1 \) space, then \((X, \tau)\) \( \Lambda_{\theta}-R_0 \).

**Definition 3.67.** Let \( A \) be a subset of a topological space \((X, \tau)\). The \( (\theta, \Lambda, \theta) \)-closure of \( A \), \( A^{(\theta, \Lambda, \theta)} \), is defined as follows:

\[
A^{(\theta, \Lambda, \theta)} = \{ x \in X \mid A \cap \Lambda y \neq \emptyset \text{ for each } V \in \Lambda_{\theta}O(X, \tau) \text{ containing } x \}.
\]

In Proposition 3.65, a topological space \((X, \tau)\) is \( \Lambda_{\theta}-R_0 \) if and only if \( \{x\} = \{x\}^{(\Lambda, \theta)} \) for each \( x \in X \). For a \( \Lambda_{\theta}-R_1 \) space, we have the following theorem.

**Theorem 3.68.** A topological space \((X, \tau)\) is \( \Lambda_{\theta}-R_1 \) if and only if \( \{x\} = \{x\}^{(\theta, \Lambda, \theta)} \) for each \( x \in X \).

**Proof.** Suppose that \((X, \tau)\) is a \( \Lambda_{\theta}-R_1 \) space. By Proposition 3.66, we have \((X, \tau)\) is \( \Lambda_{\theta}-R_0 \) and by Proposition 3.65, \( \{x\} = \{x\}^{(\Lambda, \theta)} \subseteq \{x\}^{(\theta, \Lambda, \theta)} \) for each \( x \in X \). Therefore, \( \{x\} \subseteq \{x\}^{(\theta, \Lambda, \theta)} \) for each \( x \in X \). In order to show the opposite inclusion, suppose that \( y \notin \{x\} \). Then, we have \( \{x\} \neq \{y\} \). Since \((X, \tau)\) is \( \Lambda_{\theta}-R_0 \), by Proposition 3.65, \( \{x\}^{(\Lambda, \theta)} \neq \{y\}^{(\Lambda, \theta)} \). Since \((X, \tau)\) is \( \Lambda_{\theta}-R_1 \), there exist disjoint \( U, V \in \Lambda_{\theta}O(X, \tau) \) such that \( \{x\}^{(\Lambda, \theta)} \subseteq U \) and \( \{y\}^{(\Lambda, \theta)} \subseteq V \). Since \( \{x\} \cap \Lambda y \neq \emptyset \) and \( \{y\} \cap \Lambda x \neq \emptyset \), therefore, we obtain \( \{x\}^{(\theta, \Lambda, \theta)} \subseteq \{x\} \) and hence, \( \{x\}^{(\theta, \Lambda, \theta)} = \{x\} \).

Conversely, suppose that \( \{x\}^{(\theta, \Lambda, \theta)} = \{x\} \) for each \( x \in X \). Then, we have \( \{x\} = \{x\}^{(\theta, \Lambda, \theta)} \supseteq \{x\}^{(\Lambda, \theta)} \supseteq \{x\} \) and \( \{x\} = \{x\}^{(\Lambda, \theta)} \) for each \( x \in X \). By Proposition 3.65, \((X, \tau)\) is \( \Lambda_{\theta}-R_0 \). Suppose that \( \{x\}^{(\Lambda, \theta)} \neq \{y\}^{(\Lambda, \theta)} \). Then by Corollary 3.64, \( \{x\}^{(\Lambda, \theta)} \cap \{y\}^{(\Lambda, \theta)} = \emptyset \). By Proposition 3.65, we have \( \{x\} \cap \{y\} = \emptyset \) and hence, \( \{x\}^{(\theta, \Lambda, \theta)} \cap \{y\}^{(\theta, \Lambda, \theta)} = \emptyset \). Since \( y \notin \{x\}^{(\theta, \Lambda, \theta)} \), there exists a \( (\Lambda, \theta) \)-open set \( U \) containing \( y \) such that \( \{x\} \cap \Lambda U = \emptyset \) and so \( U^{(\theta, \Lambda, \theta)} \subseteq X - \{x\} \). Put \( V = X - U^{(\theta, \Lambda, \theta)} \), then \( V \) is \( (\Lambda, \theta) \)-open set containing \( x \). Since \((X, \tau)\) is \( \Lambda_{\theta}-R_0 \), we obtain \( \{x\}^{(\Lambda, \theta)} \subseteq V \) and \( \{y\}^{(\Lambda, \theta)} \subseteq U \) such that \( U \cap V = \emptyset \). This shows that \((X, \tau)\) is \( \Lambda_{\theta}-R_1 \).

**Theorem 3.69.** A topological space \((X, \tau)\) is \( \Lambda_{\theta}-R_1 \) if and only if \( \{x\}^{(\Lambda, \theta)} = \{x\}^{(\theta, \Lambda, \theta)} \) for each \( x \in X \).

