Pure Mathematics | Research Article

Some fuzzy fixed point results for fuzzy mappings in complete b-metric spaces

Wiyada Kumam, Pakeeta Sukprasert, Poom Kumam, Abdullah Shoaib, Aqeel Shahzad and Qasim Mahmood

Abstract: In this paper, we establish some fixed point results for fuzzy mapping in a complete b-metric space. Our results unify, extend and generalize several results in the existing literature. Example is also given to support our results.

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1. Introduction and preliminaries

Fixed point theory plays an important role in the various fields of mathematics. It provides very important tools for finding the existence and uniqueness of the solutions. The Banach contraction theorem has an important role in fixed point theory and became very popular due to iterations which can be easily implemented on the computers. The idea of fuzzy set was first laid down by Zadeh (1965). Later many researcher study many direction of fuzzy for extend in some research area for example in Nashine, Vetro, Kumam and Kumam (2014), Mursaleen, Srivastava and Sharma (2016), Phiangsungnoen, Sintunavarat and Kumam (2014), Xu, Tang, Yang and Srivastava (2016) and on Weiss (1975) and Butnariu (1982) give the idea of fuzzy mapping and obtained many fixed point results. Afterward, Heilpern (1981) initiated the idea of fuzzy contraction mappings and proved a fixed point theorem for fuzzy contraction mappings which is a fuzzy analogue of Nadler (1969) fixed point theorem for multivalued mappings. In 2005, Gupta et al. (2015) obtained some existence results of fixed points for contractive mappings in fuzzy metric spaces using control function. In 2015, Aghayan, Zireh and Ebadian (2017) studied some common best proximity points for non-self-mappings.

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mappings between two subsets of complex valued \( b \)-metric spaces and generalized some well-known results that were proved in classic metric spaces on complex valued \( b \)-metric space by some new definitions. Also they presented a type of contractive condition and develop a common best proximity point theorem for non-self mappings in complex valued \( b \)-metric spaces. Very recently, Shoib, Kumam, Shahzad, Phiangsungnoen and Mahmood (2018) studied and established some fixed point results for fuzzy mappings in a complete dislocated \( b \)-metric space.

In this paper we extended and obtained an \( \alpha \)-fuzzy fixed point and an \( \alpha \)-fuzzy common fixed point for fuzzy mappings in a complete \( b \)-metric space. Example is also given which supports the proved results.

**Definition 1.1** (Aydi, Bota, Karapnar, & Mitrović, 2012) Let \( X \) be any nonempty set and \( b \geq 1 \) be any given real number. A function \( d: X \times X \rightarrow \mathbb{R}^+ \) is called a \( b \)-metric, if it satisfies the following conditions for all \( x, y, z \in X \):

1. \( d(x, y) = 0 \) if and only if \( x = y \);
2. \( d(x, y) = d(y, x) \);
3. \( d(x, z) \leq b \cdot d(x, y) + d(y, z) \).

Then, the pair \((X, d)\) is called as a \( b \)-metric space.

**Definition 1.2** (Joseph, Roselin, & Marudai, 2016) Let \((X, d)\) be a \( b \)-metric space and \( \{x_n\} \) be a sequence in \( X \). Then,

1. \( \{x_n\} \) is called as convergent sequence if and only if there exists \( x \in X \), such that for all \( \epsilon > 0 \) there exists \( n_0 \in \mathbb{N} \) such that for all \( n \geq n_0 \), we have \( d(x_n, x) < \epsilon \). So, we write \( \lim_{n \to \infty} x_n = x \);
2. \( \{x_n\} \) is called as Cauchy sequence if and only if for all \( \epsilon > 0 \) there exists \( n_0 \in \mathbb{N} \) such that for each \( m, n \geq n_0 \), we have \( d(x_m, x_n) < \epsilon \).

**Definition 1.3** (Nadler, 1969) Let \((X, d)\) be a metric space. We define the Hausdorff metric on \( CB(X) \) induced by \( d \).

\[
H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(A, y)\},
\]

for all \( A, B \in CB(X) \), where \( CB(X) \) denotes the family of closed and bounded subsets of \( X \) and

\[
d(x, A) = \inf\{d(x, a) : a \in A\},
\]

for all \( x \in X \).

