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Existence and stability for fractional parabolic integro-partial differential equations with fractional Brownian motion and nonlocal condition

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Abstract: In this paper, a nonlinear fractional parabolic stochastic integro-partial differential equations with nonlocal effects driven by a fractional Brownian motion is considered. In particular, first we have formulated the suitable solution form for the fractional partial differential equations with nonlocal effects driven by fractional Brownian motion using a parabolic transform. Next, the existence and uniqueness of solutions are obtained for the fractional stochastic partial differential equations without any restrictions on the characteristic forms when the Hurst parameter of the fractional Brownian motion is less than half. Further, we investigate the stability of the solution for the considered problem. The required result is established by means of standard Picard's iteration.

Subjects: Differential Equations; Mathematical Analysis; Stochastic Models & Processes

Keywords: nonlinear fractional integro-partial differential equations; fractional Brownian motion with Hurst parameter less than half; nonlocal Cauchy problem; stability

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PUBLIC INTEREST STATEMENT

Fractional parabolic partial differential equations are found to be an effective tool to describe certain physical phenomena such as diffusion processes, visco-elasticity theories, filtration, phase transition, electromagnetism, acoustics, electrochemistry, cosmology, and biochemistry. However, no work has been reported in the literature regarding the existence and uniqueness of solutions for nonlinear fractional parabolic integro-partial differential equations with nonlocal effects driven by a fractional Brownian motion when the Hurst parameter of the fractional Brownian motion is less than half. Motivated by these facts, in this note, we studied the existence, uniqueness and stability of solutions for the fractional stochastic partial differential equations without any restrictions on the characteristic forms when the Hurst parameter of the fractional Brownian motion is less than half.

1. Introduction

Fractional differential equations has many important applications in many areas of science and engineering. Recently, many researchers have found that it describes several physical phenomena more exactly than differential equations without fractional derivative. On the other hand, the noises arise in mathematical finance, physics, telecommunication networks, hydrology, medicine etc., can be modeled by fractional Brownian motions (Baudoin, Nualart, Ouyang, & Tindel, 2016; Grecksch & Anh, 1999; Nualart & Ouknine, 2002; Maslowski & Nualart, 2003; Tindel, Tudor, & Viens, 2003). More and more work has been devoted to the investigation of ordinary fractional differential equations driven by fractional Brownian motions (Arthi, Park, & Jung, 2016; Balasubramaniam, Vembarasan, & Senthilkumar, 2014; Boudaoui, Caraballo, & Ouahab, 2016; Diop, Ezzinbi, & Mbaye, 2015; Hamdy, 2015; Ren, Wang, & Hu, 2017; Sathiyara & Balasubramaniam, 2017; Tamilalagan & Balasubramaniam, 2017a, b). On the other hand, fractional parabolic partial differential equations are found to be an effective tool to describe certain physical phenomena such as diffusion processes, visco-elasticity theories, filtration, phase transition, electromagnetism, acoustics, electrochemistry, cosmology, and biochemistry. However, no work has been reported in the literature regarding the existence and uniqueness of solutions for nonlinear fractional parabolic integro-partial differential equations with nonlocal effects driven by a fractional Brownian motion when the Hurst parameter of the fractional Brownian motion is less than half. Motivated by these facts, in this note, we will consider the following nonlinear fractional parabolic stochastic integro-partial differential equations in the form

$$\begin{aligned}\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} &= Lu(x,t) + f_1(x,t,W(x,t)) + \int_0^t K_2(t,s) f_2(x,s,W(x,s))\,ds \\ &\quad + \int_0^t \int_{R^n} K_3(x,y,t,s) f_3(y,s,W(y,s))\,dy\,ds + g(x,t)B_H(t),\end{aligned} \tag{1.1}$$