**Proof.** Suppose that \((X, \tau)\) is a \( \Lambda_{\theta}-R_1 \) space. By Theorem 3.68, we have \( \{x\}^{(\Lambda, \theta)} \supseteq \{x\}^{(\theta, \Lambda, \theta)} \supseteq \{x\} \)

Conversely, suppose that \( \{x\}^{(\Lambda, \theta)} = \{x\}^{(\theta, \Lambda, \theta)} \) for each \( x \in X \). First, we show that \((X, \tau)\) is \( \Lambda_{\theta}-R_0 \). Let \( U \in \Lambda_{\theta}O(X, \tau) \) and \( x \in U \). Let \( y \notin U \). Then, we have \( U \cap \{y\}^{(\Lambda, \theta)} = U \cap \{y\}^{(\theta, \Lambda, \theta)} = \emptyset \) and hence, \( y \notin \{x\}^{(\theta, \Lambda, \theta)} \). There exists a \( (\Lambda, \theta) \)-open set \( V \) containing \( x \) such that \( \{y\} \cap \{x\}^{(\Lambda, \theta)} = \emptyset \) and so \( y \notin V^{(\Lambda, \theta)} \). Since \( \{x\}^{(\Lambda, \theta)} \subseteq V^{(\Lambda, \theta)} \), \( y \notin \{x\}^{(\Lambda, \theta)} \). This shows that \( \{x\}^{(\Lambda, \theta)} \subseteq U \) and \( \{x\} \) is \( \Lambda_{\theta}-R_0 \). By Proposition 3.65, \( \{x\} = \{x\}^{(\Lambda, \theta)} = \{x\}^{(\theta, \Lambda, \theta)} \) for each \( x \in X \). Hence, by Theorem 3.68 \((X, \tau)\) is \( \Lambda_{\theta}-R_1 \).

4. **Characterizations of \( \Lambda_{\theta} \)-extremally disconnected spaces**

In this section, we introduce the notion of \( \Lambda_{\theta} \)-extremally disconnected spaces. Moreover, several interesting characterizations of these spaces are investigated.

**Definition 4.1.** A topological space \((X, \tau)\) is called \( \Lambda_{\theta} \)-extremally disconnected if \( U^{(\Lambda, \theta)} \) is \( (\Lambda, \theta) \)-open in \((X, \tau)\) for every \( (\Lambda, \theta) \)-open set \( U \).

**Theorem 4.2.** A topological space \((X, \tau)\) is \( \Lambda_{\theta} \)-extremally disconnected if and only if \( U^{(\Lambda, \theta)} \cap V^{(\Lambda, \theta)} = \emptyset \) for every \( (\Lambda, \theta) \)-open sets \( U \) and \( V \) such that \( U \cap V = \emptyset \).
Proof. Suppose that $U$ and $V$ are $(\Lambda, \theta)$-open sets such that $U \cap V = \emptyset$. By Lemma 3.32, we obtain $U^{(\Lambda, \theta)} \cap V = \emptyset$ and $U^{(\Lambda, \theta)} \cap V^{(\Lambda, \theta)} = \emptyset$.

Conversely, let $U$ be any $(\Lambda, \theta)$-open set. Then $X - U$ is $(\Lambda, \theta)$-closed and hence, $[X - U]_{(\Lambda, \theta)}$ is $(\Lambda, \theta)$-open such that $U \cap [X - U]_{(\Lambda, \theta)} = \emptyset$. By hypothesis, we have $U^{(\Lambda, \theta)} \cap [X - U]^{(\Lambda, \theta)} = \emptyset$

which implies that $U^{(\Lambda, \theta)} \cap [X - U]^{(\Lambda, \theta)} = \emptyset$. Therefore, $U^{(\Lambda, \theta)} \subseteq [U^{(\Lambda, \theta)}]_{(\Lambda, \theta)}$ and so $U^{(\Lambda, \theta)} = [U^{(\Lambda, \theta)}]_{(\Lambda, \theta)}$. This shows that $U^{(\Lambda, \theta)}$ is $(\Lambda, \theta)$-open. Consequently, we obtain $(X, \tau)$ is $\Lambda_\theta$-extremally disconnected.

Lemma 4.3. Let $A$ be a subset of a topological space $(X, \tau)$. If $U$ is $(\Lambda, \theta)$-open, then $U \cap A^{(\Lambda, \theta)} \subseteq [U \cap A]^{(\Lambda, \theta)}$.

Theorem 4.4. For a topological space $(X, \tau)$, the following properties are equivalent:

1. $(X, \tau)$ is $\Lambda_\theta$-extremally disconnected.
2. $U^{(\Lambda, \theta)} \cap V^{(\Lambda, \theta)} = [U \cap V]^{(\Lambda, \theta)}$ for every $(\Lambda, \theta)$-open sets $U$ and $V$.
3. $E_{(\Lambda, \theta)} \cup F_{(\Lambda, \theta)} = [E \cup F]^{(\Lambda, \theta)}$ for every $(\Lambda, \theta)$-closed sets $E$ and $F$.

Proof. (1) $\Rightarrow$ (2): Let $U$ and $V$ be $(\Lambda, \theta)$-open sets. Then by (1), we have $U^{(\Lambda, \theta)}$ and $V^{(\Lambda, \theta)}$ are $(\Lambda, \theta)$-open sets. By Lemma 4.3,

$$U^{(\Lambda, \theta)} \cap V^{(\Lambda, \theta)} \subseteq [U \cap V]^{(\Lambda, \theta)} \subseteq [U \cap V]^{(\Lambda, \theta)} = [U \cap V]^{(\Lambda, \theta)}.$$

Consequently, we obtain $U^{(\Lambda, \theta)} \cap V^{(\Lambda, \theta)} = [U \cap V]^{(\Lambda, \theta)}$.