A fuzzy set in \( X \) is a function with domain \( X \) and values in \( [0, 1] \) in Azam (2011), \( \mathcal{F}(X) \) is the collection of all fuzzy sets in \( X \). If \( A \) is a fuzzy set and \( x \in X \), then the function values \( A(x) \) is called as the grade of membership of \( x \) in \( A \). The \( \alpha \)-level set of fuzzy set \( A_\alpha \) is denoted by \( [A]_\alpha \), and defined as:

\[
[A]_\alpha = \{x : A(x) \geq \alpha\} \quad \text{where} \quad \alpha \in (0, 1],
\]

\[
[A]_0 = \{x : A(x) > 0\}.
\]

Let \( X \) be any nonempty set and \( Y \) be a metric space. A mapping \( T \) is called as fuzzy mapping, if \( T \) is a mapping from \( X \) into \( \mathcal{F}(Y) \). A fuzzy mapping \( T \) is a fuzzy subset on \( X \times Y \) with membership function \( T(x)(y) \). The function \( T(x)(y) \) is the grade of membership of \( y \) in \( T \). For convenience, we denote the \( \alpha \)-level set of \( T \) by \( [T]_\alpha \) instead of \( (T(x))_\alpha \) (Azam, 2011).

**Definition 1.4** (Azam, 2011) A point \( x \in X \) is called an \( \alpha \)-fuzzy fixed point of a fuzzy mapping \( T: X \rightarrow \mathcal{F}(X) \) if there exists \( \alpha \in (0, 1] \) such that \( x \in [T]_\alpha \).
Definition 1.5 Let $S, T : X \to \mathcal{F}(X)$ be the two fuzzy mappings and for $x \in X$, there exist $\alpha(x), \beta(x) \in (0, 1]$. A point $x$ is said to be an $\alpha$-$\beta$-fuzzy common fixed point of $S$ and $T$ if $x \in [Sx]_{\alpha(x)} \cap [Tx]_{\beta(x)}$.

Lemma 1.1 (Azam, 2011) Let $A$ and $B$ be nonempty closed and bounded subsets of a metric space $(X, d)$. If $a \in A$, then

$$d(a, B) \leq H(A, B).$$

Lemma 1.2 (Azam, 2011) Let $A$ and $B$ be nonempty closed and bounded subsets of a metric space $(X, d)$ and $0 < \alpha \in R$. Then, for $a \in A$, there exists $b \in B$ such that

$$d(a, b) \leq H(A, B) + \alpha.$$

2. Main results

Now, we present our main results.

Theorem 2.1 Let $(X, d)$ be a complete b-metric space with constant $b \geq 1$. Let $T : X \to \mathcal{F}(X)$ be a fuzzy mapping and for $x \in X$, there exist $\alpha(x) \in (0, 1]$ satisfying the following condition:

$$H(Tx, Ty) \leq \alpha_1 d(x, Tx) + \alpha_2 d(y, Ty) + \alpha_3 d(x, Ty)$$

$$+ \alpha_4 d(y, Tx) + \alpha_5 d(x, y) + \alpha_6 \frac{d(x, Ty)(1 + d(x, Tx))}{1 + d(x, y)},$$

(2.1)

for all $x, y \in X$. Also, $\alpha_i \geq 0$, where $i = 1, 2, ..., 6$ with $\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 < 1$ and $\sum_{i=1}^{6} \alpha_i < 1$. Then, $T$ has an $\alpha$-fuzzy fixed point.

Proof Let $x_0$ be any arbitrary point in $X$, such that $x_1 \in [Tx_0]_{\alpha(x)}$. Then, by Lemma 1.2 there exists $x_2 \in [T^2x_0]_{\alpha_2(x)}$ such that

$$d(x_1, x_2) \leq H([Tx_0]_{\alpha(x)}), [T^2x_0]_{\alpha_2(x)}) + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_6)$$

$$\leq \alpha_1 d(x_0, Tx_0) + \alpha_2 d(x_1, Tx_1) + \alpha_3 d(x_0, Tx_1)$$

$$+ \alpha_4 d(x_1, Tx_0) + \alpha_5 d(x_0, x_1)$$

$$+ \alpha_6 \frac{d(x_0, Tx_0)(1 + d(x_0, Tx_1))}{1 + d(x_0, x_1)} + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_6)$$

$$\leq \alpha_1 d(x_0, x_1) + \alpha_2 d(x_1, x_2) + \alpha_3 d(x_0, x_2) + \alpha_4 d(x_1, x_1) + \alpha_5 d(x_0, x_1)$$

$$+ \alpha_6 \frac{d(x_0, x_1)(1 + d(x_0, x_2))}{1 + d(x_0, x_1)} + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_6)$$

$$\leq \frac{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_6}{1 - (2\alpha_2 + \alpha_3)} d(x_0, x_1) + \frac{(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_6)}{1 - (2\alpha_2 + \alpha_3)}.$$