with nonlocal initial condition

$$u(x,0) = \varphi(x) + \sum_{i=1}^{p} c_i u(x,t_i), \tag{1.2}$$

where $W = (w_1, \ldots, w_r), w_j$ is of the form $D^q u$, for some q, $|q| \le 2m\text{-}1$, j = 1, … , r, $0 < \alpha \le 1$, $L = \sum_{|q|=2m} a_q(x) D^q, D^q = D_1^{q_1} \ldots D_n^{q_n}, D_j = \frac{\partial}{\partial x_j}, x \in R^n, R^n$ is the n-dimensional Euclidean space, $q = (q_1, \ldots, q_n)$ is an n-dimensional multi-index, $|q| = q_1 + \ldots + q_n, t \in J,\ J = [0,T], T > 0, B_H(t)$ is a fractional Brownian motion with Hurst parameter $H \in (0, \frac{1}{2}), B_H(0) = E[B_H(t)] = 0, E[B_H(t)B_H(s)] = \frac{1}{2}\{|t|^{2H} + |s|^{2H} - |s-t|^{2H}\}$, and $E(X)$ denotes the expectation of a random variable X. It is well known that if $H = \frac{1}{2}$, then $B_H(t)$ coincides with the classical Brownian motion $B(t)$. For $H \neq \frac{1}{2}, B_H(t)$ is not a semimartingale, so one cannot use the general theory of stochastic calculus for semimartingale on $B_H(t)$, (Caraballo, Diop, & Ndiaye, 2014; Decreusefond & Ustunel, 1999; Duncan & Nualart, 2009; Elliott & Van Der Hoek, 2003; El-Borai & El-Nadi, 2017; Ren, Hou, & Sakthivel, 2015). It should be mentioned that the kind of equations given in (1.1)–(1.2) can be used to model a variety of anomalous diffusion in continuum mechanics, particularly in connection with the investigation in turbulence. In Section 2, we shall present some properties of the stochastic solutions of the nonlocal Cauchy problem (1.1), (1.2) using a parabolic transform. In Section 3, we shall prove the existence and uniqueness of solutions for the considered stochastic equations under suitable conditions. In Section 4, we shall investigate the stability of the solution for the considered problem.

2. Parabolic transform and weak solutions

In this section, we present some basic properties and some suitable solution form for the nonlinear fractional parabolic partial differential equations with nonlocal effects driven by fractional Brownian motion using a parabolic transform. In order to obtain the required result, we impose the following conditions on the functions:

- (H1) The given function φ is continuous and bounded on R^n.
- (H2) All the coefficients of a_q are bounded and satisfy a uniform Holder conditions on R^n.
- (H3) The functions f_1, f_2 and f_3 are continuous on $R^n \times J \times R^r$.
- (H4) The function g is given and bounded continuous on $R^n \times J$, also there exist two positive constants m and M, such that $m \le g(x,t) \le M$ for all $(x,t) \in R^n \times J$.
- (H5) The operator $\frac{\partial}{\partial t} - L$ is uniform parabolic. This mean that $(-1)^{m-1}\sum_{|q|=2m} a_q(x)y^q \ge c|y|^{2m}$, for all $x,\ y \in R^n,\ |y|^2 = y_1^2 + \ldots + y_n^2$, and c is a positive constant.
- (H6) The kernel K_2 and $\frac{\partial K_2}{\partial t}$ are continuous on $J \times J$.
- (H7) The kernel K_3 and $\frac{\partial K_3}{\partial t}$ are continuous on $R^n \times R^n \times J \times J$ and $\int_{R^n} |K_3| dy,\ \int_{R^n} |\frac{\partial K_3}{\partial t}| dy$ exist and continuous bounded on $R^n \times J \times J$.
- (H8) The function $\frac{\partial f_1(x,t,W)}{\partial t}$ is continuous and bounded on $R^n \times J \times R^r$.

