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*Corresponding author: P. Yiarayong,
Faculty of Science, Department of
Mathematics, Naresuan University,
Phitsanuloke 65000, Thailand
E-mail: pairote0027@hotmail.com

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Hari M. Srivastava, University of Victoria,
Canada

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On generalizations of classical primary submodules over commutative rings

P. Yiarayong^{1*} and M. Siripitukdet¹

Abstract: Let $\phi: S(M) \rightarrow S(M) \cup \{\emptyset\}$ be a function where $S(M)$ is the set of all submodules of R -module M . A proper submodule N of M is called a ϕ -classical primary submodule, if for each $m \in M$ and $a, b \in R$ with $abm \in N - \phi(N)$, then $am \in N$ or $b^n m \in N$ for some positive integer n . Some characterizations of classical primary and ϕ -classical primary submodules are obtained. It is shown that N is a ϕ -classical primary submodule of M if and only if for every $m \in M - N$, $(N:m)$ is a ϕ -primary ideal of R where $(\phi(N):m) = \phi(N:m)$. Moreover, we investigate relationships between classical primary, ϕ -classical primary and ϕ -primary submodules of modules over commutative rings. Finally, we obtain necessary and sufficient conditions of a ϕ -classical primary submodule in order to be a ϕ -primary submodule.

Subjects: Fields & Rings; Group Theory; Pure Mathematics

Keywords: classical primary submodule; ϕ -classical primary submodule; ϕ -primary submodule; ϕ -primary ideal

1. Introduction

Throughout this paper, we assume that all rings are commutative with $1 \neq 0$. Let R be a commutative ring and M be an R -module. We will denote by $(N:M)$ the residual of N by M , that is, the set of all $r \in R$ such that $rM \subseteq N$. Let I be a proper ideal of R . Then $\sqrt{I} = \{r \in R: r^n \in I, \text{ for some positive integer } n\}$ denotes the radical ideal of R . A proper ideal I of R is called a weakly primary ideal if whenever $0 \neq ab \in I$ for $a, b \in R$, then $a \in I$ or $b \in \sqrt{I}$. The notion of weakly primary ideals has been introduced and studied by Atani and Farzalipour (2005). Anderson and Badawi (2011) generalized the concept of 2-absorbing ideals to n -absorbing ideals. According to their definition, a proper ideal I of R is said to be an n -absorbing ideal of R if whenever $a_1 a_2 \dots a_{n+1} \in I$ for $a_1, a_2, \dots, a_{n+1} \in R$, then there are n of the a_i 's whose product is in I . Later, Badawi, Tekir, and Yetkin (2015) generalized the concept of weakly primary ideals to weakly 2-absorbing primary ideals. According to their definition, a proper ideal I of R is said to be a weakly 2-absorbing primary ideal of R if whenever $0 \neq abc \in I$ for $a, b, c \in R$, then $ab \in I$

ABOUT THE AUTHORS

P. Yiarayong is an assistant professor of Department of Mathematics, Faculty of Science and Technology, Pibulsongkram Rajabhat University, Phitsanuloke, Thailand. One of his research orientations deals with abstract algebra, semigroup, LA-semigroup, LA-ring and fuzzy subset.

M. Siripitukdet is an associated professor of Department of Mathematics, Faculty of Science, Naresuan University, Phitsanuloke, Thailand. He research interests include operation research, abstract algebra, semigroup and fuzzy subset.

PUBLIC INTEREST STATEMENT

In this paper, we extend the concept of classical primary submodules to the context of ϕ -classical primary submodules. Some characterizations of classical primary and ϕ -classical primary submodules are obtained. Moreover, we investigate relationships between classical primary, ϕ -classical primary and ϕ -primary submodules of modules over commutative rings.

or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. Clearly, every weakly primary ideal is a weakly 2-absorbing primary ideal. Also, Tekir, Koc, and Oral (2016) generalized the concept of quasi-primary ideals to 2-absorbing quasi-primary ideals. According to their definition, a proper ideal I of R is said to be a 2-absorbing quasi-primary ideal of R if \sqrt{I} is a 2-absorbing ideal of R . Thus, a 2-absorbing quasi-primary ideal is quasi-primary.