Let

$$\tau = \frac{(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_6)}{1 - (2\alpha_2 + \alpha_3)}.$$

then, by above 2.2, we have

$$d(x_1, x_2) \leq \tau d(x_0, x_1) + \tau.$$
Again by Lemma 1.2, \( x_n \in \{ T \} \), such that

\[
d(x_2, x_3) \leq H(\{ T \}, \{ T \}) + \frac{(a_1 + ba_2 + a_3 + a_5)^2}{1 - (a_2 + ba_3)}
\]

\[
\leq a_1 d(x_1, \{ T \}) + a_2 d(x_2, \{ T \}) + a_3 d(x_3, \{ T \})
\]

\[
+ a_4 d(x_4, \{ T \}) + \frac{d(x_1, \{ T \}) (1 + d(x_1, \{ T \}))}{1 + d(x_1, x_2)} + \frac{(a_1 + ba_2 + a_3 + a_5)^2}{1 - (a_2 + ba_3)}
\]

\[
\leq a_1 d(x_1, x_2) + a_2 d(x_2, x_3) + a_3 d(x_3, x_2) + a_4 d(x_4, x_3) + a_5 d(x_5, x_2)
\]

\[
+ a_6 d(x_6, x_5) + \frac{(a_1 + ba_2 + a_3 + a_5)^2}{1 - (a_2 + ba_3)}
\]

\[
\leq \left( \frac{(a_1 + ba_2 + a_3 + a_5)^2}{1 - (a_2 + ba_3)} \right)^2 \cdot d(x_0, x_1)
\]

By using 2.2, we get

\[
d(x_2, x_3) \leq r^2 d(x_0, x_1) + 2r^2.
\]

Continuing the same way by induction, we obtain a sequence \( \{ x_n \} \), such that \( x_{n-1} \in \{ T \} \), and \( x_n \in \{ T \} \), we have

\[
d(x_n, x_{n+1}) \leq r^n d(x_0, x_1) + nr^n.
\]

Now, for any positive integer \( m \), \( n \), and \( m > n \), we have

\[
d(x_m, x_n) \leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n)
\]

\[
\leq r^m d(x_0, x_1) + mr^n + r^{m+1} d(x_0, x_2) + (m + 1)r^{n+1} + \cdots + r^{n-1} d(x_{n-1}, x_1) + (n - 1)r^n + \sum_{i=m}^{n-1} ir^i
\]

\[
\leq r^m (1 + r + \cdots + r^{n-m-1}) d(x_0, x_1) + \sum_{i=m}^{n-1} ir^i
\]

\[
\leq \frac{r^m}{1-r} d(x_0, x_1) + \sum_{i=m}^{n-1} ir^i.
\]

Since \( r < 1 \), it follows from Cauchy root test, \( \Sigma i r^i \) is convergent, hence \( \{ x_n \} \) is a Cauchy sequence in \( X \). As \( X \) is complete. So, there exists \( z \in X \) such that \( x_n \to z \) as \( n \to \infty \).

Now, we consider

\[
d(z, [Tz]) \leq b \left[ d(z, x_{n+1}) + H(\{ T \}, \{ T \}) \right]
\]

\[
\leq b \left[ d(z, x_{n+1}) + H(\{ T \}, \{ T \}) \right].
\]

Using 2.1, with \( n \to \infty \) we get

\[
(1 - b(a_2 + a_3)) d(z, [Tz]) = 0.
\]
So, we get
\[ z \in [Tz]_{\alpha(x)}. \]

Hence, \( z \in X \) is an \( \alpha \)-fuzzy fixed point. \( \square \)

If we take \( b = 1 \) in Theorem 2.1, then we have the following corollary which is a \( b \)-metric space extension of a metric space.