Fractional stochastic nonlinear partial differential Equation (1.1), (1.2) can be transformed to the following problem

$$u(x,t) = \int_0^\infty \int_{R^n} \xi_\alpha(\theta) G(x,y,t^\alpha\theta)[\varphi(y) + \sum_{i=1}^{p} c_i u(y,t_i)]\, dy\, d\theta$$
$$+\alpha \int_0^t \int_0^\infty \int_{R^n} \theta(t-s)^{\alpha-1}\xi_\alpha(\theta) G(x,y,(t-s)^\alpha\theta) v(y,s)\, dy\, d\theta\, ds, \tag{2.1}$$

where v is given by

$$v(x,t) = f_1(x,t,W(x,t)) + F_2 + F_3 + g(x,t)B_H(t),$$
$$F_2(x,t) = \int_0^t K_2(t,s) f_2(x,s,W(x,s))\, ds, F_3(x,t) = \int_0^t \int_{R^n} K_3(t,s,x,y) f_3(y,s,W(y,s))\, dy\, ds, \tag{2.2}$$

G is the fundamental solution of the parabolic partial differential equation:

$$\frac{\partial u(x,t)}{\partial t} = Lu(x,t).$$

The proof of formula (2.1) and the definition of the function $\xi_\alpha(\theta)$ can be found in El-Borai, El-Nadi, and El-Akabawy (2010) and El-Borai, El-Nadi, and Fouad (2010). The function G satisfies the following inequalities,

$$|D^q G(x,y,t) \le \gamma t^{\nu_1} e^{-\rho\nu_2}, \tag{2.3}$$

where $\rho = |x-y|^{m_1} t^{m_2}, m_1 = \frac{2m}{2m-1}, m_2 = \frac{-1}{2m-1}, \nu_1 = -[\frac{n+|q|}{2m}], \gamma$, and ν_2 are positive constants. The function ξ_α is a probability density function defined on $(0,\infty)$. According to the properties of G, we can find a positive constant M^* such that

$$|\int_{R_n} G(x,y,t) f(y) dy| \le M^* \sup_x |f(x)|,$$

for all bounded continuous function f on R^n.

Let us suppose that $cM^* < 1$, where $c = \sum_{i=1}^{p} |c_i|$. For every $t \in (0,T)$, we define two operators $\Lambda(t)$ and $\Lambda^*(t)$ on the set of all bounded continuous function on R^n, by,

$$(\Lambda(t)f)(x) = \int_0^\infty \int_{R^n} G(x,y,t^\alpha\theta)\xi_\alpha(\theta) f(y)\, dy\, d\theta,$$
$$(\Lambda^*(t)f)(x) = \alpha \int_0^\infty \int_{R^n} \theta t^{\alpha-1} G(x,y,t^\alpha\theta)\xi_\alpha(\theta) f(y)\, dy\, d\theta. \tag{2.4}$$

According to (2.4), the inverse operator $\psi = [1 - \sum_{i=1}^{p} c_i \Lambda(t_i)]^{-1}$ exists on the set of all bounded continuous functions on R^n. From (2.1), one gets, formally,

$$\sum_{i=1}^{p} c_i u(x, t_i) = \psi \sum_{i=1}^{p} (c_i \Lambda(t_i)\varphi)(x) + \alpha\psi \sum_{i=1}^{p} c_i \int_0^{t_i} (\Lambda^*(t_i - s)v)(x, s)\, ds. \tag{2.5}$$

If we can find the stochastic process v in a suitable space, then formulas (2.1) and (2.5) will determine the stochastic process u. Let us now try to study Equation (2.2). By a weak solution of Equation (2.1), we mean a triple of adapted processes (B_H, u, v) on a filtered probability space $(\Omega, \Upsilon, P, \{\Upsilon_t : t \in J\})$, such that

(a) B_H is an F_t-fractional Brownian motion,
(b) The norm $\|v(\cdot, t)\| = \sup_x |v(x, t)|$ exists,

v satisfies Equation (2.2) and u satisfies equation (2.1). Let

$$K_H(t, s) = [\Gamma(H + \frac{1}{2})]^{-1}(t - s)^{H-\frac{1}{2}} F\left(H - \frac{1}{2}, H + \frac{1}{2}, 1 - \frac{t}{s}\right),$$

where Γ denotes the gamma function and $F(a, b, c; z)$ is the Gauss hyper geometric function. Define an operator K_H by $(K_H h) = \int_0^t K_H(t, s)h(s)\, ds$. The operator K_H is an isomorphism from the space of all square integrable functions $L_2(J)$ onto $I_{0^+}^{H+\frac{1}{2}}(L_2(J))$, where $I_{0^+}^{\alpha}(L_2(J))$ is the image of $L_2(J)$ by the fractional integral operator $I_{0^+}^{\alpha}$, where

$$(I_{0^+}^{\alpha} h)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha-1} h(s)\, ds, \qquad 0 < \alpha \le 1.$$