(2) $\Rightarrow$ (3): Let $E$ and $F$ be $(\Lambda, \theta)$-closed sets. Then $X - E$ and $X - F$ are $(\Lambda, \theta)$-open. By (2) and Lemma 3.6, we have

$$E_{(\Lambda, \theta)} \cup F_{(\Lambda, \theta)} = (X - E_{(\Lambda, \theta)} \cup F_{(\Lambda, \theta)}) = (X - (E_{(\Lambda, \theta)} \cup F_{(\Lambda, \theta)})) = (X - X_{(\Lambda, \theta)}^{(\Lambda, \theta)} \cap X_{(\Lambda, \theta)}^{(\Lambda, \theta)}) = X - (X - E_{(\Lambda, \theta)} \cup F_{(\Lambda, \theta)}) = X - (X - E_{(\Lambda, \theta)} \cup F_{(\Lambda, \theta)}) = X - [E \cup F]^{(\Lambda, \theta)} = X - [E \cup F]^{(\Lambda, \theta)}.$$ 

(3) $\Rightarrow$ (1): The proof is similar to that of (2) $\Rightarrow$ (3).

Theorem 4.5. For a topological space $(X, \tau)$, the following properties are equivalent:

1. $(X, \tau)$ is $\Lambda_\theta$-extremally disconnected.
2. $U^{(\Lambda, \theta)} \cap V^{(\Lambda, \theta)} = [U \cap V]^{(\Lambda, \theta)}$ for every $(\Lambda, \theta)$-open sets $U$ and $V$.
3. $U^{(\Lambda, \theta)} \cap V^{(\Lambda, \theta)} = \emptyset$ for every $(\Lambda, \theta)$-open sets $U$ and $V$ such that $U \cap V = \emptyset$.

Proof. This follows from Theorem 4.2 and Theorem 4.4.

Theorem 4.6. For a topological space $(X, \tau)$, the following properties are equivalent:

1. $(X, \tau)$ is $\Lambda_\theta$-extremally disconnected.
2. The $(\Lambda, \theta)$-closure of every $\beta(\Lambda, \theta)$-open set of $X$ is $(\Lambda, \theta)$-open.
3. The $(\Lambda, \theta)$-closure of every $\rho(\Lambda, \theta)$-open set of $X$ is $(\Lambda, \theta)$-open.

Proof. This follows immediately from Proposition 3.26.
Theorem 4.7. For a topological space \((X, r)\), the following properties are equivalent:

1. \((X, r)\) is \(\Lambda_r\)-extremally disconnected.
2. For each \(U \in \beta_{\Lambda_r}O(X, r)\) and each \(V \in s_{\Lambda_r}O(X, r)\) such that \(U \cap V = \emptyset\), \(U^{(\Lambda_r)} \cap V^{(\Lambda_r)} = \emptyset\).
3. For each \(A \in b_{\Lambda_r}O(X, r)\) and each \(B \in s_{\Lambda_r}O(X, r)\) such that \(A \cap B = \emptyset\), \(A^{(\Lambda_r)} \cap B^{(\Lambda_r)} = \emptyset\).
4. For each \(U \in p_{\Lambda_r}O(X, r)\) and each \(V \in s_{\Lambda_r}O(X, r)\) such that \(U \cap V = \emptyset\), \(U^{(\Lambda_r)} \cap V^{(\Lambda_r)} = \emptyset\).
5. For each \(A \in \Lambda_rO(X, r)\) and each \(B \in s_{\Lambda_r}O(X, r)\) such that \(A \cap B = \emptyset\), \(A^{(\Lambda_r)} \cap B^{(\Lambda_r)} = \emptyset\).

Proof. (1) \(\Rightarrow\) (2): Suppose that \(U \in \beta_{\Lambda_r}O(X, r)\) and \(V \in s_{\Lambda_r}O(X, r)\) such that \(U \cap V = \emptyset\). Therefore, we have \(U \cap V^{(\Lambda_r)} = \emptyset\) and by Lemma 3.32, \(U^{(\Lambda_r)} \cap V^{(\Lambda_r)} = \emptyset\). By Theorem 4.6, we obtain \(U^{(\Lambda_r)}\) is \((\Lambda, \theta)\)-open and so

\[
U^{(\Lambda_r)} \cap V^{(\Lambda_r)} = U^{(\Lambda_r)} \cap V^{(\Lambda_r)} = \emptyset
\]

since \(V \in s_{\Lambda_r}O(X, r)\).

(2) \(\Rightarrow\) (3) and (3) \(\Rightarrow\) (4) follows from Remark 3.19.

(4) \(\Rightarrow\) (5): This is obvious since every \((\Lambda, \theta)\)-open set is \(p(\Lambda, \theta)\)-open.

(5) \(\Rightarrow\) (1): This is obvious since every \((\Lambda, \theta)\)-open set is \(s(\Lambda, \theta)\)-open.

Theorem 4.8. For a topological space \((X, r)\), the following properties are equivalent:

1. \((X, r)\) is \(\Lambda_r\)-extremally disconnected.
2. For each \(U \in s_{\Lambda_r}O(X, r)\) and each \(V \in \beta_{\Lambda_r}O(X, r)\), \(U^{(\Lambda_r)} \cap V^{(\Lambda_r)} = [U \cap V]^{(\Lambda_r)}\).

Proof. (1) \(\Rightarrow\) (2): Let \(U \in s_{\Lambda_r}O(X, r)\) and \(V \in \beta_{\Lambda_r}O(X, r)\). By Theorem 4.6 and Lemma 4.3, we have

\[
U^{(\Lambda_r)} \cap V^{(\Lambda_r)} = U^{(\Lambda_r)} \cap V^{(\Lambda_r)} \subseteq [U^{(\Lambda_r)} \cap V^{(\Lambda_r)}]^{(\Lambda_r)} \subseteq [(U \cap V)^{(\Lambda_r)}]^{(\Lambda_r)} = [U \cap V]^{(\Lambda_r)}.
\]

Consequently, we obtain \(U^{(\Lambda_r)} \cap V^{(\Lambda_r)} = [U \cap V]^{(\Lambda_r)}\).