**Corollary 2.1** Let \( (X, d) \) be a complete metric space. Let \( T : X \to \mathcal{F}(X) \) be a fuzzy mapping and for \( x \in X \), there exist \( \alpha(x) \in (0, 1] \) satisfying the following condition:

\[
H([Tx]_{\alpha(x)}, [Ty]_{\alpha(y)}) \leq a_1 d(x, [Tx]_{\alpha(x)}) + a_2 d(y, [Ty]_{\alpha(y)}) + a_3 d(x, [Ty]_{\alpha(y)}) + a_4 d(y, [Tx]_{\alpha(x)}) + a_5 d(x, y) + a_6 d(x, [Tx]_{\alpha(x)})(1 + d(x, [Tx]_{\alpha(x)})) \tag{2.6}
\]

for all \( x, y \in X \). Also, \( a_i \geq 0 \), where \( i = 1, 2, \ldots, 6 \) with \( a_1 + a_2 + 2ba_3 + a_4 + a_5 + a_6 < 1 \) and \( \sum_{i=1}^{6} a_i < 1 \). Then, \( T \) has an \( \alpha \)-fuzzy fixed point.

Next, we replace the another condition in Theorem 2.1; we get the following result which is a \( b \)-metric space extension of fixed point theorem given by Heilpern (1981).

**Corollary 2.2** Let \( (X, d) \) be a complete metric space. Let \( T : X \to \mathcal{F}(X) \) be a fuzzy mapping and for \( x \in X \), there exist \( \alpha(x) \in (0, 1] \) satisfying the following condition:

\[
H([Tx]_{\alpha(x)}, [Ty]_{\alpha(y)}) \leq k(d(x, y)), \tag{2.7}
\]

for all \( x, y \in X \), where \( k \in (0, 1) \). Then, \( T \) has an \( \alpha \)-fuzzy fixed point.

**Theorem 2.2** Let \( (X, d) \) be a complete \( b \)-metric space with constant \( b \geq 1 \). Let \( S, T : X \to \mathcal{F}(X) \) be the two fuzzy mappings and for \( x \in X \), there exist \( a_i(x), \alpha_i(x) \in (0, 1] \) satisfying the following condition:

\[
H([Tx]_{\alpha_i(x)}, [Sy]_{\alpha_i(y)}) \leq a_1 d(x, [Tx]_{\alpha_i(x)}) + a_2 d(y, [Sy]_{\alpha_i(y)}) + a_3 d(x, [Sy]_{\alpha_i(y)}) + a_4 d(y, [Tx]_{\alpha_i(x)}) + a_5 d(x, y), \tag{2.8}
\]

for all \( x, y \in X \). Also \( a_i \geq 0 \), where \( i = 1, 2, \ldots, 6 \) with \( a_1 + a_2 + (b + 1)(a_3 + a_4 + a_5) + (b^2 + 2ba_5 < 2 \) and \( \sum_{i=1}^{5} a_i < 1 \). Then, \( S \) and \( T \) have an \( \alpha \)-fuzzy common fixed point.

**Proof** Let \( x_0 \) be any arbitrary point in \( X \), such that \( x_0 \in [Tx_0]_{\alpha_{x_0}} \). Then, by Lemma 1.2 there exists \( x_1 \in [Sx_0]_{\alpha_{x_1}} \) such that

\[
d(x_1, x_2) \leq H([Tx_0]_{\alpha_{x_0}}, [Sx_0]_{\alpha_{x_1}}) + (a_1 + ba_3 + a_6)
\]

\[
\leq a_1 d(x_0, [Tx_0]_{\alpha_{x_0}}) + a_2 d(x_1, [Sx_1]_{\alpha_{x_1}}) + a_3 d(x_0, [Sx_1]_{\alpha_{x_1}})
\]

\[
+ a_4 d(x_1, [Tx_1]_{\alpha_{x_1}}) + a_5 d(x_0, x_1) + (a_1 + ba_3 + a_6)
\]

\[
\leq a_1 d(x_0, x_1) + a_2 d(x_1, x_2) + a_3 d(x_2, x_2) + a_4 d(x_1, x_1) + a_5 d(x_0, x_1)
\]

\[
+ (a_1 + ba_3 + a_6)
\]

\[
\leq a_1 d(x_0, x_1) + a_2 d(x_1, x_2) + ba_1 d(x_0, x_1) + d(x_1, x_2)
\]

\[
+ a_4 d(x_1, x_1) + (a_1 + ba_3 + a_6)
\]

\[
\leq a_1 + ba_4 + a_5 d(x_0, x_1) + (a_1 + ba_3 + a_6)
\]