The inverse operator K_H^{-1} exists and can be defined on the set of all functions $h \in I_{0^+}^{H+\frac{1}{2}}(L_2(J))$. It is well known that there exists a Brownian motion $B(t)$ such that the fractional Brownian motion $B_H(t)$ can be represented by $B_H(t) = \int_0^t K_H(t, s) dB(s)$ (Nualart & Ouknine, 2002).

THEOREM 2.1 Let $H \in (0, \frac{1}{2})$ and v be a weak solution of Equation (2.2). If f_1, f_2, and f_3 are Borel functions on $R^n \times J \times R^r$ and satisfy the linear growth condition:

$$|f_i(x, t, W)| \le M_1[1 + \sum_{j=1}^{r} |w_j|],\ i = 1, 2, 3 \tag{2.6}$$

for all $x \in R^n, t \in J, W \in R^r$, where M_1 is a positive constant, then $f_i(x, t, W) \in I_{0^+}^{H+\frac{1}{2}}(L_2(J)), i = 1, 2, 3$ almost surely for every $x \in R^n$ and $|q| = 2m - 1$.

Proof From (2.1), (2.2), (2.3), (2.5) and conditions (H4), (H6), (H7), one gets,

$$V(t) \le M|B_H(t)| + M_2 \int_0^t V(s)\, ds + M_2,$$

for some positive constant M_2, $V(t) = \sup_x |v(x, t)|$.

Thus for some positive constant M_3, we have

$$V(t) \le M|B_H(t)| + M_2 \int_0^t e^{M_2(t-s)} |B_H(s)|\, ds + M_3 e^{M_2 t}. \tag{2.7}$$

From (2.6) and (2.7), one gets, for some positive constant M_3;

$$\int_0^T f_i^2(x,s,W)\,ds \le M_3[T + \int_0^T B_H^2(s)\,ds + 1]. \tag{2.8}$$

For some positive constant M_3, we have

$$|I_{0^+}^{H+\frac{1}{2}} f_i| = \frac{1}{\Gamma(H+\frac{1}{2})}|\int_0^t (t-s)^{H+\frac{1}{2}} f_i(x,s,W)\,ds| \le M_3 \int_0^T f_i^2(x,s,W)\,ds. \tag{2.9}$$

Hence the required result. According to the conditions (H6), (H7) and the conditions and results of Theorem 2.1, we can find also that f_1, F_2 and F_3 are elements of $I_{0^+}^{H+\frac{1}{2}}(L_2(J))$, for every $x \in R^n, W \in R^r$. For every $x \in R^n$, let us define an operator Q_H from $L_2(J)$ onto $I_{0^+}^{H+\frac{1}{2}}(L_2(J))$, by:

$$(Q_H h)(x,t) = \int_0^t Q_H(x,t,s)h(s)\,ds,$$

where $Q_H(x,t,s) = g(x,t)K_H(t,s)$.

Using conditions (H6), (H7) and that the functions F_2, F_3 are elements of $I_{0^+}^{H+\frac{1}{2}}(L_2(J))$, one gets that $K_H^{-1}F_2$ and $K_H^{-1}F_3$ are defined and can be represented by:

$$(K_H^{-1}F)(x,t) = t^{H-\frac{1}{2}} I_{0^+}^{\frac{1}{2}-H} t^{\frac{1}{2}-H} F_i{}', i = 1, 2$$

where

$$F_2{}'(x,t) = K_2(t,t) f_2(x,s,W(x,s))\,ds,$$

$$F_3{}'(x,t) = \int_{R^n} K_3(t,t,x,y) f_3(y,s,W(y,s))\,dy + \int_0^t \int_{R^n} \frac{\partial K_3}{\partial t}(t,s,x,y) f_3(y,s,W(y,s))\,dy\,ds.$$