(2) \(\Rightarrow\) (1): This is obvious since every \((\Lambda, \theta)\)-open set is \(s(\Lambda, \theta)\)-open and \(p(\Lambda, \theta)\)-open.

Theorem 4.9. A topological space \((X, r)\) is \(\Lambda_r\)-extremally disconnected if and only if \(r_{\Lambda_r}O(X, r) = r_{\Lambda_r}C(X, r)\).

Proof. Suppose that \((X, r)\) is \(\Lambda_r\)-extremally disconnected. Let \(V \in r_{\Lambda_r}O(X, r)\). Then, we have \(V = [V^{(\Lambda_r)}]^{(\Lambda_r)}\). Since \((X, r)\) is \(\Lambda_r\)-extremally disconnected,

\[
[V^{(\Lambda_r)}]^{(\Lambda_r)} = [V^{(\Lambda_r)}]^{(\Lambda_r)}_{\Lambda_r} = [[[V^{(\Lambda_r)}]^{(\Lambda_r)}]^{(\Lambda_r)}_{\Lambda_r}]^{(\Lambda_r)} = [V^{(\Lambda_r)}]^{(\Lambda_r)}_{\Lambda_r} = V
\]

and so \(V \in r_{\Lambda_r}C(X, r)\). Therefore, we obtain \(r_{\Lambda_r}O(X, r) \subseteq r_{\Lambda_r}C(X, r)\). On the other hand, let \(V \in r_{\Lambda_r}C(X, r)\). Then, we have \(V = [V^{(\Lambda_r)}]^{(\Lambda_r)}\). Since \((X, r)\) is \(\Lambda_r\)-extremally disconnected,

\[
[V^{(\Lambda_r)}]^{(\Lambda_r)} = [V^{(\Lambda_r)}]^{(\Lambda_r)}_{\Lambda_r} = [[[V^{(\Lambda_r)}]^{(\Lambda_r)}]^{(\Lambda_r)}_{\Lambda_r}]^{(\Lambda_r)} = [V^{(\Lambda_r)}]^{(\Lambda_r)}_{\Lambda_r} = V
\]

and hence, \(V \in r_{\Lambda_r}O(X, r)\). Therefore, \(r_{\Lambda_r}O(X, r) \subseteq r_{\Lambda_r}C(X, r)\). Consequently, we obtain \(r_{\Lambda_r}O(X, r) = r_{\Lambda_r}C(X, r)\).

Conversely, suppose that \(r_{\Lambda_r}O(X, r) = r_{\Lambda_r}C(X, r)\). Let \(V\) be any \((\Lambda, \theta)\)-open set. Then, we have \([V^{(\Lambda_r)}]^{(\Lambda_r)} \in r_{\Lambda_r}C(X, r)\) and so \([V^{(\Lambda_r)}]^{(\Lambda_r)} \in r_{\Lambda_r}O(X, r)\). Therefore, we obtain

\[
[V^{(\Lambda_r)}]^{(\Lambda_r)} = [V^{(\Lambda_r)}]^{(\Lambda_r)}_{\Lambda_r} = [[[V^{(\Lambda_r)}]^{(\Lambda_r)}]^{(\Lambda_r)}_{\Lambda_r}]^{(\Lambda_r)} = [V^{(\Lambda_r)}]^{(\Lambda_r)}_{\Lambda_r} = V^{(\Lambda_r)}.
\]

This shows that \(V^{(\Lambda_r)}\) is a \((\Lambda, \theta)\)-open set. Hence, \((X, r)\) is \(\Lambda_r\)-extremally disconnected.

Theorem 4.10. For a topological space \((X, r)\), the following properties are equivalent:

1. \((X, r)\) is \(\Lambda_r\)-extremally disconnected.
(2) For each $U \in s\Lambda\omega(O(X, r), U^{(\Lambda, \theta)} \in \Lambda\omega(O(X, r))$.

(3) For each $U, V \in s\Lambda\omega(O(X, r))$, $|U \cap V^{(\Lambda, \theta)} = U^{(\Lambda, \theta)} \cap V^{(\Lambda, \theta)}$.

(4) For each $U, V \in \Lambda\omega(O(X, r))$, $|U \cap V^{(\Lambda, \theta)} = U^{(\Lambda, \theta)} \cap V^{(\Lambda, \theta)}$.

Proof. (1) $\Rightarrow$ (2): Let $U \in s\Lambda\omega(O(X, r))$. Then, we have $U \subseteq |U^{(\Lambda, \theta)}$. Since $(X, r)$ is $\Lambda\omega$-extremely disconnected,

$$U^{(\Lambda, \theta)} \subseteq |U^{(\Lambda, \theta)}|^{(\Lambda, \theta)} = \|U^{(\Lambda, \theta)}\|^{(\Lambda, \theta)} \subseteq \|U^{(\Lambda, \theta)}\|.$$  

Consequently, we obtain $U^{(\Lambda, \theta)} \in \Lambda\omega(O(X, r))$.