\[
\leq 1 - (a_2 + ba_3) \cdot d(x_0, x_1) + \frac{a_1 + ba_3 + a_6}{1 - (a_2 + ba_3)}. \tag{2.9}
\]
Continuing the same way, by induction, we have a sequence $x_n$ in $\mathbb{R}^n$. Then, by above 2.11, we have
\[
d(x_n, x_{n+1}) \leq a_1 d(x_{n-1}, x_n) + a_2 d(x_n, x_{n+1}) + a_3 d(x_{n+1}, x_{n+2}) + \cdots + a_n d(x_{n-1}, x_n)
\]
Adding 2.10 and 2.11, we get
\[
d(x_n, x_{n+1}) \leq \frac{a_1 + a_2 + a_3 + a_n}{2 - (a_1 + a_2 + a_3 + a_n)} d(x_{n-1}, x_n) + \frac{a_1 + a_2 + a_3 + 2a_n}{2 - (a_1 + a_2 + a_3 + a_n)}
\]
Let,
\[
\tau = \frac{a_1 + a_2 + a_3 + 2a_n}{2 - (a_1 + a_2 + a_3 + a_n)} < \frac{1}{b'}
\]
then, by above 2.11, we have
\[
d(x_n, x_{n+1}) \leq \tau d(x_{n-1}, x_n) + \tau.
\]
Again by Lemma 1.2, $x_3 \in \{x_2 \}_{x_2}$ such that
\[
d(x_2, x_3) \leq H(\mathbb{R}^n, \{x_2 \}_{x_2}) + (a_1 + a_2 + a_3 + 2a_n)(x_2, x_3) + \frac{a_1 + a_2 + a_3 + 2a_n}{2 - (a_1 + a_2 + a_3 + a_n)}
\]
Continuing the same way, by induction, we have a sequence $\{x_n\}$ such that $x_{2n+1} \in \{x_{2n} \}_{x_{2n}}$ and $x_{2n+2} \in \{x_{2n+1} \}_{x_{2n+1}}$ with
\[
d(x_{2n+1}, x_{2n+2}) \leq \frac{a_1 + a_2 + a_3, 2^{2n+1}}{(1 - (a_2 + a_3))^{2n}}
\]
Using (2.8), with $A_s$, $X$ is complete. So, there exists $z \in X$ such that $x_n \rightarrow z$ as $n \rightarrow \infty$. Now, for any positive integer $m$, $n$ and $(n > m)$, we have

\[
d(x_n, x_m) \leq d(x_n, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{m-1}, x_n) \\
\leq \tau \cdot d(x_n, x_1) + m \tau + m \tau^2 + (m + 1) \tau^3 + \cdots + n \tau^{n-1} \\
\leq \tau^m (1 + \tau + \cdots + \tau^{n-1}) d(x_n, x_1) + \sum_{i=m}^{n-1} i \tau^i \\
\leq \frac{\tau^m}{1 - \tau} d(x_n, x_1) + \sum_{i=m}^{n-1} i \tau^i.
\]

Since $\tau < 1$, it follows from Cauchy root test, $\sum i \tau^i$ is convergent, hence $\{x_n\}$ is a Cauchy sequence in $X$. As, $X$ is complete. So, there exists $z \in X$ such that $x_n \rightarrow z$ as $n \rightarrow \infty$.

Now, we prove $z \in X$ be the $\alpha$-fuzzy common fixed point of $S$ and $T$.

\[
d(z, [Sz]_{\alpha z}) \leq b \cdot d(z, x_{2n+1}) + d(x_{2n+1}, [Sz]_{\alpha z}) \\
\leq b \cdot [d(z, x_{2n+1}) + H([Tx_{2n}]_{\alpha x_{2n}}, [Tz]_{\alpha z})].
\]

Using (2.8), with $n \rightarrow \infty$ we get

\[
(1 - b(a_2 + a_1))d(z, [Sz]_{\alpha z}) \leq 0.
\]

So, we get

\[
z \in [Sz]_{\alpha z}.
\]
This implies that \( z \in X \) is an \( \alpha \)-fuzzy fixed point for \( S \). Similarly, we can show that \( z \in |Tz|_{w} \) Hence, \( z \in X \), be an \( \alpha \)-fuzzy common fixed point.

If we take \( b = 1 \) in Theorem 2.2, then we have the following corollary which is a \( b \)-metric space extension of a metric space.