Notice that $F_2{}'$ and $F_3{}'$ are elements of $I_{0^+}^{H+\frac{1}{2}}(L_2(J))$. Using condition (H4), we can see that Q_H^{-1} exists and is defined on $I_{0^+}^{H+\frac{1}{2}}(L_2(J))$. Now according to Theorem 2.1 and the last discussions, the weak solution v of Equation (2.2) can be represented by

$$v(x,t) = \int_0^t Q_H(x,t,s) d\tilde{B}(x,s) + \varphi^*(x),$$

where $\tilde{B}(x,t) = B(t) + \int_0^t \eta(x,s)\,ds, \eta = \eta_1 + \eta_2 + \eta_3, \eta_1 = Q_H^{-1} f_1, \eta_i = Q_H^{-1} F_i, i = 2, 3$ and $\varphi^*(x) = f(x,0,W(x,0))$. Notice that η_1 exists according to condition (H8).

3. Existence and uniqueness of solutions

Formula (2.10) leads to the fact that two weak solutions of Equation (2.2) must have the same distributions. We can also conclude that if two weak solutions of Equation (2.2) defined on the same filtered probability space must coincide almost surely, (El-Borai & El-Said, 2015).

THEOREM 3.1 *If f_1, f_2 and f_3 are continuous on $R^n \times J \times R^r$ and satisfy a Lipschitz condition;*

$$|f_i(x,t,W) - f_i(x,t,W^*)| \le M \sum_{j=1}^r |w_j - w_j^*|, i = 1, 2, 3 \tag{3.1}$$

for all $x \in R^n, W, W^* \in R^r, t \in J, W = (w_1, \dots, w_r), W^* = (w_1^*, \dots, w_r^*)$, then there is a weak solution of Equation (2.1). Moreover, $E[u^2(x,t)] < \infty$.

Proof Let us use the method of successive approximations. set,

$$\begin{aligned} v_k(x,t) =& g(x,t)B_H(t) + f_1(x,t,W_k) + \int_0^t K_2(t,s) f_2(x,s,W_k(x,s))\, ds \\ &+ \int_0^t \int_{R^n} K_3(x,y,t,s) f_3(y,s,W_k(y,s))\, dy\, ds, \end{aligned} \tag{3.2}$$

where $W_k = (w_{1k}, \ldots, w_{rk})$ and every w_{jk} is of the form $D^q u_k$ for some $q, |q| \le 2m-1$,

$$\begin{aligned} u_k(x,t) &= \int_0^\infty \int_{R^n} \xi_\alpha(\theta) G(x,y,t^\alpha \theta)[\varphi(y) + \sum_{i=1}^p c_i u_k(y,t_i)]\, dy\, d\theta \\ &+\alpha \int_0^t \int_0^\infty \int_{R^n} \theta (t-s)^{\alpha-1} \xi_\alpha(\theta) G(x,y,(t-s)^\alpha \theta) v_k(y,s)\, dy\, d\theta\, ds, \end{aligned} \tag{3.3}$$

$$\sum_{i=1}^p c_i u_k(x,t_i) = \psi \sum_{i=1}^p (c_i \Lambda(t_i)) \varphi(x) + \alpha \psi \sum_{i=1}^p c_i \int_0^{t_i} (\Lambda^*(t_i - s) v_k(x,s)\, ds.$$

Suppose that the zero approximation $v_0(x,t) = 0$. Using (2.3) and (3.1)–(3.4), one gets, for some constant $M > 0$,

$$|v_{k+1}(x,t) - v_k(x,t)| \le \frac{M^k}{k!} \int_0^t (t-s)^k |B_H(s)|\, ds.$$

The last inequality leads to the fact that the sequence $\{v_k\}$ uniformly converges to a stochastic process v on $R^n \times J$. It is clear that,

$$E[v^2(x,t)] \le [\sum_{k=0}^\infty \frac{1}{(k+1)^2}][\sum_{k=0}^\infty E(k+1)^2 \{v_{k+1}(x,t) - v_k(x,t)\}^2]. \tag{3.4}$$

From (3.5) and the fact that $E[B_H^2(t)] = t^{2H}$, we get $E[v^2(x,t)] < \infty$. Using (2.1) and (2.5), we get also $E[u^2(x,t)] < \infty$. This complete the proof of the theorem, (El-Borai, 2002, 2004; El-Borai, El-Nadi, Labib, & Ahmed, 2004; El-Nadi, 2005).