Let $\phi: \mathcal{I}(R) \rightarrow \mathcal{I}(R) \cup \{\emptyset\}$ be a function where $\mathcal{I}(R)$ is a set of ideals of R . A proper ideal I of R is called a ϕ -prime ideal of R as in Anderson and Bataineh (2008) if whenever $ab \in I - \phi(I)$ for $a, b \in R$, then $a \in I$ or $b \in I$. Darani (2012) generalized the concept of primary and weakly primary ideals to ϕ -primary ideals. A proper ideal I of R is said to be a ϕ -primary ideal of R if whenever $ab \in I - \phi(I)$ for $a, b \in R$, then $a \in I$ or $b \in \sqrt{I}$. Clearly, every ϕ -prime ideal is a ϕ -primary ideal. Later, Badawi, Tekir, Ugurlu, Ulucak, and Celikel (2016) generalized the concept of 2-absorbing primary ideals to ϕ -2-absorbing primary ideals. According to their definition, a proper ideal I of R is said to be a ϕ -2-absorbing primary ideal of R if whenever $abc \in I - \phi(I)$ for $a, b, c \in R$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. Thus, a ϕ -primary ideal is ϕ -2-absorbing primary.

In 2004, Behboodi introduced the concepts of a classical prime submodule. A proper submodule N of an R -module M is said to be a classical prime submodule of M if whenever $abm \in N$ for $a, b \in R, m \in M$, then $am \in N$ or $bm \in N$. (see also Azizi, 2006; Azizi, 2008; Behboodi, 2006, in which, the notion of classical prime submodules is named “weakly prime submodules”). For more information on classical prime submodules, the reader is referred to (Arabi-Kakavand & Behboodi, 2014; Behboodi, 2007; Behboodi & Shojaei, 2010; Yilmaz & Cansu, 2014). Later, Baziar and Behboodi (2009) introduced the concepts of a classical primary submodule. According to their definition, a proper submodule N of M is said to be a classical primary submodule of M if whenever $abm \in N$ for $a, b \in R, m \in M$, then $am \in N$ or $b^n m \in M$ for some positive integer n . Clearly, every classical prime submodule is a classical primary. Also, Behboodi, Jahani-Nezhad, and Naderi (2011) introduced the concepts of a classical quasi-primary submodule. According to their definition, a proper submodule N of M is said to be a classical quasi-primary submodule of M if whenever $abm \in N$ for $a, b \in R, m \in M$, then $a^n m \in N$ or $b^n m \in N$ for some positive integer n . Thus, a classical primary submodule is classical quasi-primary. The notion of weakly classical primary submodules has been introduced and studied by Mostafanasab (2015). A proper submodule N of an R -module M is said to be a weakly classical primary submodule of M if whenever $0 \neq abm \in N$ for $a, b \in R, m \in M$, then $am \in N$ or $b^n m \in N$ for some positive integer n . Mostafanasab, Tekir, and Oral (2016) introduced the concepts of a weakly classical prime submodule. According to their definition, a proper submodule N of M is said to be a weakly classical prime submodule of M if whenever $0 \neq abm \in N$ for $a, b \in R, m \in M$, then $am \in N$ or $bm \in N$.

Zamani (2010), generalized the concept of prime and weakly prime submodules to ϕ -prime submodules. Let $\phi: \mathcal{S}(M) \rightarrow \mathcal{S}(M) \cup \{\emptyset\}$ be a function where $\mathcal{S}(M)$ is the set of all submodules of M . Recall that a proper submodule N of M is called a ϕ -prime submodule of M as in Zamani (2010) if whenever $am \in N - \phi(N)$ for $a \in R, m \in M$, then $m \in N$ or $a \in (N:M)$. Also, Ebrahimpour and Mirzaee in (2017), generalized the concept of semiprime and weakly semiprime submodules to ϕ -semiprime submodules. According to their definition, a proper submodule N of M is said to be a ϕ -semiprime submodule of M if whenever $a^2 m \in N - \phi(N)$ for $a \in R, m \in M$, then $am \in N$.

Motivated and inspired by the above works, the purposes of this paper are to introduce generalizations of classical primary submodule to the context of ϕ -classical primary submodule. A proper submodule N of M is said to be a ϕ -classical primary submodule of M if whenever $abm \in N - \phi(N)$ for $a, b \in R, m \in M$, then $am \in N$ or $b^n m \in N$ for some positive integer n . Some characterizations of classical primary and ϕ -classical primary submodules are obtained. We show that N is a ϕ -classical primary submodule of M if and only if for every $m \in M - N$, $(N:m)$ is a ϕ -primary ideal of R with $(\phi(N):m) = \phi(N:m)$. Moreover, we investigate relationships between classical primary, ϕ -classical primary and ϕ -primary submodules of modules over commutative rings. Finally, we obtain necessary and sufficient conditions of a ϕ -classical primary submodule in order to be a ϕ -primary submodule.