**Corollary 2.3** Let \((X, d)\) be a complete metric space. Let \( S, T : X \rightarrow \mathcal{F}(X) \) be the two fuzzy mappings and for \( x \in X \), there exist \( a_i(x), a_i(x) \in (0, 1] \) satisfying the following condition:

\[
H(|Tx|_{a_i(x)}, |Sy|_{a_i(y)}) \leq a_1d(x, |Tx|_{a_i(x)}) + a_2d(y, |Sy|_{a_i(y)}) + a_3d(x, |Sy|_{a_i(y)})
+ a_4d(y, |Tx|_{a_i(x)}) + a_5d(x, y),
\]

(2.16)

for all \( x, y \in X \). Also \( a_i \geq 0 \), where \( i = 1, 2, \ldots, 5 \) with \( (a_1 + a_2)(2) + (a_3 + a_4)(2) + 2a_5 < 2 \) and \( \sum a_i < 1 \). Then, \( S \) and \( T \) have \( \alpha \)-fuzzy common fixed point.

In Theorem 2.2, if \( T = S \) where \( S, T : X \rightarrow \mathcal{F}(X) \) be the two fuzzy mappings and \( a_i(x) = a_j(x) = \alpha(x) \), then we have the following corollary.

**Corollary 2.4** Let \((X, d)\) be a complete metric space. Let \( T : X \rightarrow \mathcal{F}(X) \) be the fuzzy mappings and for \( x \in X \), there exist \( a_i(x) \in (0, 1] \) satisfying the following condition:

\[
H(|Tx|_{a_i(x)}, |Ty|_{a_i(y)}) \leq a_1d(x, |Tx|_{a_i(x)}) + a_2d(y, |Ty|_{a_i(y)}) + a_3d(x, |Ty|_{a_i(y)})
+ a_4d(y, |Tx|_{a_i(x)}) + a_5d(x, y),
\]

(2.17)

for all \( x, y \in X \). Also \( a_i \geq 0 \), where \( i = 1, 2, \ldots, 5 \) with \( (a_1 + a_2)(2) + (a_3 + a_4)(2) + 2a_5 < 2 \) and \( \sum a_i < 1 \). Then, \( T \) has an \( \alpha \)-fuzzy fixed point.

**Proof** Let \( x_0 \) be any arbitrary point in \( X \), such that \( x_1 \in |Ix_0|_{a_i(x_0)} \). Then, by Lemma 1.2 there exists \( x_2 \in |Tx_1|_{a_i(x_1)} \), such that

\[
d(x_1, x_2) \leq H(|Tx_0|_{a_i(x_0)}, |Tx_1|_{a_i(x_1)}) + (a_1 + ba_3 + a_5)
\]

\[
\leq a_1d(x_0, |Tx_0|_{a_i(x_0)}) + a_2d(x_1, |Tx_1|_{a_i(x_1)}) + a_3d(x_0, |Tx_1|_{a_i(x_1)})
+ a_4d(x_1, |Tx_0|_{a_i(x_0)}) + (a_1 + ba_3 + a_5)
\]

\[
\leq a_1d(x_0, x_1) + a_2d(x_1, x_2) + a_3d(x_0, x_2) + a_4d(x_1, x_1) + a_5d(x_0, x_1)
+ (a_1 + ba_3 + a_5)
\]

\[
\leq a_1d(x_0, x_1) + a_2d(x_1, x_2) + ba_3[d(x_0, x_1) + d(x_1, x_2)]
+ a_4d(x_0, x_1) + (a_1 + ba_3 + a_5)
\]

\[
\leq a_1 + ba_3 + a_5 \frac{d(x_0, x_1)}{1 - (a_2 + ba_3)} + \frac{(a_1 + ba_3 + a_5)}{1 - (a_2 + ba_3)}.
\]

(2.18)
Similarly, by symmetry, we have
\[
d(x_2, x_1) \leq H([Tx_1]_{[x_0,x_1]}, [Tx_0]_{[x_0,x_1]}) + (a_2 + ba_4 + a_5)
\]
\[
\leq a_1d(x_1, [Tx_1]_{[x_0,x_1]}) + a_d(x_0, [Tx_0]_{[x_0,x_1]}) + a_2d(x_2, [Tx_0]_{[x_0,x_1]})
+ a_d(x_2, [Tx_1]_{[x_0,x_1]}) + a_2d(x_1, x_0) + (a_2 + ba_4 + a_5)
\]
\[
\leq a_1d(x_1, x_2) + a_2d(x_0, x_1) + a_3d(x_1, x_0) + a_2d(x_0, x_1) + (a_2 + ba_4 + a_5)
+ (a_2 + ba_4 + a_5)
\]
\[
\leq a_1d(x_1, x_2) + a_2d(x_0, x_1) + ba_a[d(x_0, x_1) + d(x_1, x_2)]
+ a_2d(x_1, x_0) + (a_2 + ba_4 + a_5)
\]
\[
\leq a_1d(x_1, x_2) + a_2 + ba_a + a_5
\]
\[
\frac{1}{1 - (a_1 + ba_4)}d(x_0, x_1) + \frac{a_2 + ba_4 + a_5}{1 - (a_1 + ba_4)}.
\]