4. Stability of solutions

In order to study the stability results for problem (1.1), (1.2), we shall prove that the weak solutions of the Cauchy problem (1.1), (1.2) depends continuously on the part of the initial condition $\varphi(x)$. Let $u_k, k = 1, 2$ be weak solutions of the equations

$$\begin{aligned} \frac{\partial^\alpha u_k(x,t)}{\partial t^\alpha} &= L u_k(x,t) + f_1(x,t,W_k(x,t)) + \int_0^t K_2(t,s) f_2(x,s,W_k(x,s))\, ds \\ &+ \int_0^t \int_{R^n} K_3(x,y,t,s) f_3(y,s,W_k(y,s))\, dy\, ds + g(x,t) B_H(t), \end{aligned} \tag{3.5}$$

with initial conditions

$$u_k(x,0) = \varphi_k(x) + \sum_{i=1}^p c_i u_k(x,t_i), k = 1, 2, \tag{4.1}$$

where $W_k = (w_{1k}, \ldots, w_{rk})$, w_{jk} is of the form $D^q u_k$, for some $q, |q| \le 2m-1, j = 1, \ldots, r$. It is supposed that $\varphi_1(x)$ and $\varphi_2(x)$ are given bounded continuous functions on R^n.

THEOREM 4.1 *If for sufficiently small positive number* ϵ, $\sup_x |\varphi_1(x) - \varphi_2(x)| \le \epsilon$, *then* $\sup_x |u_1(x,t) - u_2(x,t)| \le M\epsilon$, *for some positive constant M.*

Proof We have

$$
\begin{aligned}
u_k(x,t) = & \int_0^\infty \int_{R^n} \xi_\alpha(\theta) G(x,y,t^\alpha\theta)[\varphi_k(y) + \sum_{i=1}^p c_i u_k(y,t_i)]\,dy\,d\theta \\
& + \alpha \int_0^t \int_0^\infty \int_{R^n} \theta(t-s)^{\alpha-1}\xi_\alpha(\theta) G(x,y,(t-s)^\alpha\theta) v_k(y,s)\,dy\,d\theta\,ds, k=1,2, \\
& \sum_{i=1}^p c_i u_k(x,t_i) = \psi \sum_{i=1}^p (c_i \Lambda(t_i))\varphi_k(x) + \alpha\psi \sum_{i=1}^p c_i \int_0^{t_i} \Lambda^*(t_i-s) v_k(x,s)\,ds,
\end{aligned}
\tag{4.2}
$$

where

$$
\begin{aligned}
v_k(x,t) = & g(x,t)B_H(t) + f_1(x,t,W_k) + \int_0^t K_2(t,s) f_2(x,s,W_k(x,s))\,ds \\
& + \int_0^t \int_{R^n} K_3(x,y,t,s) f_3(y,s,W_k(y,s))\,dy\,ds, k=1,2.
\end{aligned}
\tag{4.3}
$$

Using (4.3), (4.4), (4.5) and remembering that f_1, f_2, f_3 satisfy Lipschitz condition, we get

$$
\sup_x |v_1(x,t) - v_2(x,t)| \le M \int_0^t \sup_x |v_1(x,s) - v_2(x,s)|\,ds + M\epsilon,
$$

consequently

$$
\sup_x |v_1(x,t) - v_2(x,t)| \le M e^{Mt}\epsilon.
$$

From (4.3) and (4.4), we get the required result.

5. Conclusion

In this paper, we discussed the existence, uniqueness, and stability of solutions for the fractional stochastic partial differential equations without any restrictions on the characteristic forms when the Hurst parameter of the fractional Brownian motion is less than half. Our future work will be focused on investigate the approximate controllability for Hilfer fractional stochastic partial differential equations with fractional Brownian motion and Poisson jumps.

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