(2) $\Rightarrow$ (3): Let $U, V \in s\Lambda\omega(O(X, r))$. By (2), we have $U^{(\Lambda, \theta)}, V^{(\Lambda, \theta)} \in \Lambda\omega(O(X, r))$ and hence,

$$U^{(\Lambda, \theta)} \cap V^{(\Lambda, \theta)} = [U^{(\Lambda, \theta)} \cap V^{(\Lambda, \theta)}]^{(\Lambda, \theta)} \subseteq \|U^{(\Lambda, \theta)} \cap V^{(\Lambda, \theta)}\|^{(\Lambda, \theta)} \subseteq \|U^{(\Lambda, \theta)} \cap V^{(\Lambda, \theta)}\|^{(\Lambda, \theta)}\cap V^{(\Lambda, \theta)}\|^{(\Lambda, \theta)} = \|U^{(\Lambda, \theta)} \cap V^{(\Lambda, \theta)}\|^{(\Lambda, \theta)} \subseteq \|U^{(\Lambda, \theta)} \cap V^{(\Lambda, \theta)}\|^{(\Lambda, \theta)} \subseteq \|U^{(\Lambda, \theta)} \cap V^{(\Lambda, \theta)}\|^{(\Lambda, \theta)}.$$  

Therefore, we obtain $|U \cap V^{(\Lambda, \theta)} = U^{(\Lambda, \theta)} \cap V^{(\Lambda, \theta)}$.

(3) $\Rightarrow$ (4): This is obvious since every $(\Lambda, \theta)$-open set is $s(\Lambda, \theta)$-open.

(4) $\Rightarrow$ (1): The proof is obvious from Theorem 4.4.

Theorem 4.11. For a topological space $(X, r)$, the following properties are equivalent:

(1) $(X, r)$ is $\Lambda\omega$-extremely disconnected.

(2) For each $V \in \beta\Lambda\omega(O(X, r)), V^{(\Lambda, \theta)} \in r\Lambda\omega(O(X, r))$.

(3) For each $V \in b\Lambda\omega(O(X, r)), V^{(\Lambda, \theta)} \in r\Lambda\omega(O(X, r))$.

(4) For each $V \in s\Lambda\omega(O(X, r)), V^{(\Lambda, \theta)} \in r\Lambda\omega(O(X, r))$.

(5) For each $V \in a\Lambda\omega(O(X, r)), V^{(\Lambda, \theta)} \in r\Lambda\omega(O(X, r))$.

(6) For each $V \in \Lambda\omega(O(X, r)), V^{(\Lambda, \theta)} \in r\Lambda\omega(O(X, r))$.

(7) For each $V \in r\Lambda\omega(O(X, r)), V^{(\Lambda, \theta)} \in r\Lambda\omega(O(X, r))$.

(8) For each $V \in p\Lambda\omega(O(X, r)), V^{(\Lambda, \theta)} \in r\Lambda\omega(O(X, r))$.

Proof. (1) $\Rightarrow$ (2): Suppose that $V \in \beta\Lambda\omega(O(X, r))$. By (1) and Theorem 4.6, we have $V^{(\Lambda, \theta)} = |V^{(\Lambda, \theta)}|$ and so

$$V^{(\Lambda, \theta)} = |V^{(\Lambda, \theta)}| = |V^{(\Lambda, \theta)}|^{(\Lambda, \theta)}.$$  

Consequently, we obtain $V^{(\Lambda, \theta)} \in r\Lambda\omega(O(X, r))$.

(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) follows from Remark 3.19.

(4) $\Rightarrow$ (5) and (5) $\Rightarrow$ (6) follows from Proposition 3.3(1).

(6) $\Rightarrow$ (7): This is obvious since every $r(\Lambda, \theta)$-open set is $(\Lambda, \theta)$-open.

(7) $\Rightarrow$ (8): Let $V \in p\Lambda\omega(O(X, r))$. Then $V^{(\Lambda, \theta)} = |V^{(\Lambda, \theta)}|$ is $r(\Lambda, \theta)$-open. By (7), $|V^{(\Lambda, \theta)}|$ is $r(\Lambda, \theta)$-open and so

$$|V^{(\Lambda, \theta)}|^{(\Lambda, \theta)} = |V^{(\Lambda, \theta)}|^{(\Lambda, \theta)}.$$  

Since every $p(\Lambda, \theta)$-open set is $\beta(\Lambda, \theta)$-open. By Proposition 3.8 and Proposition 3.26, we have $V^{(\Lambda, \theta)} = |V^{(\Lambda, \theta)}|$ and hence, $V^{(\Lambda, \theta)} \in r\Lambda\omega(O(X, r))$.

(8) $\Rightarrow$ (1): Let $V$ be a $(\Lambda, \theta)$-open set. Then $V$ is $p(\Lambda, \theta)$-open. By (8), $V^{(\Lambda, \theta)} \in r\Lambda\omega(O(X, r))$ and hence, $V^{(\Lambda, \theta)} \in \Lambda\omega(O(X, r))$. This shows that $(X, r)$ is $\Lambda\omega$-extremely disconnected.

Theorem 4.12. For a topological space $(X, r)$, the following properties are equivalent:

(1) $(X, r)$ is $\Lambda\omega$-extremely disconnected.
\[(2) \ r\Lambda_\omega C(X, r) \subseteq \Lambda_\omega O(X, r). \]
\[(3) \ r\Lambda_\omega C(X, r) \subseteq \alpha\Lambda_\omega O(X, r). \]
\[(4) \ r\Lambda_\omega C(X, r) \subseteq \beta\Lambda_\omega O(X, r). \]
\[(5) \ s\Lambda_\omega O(X, r) \subseteq \alpha\Lambda_\omega O(X, r). \]
\[(6) \ s\Lambda_\omega C(X, r) \subseteq \alpha\Lambda_\omega C(X, r). \]
\[(7) \ s\Lambda_\omega C(X, r) \subseteq \beta\Lambda_\omega C(X, r). \]
\[(8) \ s\Lambda_\omega O(X, r) \subseteq \beta\Lambda_\omega O(X, r). \]
\[(9) \ \beta\Lambda_\omega O(X, r) \subseteq \beta\Lambda_\omega O(X, r). \]
\[(10) \ \beta\Lambda_\omega C(X, r) \subseteq \beta\Lambda_\omega C(X, r). \]
\[(11) \ \beta\Lambda_\omega C(X, r) \subseteq \beta\Lambda_\omega C(X, r). \]
\[(12) \ \beta\Lambda_\omega O(X, r) \subseteq \beta\Lambda_\omega O(X, r). \]
\[(13) \ r\Lambda_\omega O(X, r) \subseteq \beta\Lambda_\omega C(X, r). \]
\[(14) \ r\Lambda_\omega O(X, r) \subseteq \beta\Lambda_\omega C(X, r). \]
\[(15) \ r\Lambda_\omega O(X, r) \subseteq \beta\Lambda_\omega O(X, r). \]

**Proof.** \(1 \Rightarrow 2\): Let \( A \in r\Lambda_\omega C(X, r) \). Then, we have \( A = [A_{(\Lambda, \theta)}]_{(A, \theta)} \) and by \(1\),
\( A = [A_{(\Lambda, \theta)}]_{(A, \theta)} = [A_{(\Lambda, \theta)}]_{(A, \theta)} = A_{(\Lambda, \theta)} \). Therefore, \( A \in \Lambda_\omega O(X, r) \). Consequently, we obtain \( r\Lambda_\omega C(X, r) \subseteq \Lambda_\omega O(X, r) \).

\(2 \Rightarrow 3\): Follows from Proposition 3.3.1.

\(3 \Rightarrow 4\): Follows from Proposition 3.3.2.

\(4 \Rightarrow 5\): Let \( A \in s\Lambda_\omega O(X, r) \). Then \( A \subseteq [A_{(\Lambda, \theta)}]_{(A, \theta)} \). Since \( [A_{(\Lambda, \theta)}]_{(A, \theta)} \) is a \( r(\Lambda, \theta) \)-closed set. By \( 4\),
\( [A_{(\Lambda, \theta)}]_{(A, \theta)} \) is \( p(\Lambda, \theta) \)-open and hence,
\( [A_{(\Lambda, \theta)}]_{(A, \theta)} \subseteq [A_{(\Lambda, \theta)}]_{(A, \theta)} \). Hence, \( A \subseteq [A_{(\Lambda, \theta)}]_{(A, \theta)} \). Therefore, \( A \subseteq \alpha\Lambda_\omega O(X, r) \). This shows that \( s\Lambda_\omega O(X, r) \subseteq \alpha\Lambda_\omega O(X, r) \).

\(5 \Rightarrow 6\): Let \( A \in s\Lambda_\omega C(X, r) \). Then, we have \( X - A \in s\Lambda_\omega O(X, r) \). By \(5\), \( X - A \in \alpha\Lambda_\omega O(X, r) \) and so \( A \in \alpha\Lambda_\omega C(X, r) \). Therefore, we obtain \( s\Lambda_\omega C(X, r) \subseteq \alpha\Lambda_\omega C(X, r) \).

\(6 \Rightarrow 7\): This is obvious since every \( p(\Lambda, \theta) \)-closed set is \( a(\Lambda, \theta) \)-closed.

\(7 \Rightarrow 8\): The proof is similar to that of \( 5 \Rightarrow 6 \).

\(8 \Rightarrow 9\): Let \( A \in \beta\Lambda_\omega O(X, r) \). By Proposition 3.26.4, \( A_{(\Lambda, \theta)} \) is \( s(\Lambda, \theta) \)-open and by \( 8\), \( A_{(\Lambda, \theta)} \) is \( p(\Lambda, \theta) \)-open. Therefore, we have
\( A_{(\Lambda, \theta)} \subseteq [A_{(\Lambda, \theta)}]_{(A, \theta)} \).

and so \( A \subseteq [A_{(\Lambda, \theta)}]_{(A, \theta)} \). Hence, \( A \in p\Lambda_\omega O(X, r) \). Consequently, we obtain \( \beta\Lambda_\omega O(X, r) \subseteq p\Lambda_\omega O(X, r) \).

\(9 \Rightarrow 10\): The proof is similar to that of \( 5 \Rightarrow 6 \).

\(10 \Rightarrow 11\): This is obvious since every \( b(\Lambda, \theta) \)-closed set is \( \beta(\Lambda, \theta) \)-closed.

\(11 \Rightarrow 12\): The proof is similar to that of \( 5 \Rightarrow 6 \).

\(12 \Rightarrow 13\): Let \( A \in \beta\Lambda_\omega O(X, r) \). Then \( A \) is a \( \beta(\Lambda, \theta) \)-open set. By Proposition 3.26.5, \( A_{(\Lambda, \theta)} \) is a \( b(\Lambda, \theta) \)-open set. Then by \( 12\), \( A_{(\Lambda, \theta)} \) is \( p(\Lambda, \theta) \)-open. So \( A_{(\Lambda, \theta)} = [A_{(\Lambda, \theta)}]_{(A, \theta)} = [A_{(\Lambda, \theta)}]_{(A, \theta)} = A \) and hence, \( A \) is \( (\Lambda, \theta) \)-open. This implies that \( A \) is \( p(\Lambda, \theta) \)-closed. Therefore, \( A \in p\Lambda_\omega C(X, r) \). Hence, \( \beta\Lambda_\omega O(X, r) \subseteq \beta\Lambda_\omega C(X, r) \).
(13) \( \Rightarrow \) (14): Let \( A \in r\Lambda\Omega(X, r) \). Then by (13), \( A \) is a \((\Lambda, \theta)\)-open. So \( A_{(\Lambda, \theta)} \subseteq A \). Since \( A \) is \((\Lambda, \theta)\)-open, \( A_{(\Lambda, \theta)} \subseteq A \) and hence \( A_{(\Lambda, \theta)} = A \). This means that \( A \) is \((\Lambda, \theta)\)-closed. Therefore, \( A \in \Lambda\Omega(X, r) \). Hence, \( r\Lambda\Omega(X, r) \subseteq \Lambda\Omega(X, r) \).