Adding 2.19 and 2.20, we get
\[
d(x_1, x_2) \leq \frac{a_1 + a_2 + ba_4 + ba_a + 2a_5}{2 - (a_1 + a_2 + ba_a + ba_4)}d(x_0, x_1) + \frac{a_1 + a_2 + ba_4 + ba_a + 2a_5}{2 - (a_1 + a_2 + ba_a + ba_4)}
\]

Let,
\[
\tau = \frac{a_1 + a_2 + ba_4 + ba_a + 2a_5}{2 - (a_1 + a_2 + ba_a + ba_4)} < \frac{1}{b}.
\]

then, by above 2.20, we have
\[
d(x_1, x_2) \leq \tau d(x_0, x_1) + \tau.
\]

Again by Lemma 1.2, \(x_3 \in [Ix_2]_{[x_0,x_1]}\) such that
\[
d(x_3, x_4) \leq H([Tx_1]_{[x_0,x_1]}, [Tx_2]_{[x_0,x_1]}) + \frac{(a_1 + a_2 + ba_a + ba_4 + 2a_5)}{2 - (a_1 + a_2 + ba_a + ba_4)}
\]
\[
\leq \tau^2 d(x_0, x_1) + 2\tau^2.
\]

Continuing the same way, by induction, we have a sequence \(\{x_n\}\) such that \(x_{2n+1} \in [Tx_n]_{[x_0,x_1]}\) and
\(x_{2n+2} \in [Tx_{2n+1}]_{[x_0,x_1]}\) with
\[
d(x_{2n+1}, x_{2n+2}) \leq H([Tx_{2n+1}]_{[x_0,x_1]}, [Tx_{2n+2}]_{[x_0,x_1]}) + \frac{(a_1 + ba_4 + a_5 + 2a_5)^{2n+1}}{(1 - (a_2 + ba_a))^n}
\]
\[
\leq a_1d(x_{2n}, x_{2n+1}) + a_2d(x_{2n+1}, x_{2n+2})
+ a_3d(x_{2n}, x_{2n+1}) + a_4d(x_{2n+1}, x_{2n+2})
+ a_5d(x_{2n}, x_{2n+1}) + \frac{(a_1 + ba_4 + a_5 + 2a_5)^{2n+1}}{(1 - (a_2 + ba_a))^n}
\]
\[
\leq a_1 + ba_4 + a_5
d(x_{2n}, x_{2n+1}) + \frac{(a_1 + ba_4 + a_5 + 2a_5)^{2n+1}}{(1 - (a_2 + ba_a))^n}.
\]
Similarly,

\[
d(x_{2n+2}, x_{2n+1}) \leq H(Tx_{2n+1}, Tx_{2n+1}) + \frac{(a_2 + ba_n + a_2)^{2n+1}}{1 - (a_1 + ba_n)2^n} \\
\leq a_1d(x_{2n+1}, T(x_{2n+1})) + a_2 d(x_{2n+1}, Tx_{2n+1}) \\
+ a_3d(x_{2n+1}, (Tx_{2n+1})) + a_4d(x_{2n+1}, T(x_{2n+1})) \\
+ \frac{(a_2 + ba_n + a_2)^{2n+1}}{1 - (a_1 + ba_n)2^n} \\
\leq \frac{(a_2 + ba_n + a_2)^{2n+1}}{1 - (a_1 + ba_n)2^n} d(x_{2n+1}, x_{2n+1}) + \frac{(a_2 + ba_n + a_2)^{2n+1}}{1 - (a_1 + ba_n)2^n}.
\]

Adding 2.22 and 2.23, we get

\[
d(x_{2n+1}, x_{2n+2}) \leq \tau d(x_{2n}, x_{2n+1}) + \tau^{2n+1}.
\]