(14) \( \Rightarrow \) (15): This is obvious since every \((\Lambda, \theta)\)-closed set is \((\Lambda, \theta)\)-open.

(15) \( \Rightarrow \) (1): Let \( V \) be a \((\Lambda, \theta)\)-open set. Then, we have \( [V_{(\Lambda, \theta)}]_{(\Lambda, \theta)} = r(\Lambda, \theta) \)-open. By (15), \( [V_{(\Lambda, \theta)}]_{(\Lambda, \theta)} \) is \((\Lambda, \theta)\)-open and so

\[
[[V_{(\Lambda, \theta)}]_{(\Lambda, \theta)}]_{(\Lambda, \theta)} \subseteq [V_{(\Lambda, \theta)}]_{(\Lambda, \theta)}.
\]

By Proposition 3.8, we have \( V_{(\Lambda, \theta)} \subseteq [V_{(\Lambda, \theta)}]_{(\Lambda, \theta)} \) and hence, \( V_{(\Lambda, \theta)} = [V_{(\Lambda, \theta)}]_{(\Lambda, \theta)} \). This shows that \( V_{(\Lambda, \theta)} \) is \((\Lambda, \theta)\)-open. Consequently, we obtain \((X, r)\) is \(\Lambda_\theta\)-extremally disconnected.

### 5. Characterizations of \(\Lambda_\theta\)-hyperconnected spaces

In this section, we introduce the notion of \(\Lambda_\theta\)-hyperconnected spaces and investigate some characterizations of \(\Lambda_\theta\)-hyperconnected spaces.

**Definition 5.1.** A topological space \((X, r)\) is called \(\Lambda_\theta\)-hyperconnected if every non-empty \((\Lambda, \theta)\)-open set is \(\Lambda_\theta\)-dense.

**Definition 5.2.** A subset \(N\) of a topological space \((X, r)\) is said to be \(\Lambda_\theta\)-nowhere dense if \(N_{(\Lambda, \theta)} = \emptyset\).

**Theorem 5.3.** For a topological space \((X, r)\), the following properties are equivalent:

1. \((X, r)\) is \(\Lambda_\theta\)-hyperconnected.
2. \(A\) is \(\Lambda_\theta\)-dense or \(\Lambda_\theta\)-nowhere dense for every subset \(A\) of \(X\).
3. \(U \cap V \neq \emptyset\) for every non-empty \((\Lambda, \theta)\)-open sets \(U\) and \(V\) of \(X\).
4. \(U \cap V \neq \emptyset\) for every non-empty \((\Lambda, \theta)\)-open sets \(U\) and \(V\) of \(X\).

**Proof.** (1) \( \Rightarrow \) (2): Suppose that \(A\) is not \(\Lambda_\theta\)-nowhere dense. Then, we have \(A_{(\Lambda, \theta)} \neq \emptyset\) and by (1), \([A_{(\Lambda, \theta)}]_{(\Lambda, \theta)} = X\). Since \(X = [A_{(\Lambda, \theta)}]_{(\Lambda, \theta)} \subseteq A_{(\Lambda, \theta)}\), we obtain \(A_{(\Lambda, \theta)} = X\). Therefore, \(A\) is \(\Lambda_\theta\)-dense.

(2) \( \Rightarrow \) (3): Suppose that \(U \cap V = \emptyset\) for some non-empty \((\Lambda, \theta)\)-open sets \(U\) and \(V\) of \(X\). This implies that \(U_{(\Lambda, \theta)} \cap V_{(\Lambda, \theta)} = \emptyset\) and so \(U\) is not \(\Lambda_\theta\)-dense. Moreover, since \(U_{(\Lambda, \theta)}\) is \((\Lambda, \theta)\)-open, \(\emptyset \neq U \subseteq U_{(\Lambda, \theta)}\) and so \(U\) is not \(\Lambda_\theta\)-nowhere dense. This is a contradiction. Therefore, \(U \cap V \neq \emptyset\) for every non-empty \((\Lambda, \theta)\)-open sets \(U\) and \(V\) of \(X\).

(3) \( \Rightarrow \) (4): Suppose that \(U \cap V = \emptyset\) for some non-empty \((\Lambda, \theta)\)-open sets \(U\) and \(V\) of \(X\). By Proposition 3.35, there exist \(M, N \in \Lambda\Omega(X, r)\) such that \(M \subseteq U \subseteq M_{(\Lambda, \theta)}\) and \(N \subseteq V \subseteq N_{(\Lambda, \theta)}\). Since \(U\) and \(V\) are non-empty, \(M\) and \(N\) are non-empty. Moreover, we have \(M \cup N \subseteq U \cap V = \emptyset\). This is a contradiction. Hence, \(U \cap V \neq \emptyset\) for every non-empty \((\Lambda, \theta)\)-open sets \(U\) and \(V\) of \(X\).