Therefore,

\[
d(x_n, x_{n+1}) \leq \frac{a_1 + a_2 + ba_n + 2a_3}{2 - (a_1 + a_2 + ba_n + 2a_3)} d(x_{n-1}, x_n) + \left( \frac{a_1 + a_2 + ba_n + 2a_3}{2 - (a_1 + a_2 + ba_n + 2a_3)} \right)^n d(x_n, x_{n+1}) + \left( \frac{a_1 + a_2 + ba_n + 2a_3}{2 - (a_1 + a_2 + ba_n + 2a_3)} \right)^n d(x_{n+1}, x_n) + \tau^n
\]

\[
\leq \tau d(x_{n-2}, x_{n-1}) + \tau^n = \tau^2 d(x_{n-2}, x_{n-1}) + 2\tau^n
\]

\[
\leq \cdots \cdots
\]

\[
d(x_n, x_{n+1}) \leq \tau^n d(x_0, x_1) + nr^n.
\]

Now, for any positive integer \(m, n\) and \(n > m\), we have

\[
d(x_m, x_n) \leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n)
\]

\[
\leq \tau^n d(x_0, x_1) + \tau^{m+1} d(x_0, x_1) + (m+1)\tau^{m+1} + \cdots
\]

\[
= \tau^n(1 + \tau + \cdots + \tau^{n-1})d(x_0, x_1) + \sum_{i=m}^{n-1} \tau^i
\]

\[
\leq \frac{\tau^m}{1 - \tau} d(x_0, x_1) + \sum_{i=m}^{n-1} \tau^i.
\]

Since \(\tau < 1\), it follows from Cauchy root test, \(\Sigma \tau^i\) is convergent, hence \(\{x_n\}\) is a Cauchy sequence in \(X\). As, \(X\) is complete. So, there exists \(z \in X\) such that \(x_n \rightarrow z\) as \(n \rightarrow \infty\).
This implies that $z \in X$ is an $\alpha$-fuzzy fixed point for $T$.

**Example 2.1** Let $X=(\text{Azam, 2011})$ and $d(x,y) = |x - y|$, whenever $x, y \in X$, then $(X, d)$ is a complete $b$-metric space. Define a fuzzy mapping $T : X \to T(X)$ by

$$T(x,t) = \begin{cases} 1, & 0 \leq t \leq x/4; \\ 1/2, & x/4 < t \leq x/3; \\ 1/4, & x/3 < t \leq x/2; \\ 0, & x/2 < t \leq 1. \end{cases}$$

For all $x \in X$, there exists $\alpha(x) = 1$, such that

$$[Tx]_{\alpha(x)} = [0, x/4].$$

Then,

$$H([Tx]_{\alpha(x)}, [Ty]_{\alpha(y)}) \leq \frac{1}{5} |x - x| + \frac{1}{10} |y - y| + \frac{1}{15} |x - y| + \frac{1}{20} |y - x| + \frac{1}{25} |x - y| + \frac{1}{30} \left( \frac{|x - x|((1 + |x - x|))}{1 + |x - y|} \right).$$

Since, all the conditions of Theorem 2.1 are satisfied. So, there exists $0 \in X$ is an $\alpha$-fuzzy fixed point of $T$.

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**Competing interests**

The authors declare that they have no competing interests.

**Author details**

Wiyada Kumam
E-mail: wiyada.kum@kmutt.ac.th
ORCID ID: http://orcid.org/0000-0001-8773-4821

Pakeeta Sukprasert
E-mail: happy_t_ik@hotmail.com

Poom Kumam
E-mail: poom.kum@kmutt.ac.th
ORCID ID: http://orcid.org/0000-0002-5463-4581

Abdullah Shoaib
E-mail: abdullahshoaib15@yahoo.com

Aqeel Shahzad
E-mail: aqeel4all84@gmail.com

Qasim Mahmood
E-mail: qasim_math@yahoo.com

**Affiliations**

1. Program in Applied Statistics, Faculty of Science and Technology, Department of Mathematics and Computer Science, Rajamangala University of Technology Thanyaburi (RMITT), Thanyaburi, Pathumthani 12110, Thailand.  

2. KMITT - Fixed Point Theory and Applications Research Group (KMITT-FPTARG), Theoretical and Computational Science Center (TaCS), Science Laboratory Building, Faculty of Science, King Mongkut’s University of Technology Thonburi (KMITT), 126 Pracha-Uttith Road, Bang Mod, Thung Khru, Bangkok 10140, Thailand.  

3. KMITT-Fixed Point Research Laboratory, Department of Mathematics, Room SCL 802 Fixed Point Laboratory, Science Laboratory Building, Faculty of Science, King Mongkut’s University of Technology Thonburi (KMITT), 126 Pracha-Uttith Road, Bang Mod, Thung Khru, Bangkok 10140, Thailand.  


**Authors Contribution**

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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