(4) \( \Rightarrow \) (1): Suppose that \((X, r)\) is not \(\Lambda_\theta\)-hyperconnected. Then, there exists a non-empty \((\Lambda, \theta)\)-open set \(S\) such that \(S_{(\Lambda, \theta)} \neq \emptyset\) and so \(S_{(\Lambda, \theta)} = S\). This implies that \(S_{(\Lambda, \theta)} = S\) and \(V\) are non-empty \((\Lambda, \theta)\)-open sets such that \(X - V_{(\Lambda, \theta)} \cap V = \emptyset\). This is a contradiction. Consequently, we obtain \((X, r)\) is \(\Lambda_\theta\)-hyperconnected.

**Corollary 5.4.** For a topological space \((X, r)\), the following properties are equivalent:

1. \((X, r)\) is \(\Lambda_\theta\)-hyperconnected.
2. \(U \cap V \neq \emptyset\) for every non-empty \((\Lambda, \theta)\)-open set \(U\) and every non-empty \((\Lambda, \theta)\)-open set \(V\) of \(X\).
3. \(U \cap V \neq \emptyset\) for every non-empty \((\Lambda, \theta)\)-open set \(U\) and every non-empty \((\Lambda, \theta)\)-open set \(V\) of \(X\).

**Proof.** The proof is obvious and follows from Theorem 5.3.
Theorem 5.5. For a topological space \((X, \tau)\), the following properties are equivalent:

1. \((X, \tau)\) is \(\Lambda_\theta\)-hyperconnected.
2. \(V\) is \(\Lambda_\theta\)-dense for every non-empty set \(V \in \beta\Lambda_\theta O(X, \tau)\).
3. \(V \cup [V(\Lambda, \theta)|_{\Lambda, \theta}] = X\) for every non-empty set \(V \in \beta\Lambda_\theta O(X, \tau)\).
4. \(V \cup [V(\Lambda, \theta)|_{\Lambda, \theta}] = X\) for every non-empty set \(V \in s\Lambda_\theta O(X, \tau)\).
5. \(V \cup [V(\Lambda, \theta)|_{\Lambda, \theta}] = X\) for every non-empty set \(V \in s\Lambda_\theta O(X, \tau)\).

Proof. (1) \(\Rightarrow\) (2): Let \(V\) be any non-empty \(\beta(\Lambda, \theta)\)-open set. Then, we have \([V(\Lambda, \theta)|_{\Lambda, \theta}] \neq \emptyset\) and hence, \(V(\Lambda, \theta)|_{\Lambda, \theta} = X\).

(2) \(\Rightarrow\) (3): Let \(V\) be any non-empty \(\beta(\Lambda, \theta)\)-open set. By (2), \(V(\Lambda, \theta) = X\) and so \(V \cup [V(\Lambda, \theta)|_{\Lambda, \theta}] = V \cup X(\Lambda, \theta) = V \cup X = X\).

(3) \(\Rightarrow\) (4): Let \(V\) be any non-empty \(s(\Lambda, \theta)\)-open set. Then, we obtain \(V(\Lambda, \theta) = [V(\Lambda, \theta)|_{\Lambda, \theta}]\). By (3), \(X = V \cup [V(\Lambda, \theta)|_{\Lambda, \theta}] = V \cup [V(\Lambda, \theta)|_{\Lambda, \theta}]\).

(4) \(\Rightarrow\) (5): Let \(V\) be any non-empty \(s(\Lambda, \theta)\)-open set. By (4), we have \(X = V \cup [V(\Lambda, \theta)|_{\Lambda, \theta}] \subseteq V \cup [V(\Lambda, \theta)|_{\Lambda, \theta}]\).

Consequently, we obtain \(V \cup [V(\Lambda, \theta)|_{\Lambda, \theta}] = X\).

(5) \(\Rightarrow\) (1): Let \(V\) be any non-empty \((\Lambda, \theta)\)-open set. Then \(V\) is \(s(\Lambda, \theta)\)-open. By (5), \(V \cup [V(\Lambda, \theta)|_{\Lambda, \theta}] = X\) and so \(V(\Lambda, \theta) = X\). Hence, \((X, \tau)\) is \(\Lambda_\theta\)-hyperconnected.

Theorem 5.6. For a topological space \((X, \tau)\), the following properties are equivalent:

1. \((X, \tau)\) is \(\Lambda_\theta\)-hyperconnected.
2. \(V\) is \(\Lambda_\theta\)-dense for every non-empty set \(V \in p\Lambda_\theta O(X, \tau)\).
3. \(V \cup [V(\Lambda, \theta)|_{\Lambda, \theta}] = X\) for every non-empty set \(V \in p\Lambda_\theta O(X, \tau)\).
4. \(V \cup [V(\Lambda, \theta)|_{\Lambda, \theta}] = X\) for every non-empty set \(V \in s\Lambda_\theta O(X, \tau)\).

Proof. It is similar to that of Theorem 5.5.

Theorem 5.7. For a topological space \((X, \tau)\), the following properties are equivalent:

1. \((X, \tau)\) is \(\Lambda_\theta\)-hyperconnected.
2. \(V\) is \(\Lambda_\theta\)-dense for every non-empty set \(V \in s\Lambda_\theta O(X, \tau)\).
3. \(V \cup [V(\Lambda, \theta)|_{\Lambda, \theta}] = X\) for every non-empty set \(V \in s\Lambda_\theta O(X, \tau)\).

Proof. It is similar to that of Theorem 5.5.


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