2. Some basic properties of ϕ -classical primary submodules

The results of the following theorems seem to play an important role to study ϕ -classical primary submodules of modules over commutative rings; these facts will be used frequently and normally, we shall make no reference to this definition.

Definition 2.1. Let M be an R -module and let $\phi: S(M) \rightarrow S(M) \cup \{\emptyset\}$ be a function where $S(M)$ be a set of all submodules of M . A proper submodule N of M is called a ϕ -classical primary submodule, if for each $m \in M$, $a, b \in R$ with $abm \in N - \phi(N)$, then $am \in N$ or $b^n m \in N$ for some positive integer n .

Remark 2.2. It is easy to see that every classical primary submodule is ϕ -classical primary.

The following example shows that the converse of Remark 2.2 is not true.

Example 2.3. Let $R = \mathbf{Z}$ and $M = \mathbf{Z}_2 \times \mathbf{Z}_3 \times \mathbf{Z}$. Clearly, M is an R -module. Consider the submodule $N = \{([0], [0], 0)\}$ of an R -module M . Define $\phi: S(M) \rightarrow S(M) \cup \{\emptyset\}$ by $\phi(K) = \{([0], [0], 0)\}$ for every submodule K of M . It is easy to see that N is a ϕ -classical primary submodule of M . Notice that $2 \cdot 3([1], [1], 0) \in \{([0], [0], 0)\}$, but $2([1], [1], 0) \notin \{([0], [0], 0)\}$ and $3^n([1], [1], 0) \notin \{([0], [0], 0)\}$ for all positive integer n . Therefore, N is not a classical primary submodule of M .

Throughout the rest of this paper, M is an R -module and $\phi: S(M) \rightarrow S(M) \cup \{\emptyset\}, \varphi: I(R) \rightarrow I(R) \cup \{\emptyset\}$ are functions. Since $N - \phi(N) = N - (N \cap \phi(N))$ and $I - \phi(I) = I - (I \cap \phi(I))$ for $N \in S(M), I \in I(R)$, without loss of generality, we will assume that $\phi(N) \subseteq N$ and $\phi(I) \subseteq I$. Theorem 2.4. Let M be an R -module. Then, the following statements hold:

- (1) If N is a ϕ -classical primary submodule of M , then $(N:m)$ is a ϕ -primary ideal of R for every $m \in M - N$ with $(\phi(N):m) \subseteq \phi(N:m)$.
- (2) If $(N:m) \subseteq (\phi(N):m)$ and $(N:m)$ is a ϕ -primary ideal of R for every $m \in M - N$, then N is a ϕ -primary submodule of M .

Proof 1. Let $a, b \in R$ such that $ab \in (N:m) - \phi(N:m)$. Then $abm \in N$ and $ab \notin \phi(N:m)$. Since $(\phi(N):m) \subseteq \phi(N:m)$, we have $ab \notin (\phi(N):m)$. Clearly, $abm \in N - \phi(N)$. By assumption, $am \in N$ or $b^n m \in N$ for some positive integer n . Therefore, $a \in (N:m)$ or $b^n \in (N:m)$ for some positive integer n . Hence, $(N:m)$ is a ϕ -primary ideal of R .

2. Let $a, b \in R$ and $m \in M - N$ such that $abm \in N - \phi(N)$. Then $ab \in (N:m)$ and $ab \notin (\phi(N):m)$. Since $(\phi(N):m) \subseteq (\phi(N):m)$, we have $ab \notin \phi(N:m)$. It is clear that $ab \in (N:m) - \phi(N:m)$. By hypothesis, $a \in (N:m)$ or $b^n \in (N:m)$ for some positive integer n . Clearly, $am \in N$ or $b^n m \in N$ for some positive integer n . Hence, N is a ϕ -classical primary submodule of M . \square

The following example shows that the converse of Theorem 2.4 is not true.

Example 2.5 1. Let $M = \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}$ be a \mathbf{Z} -module. Define $\varphi: I(R) \rightarrow I(R) \cup \{\emptyset\}$ by $\varphi(I) = \begin{cases} \{0\}; I = \{0\} \\ 2\mathbf{Z}; I = 2\mathbf{Z} \\ 3\mathbf{Z}; I = 3\mathbf{Z} \\ \emptyset; \text{otherwise} \end{cases}$

for every ideal I of R . Consider the submodule $N = \{0\} \times 2\mathbf{Z} \times 3\mathbf{Z}$ of M . Clearly, $(N:(m_1, m_2, m_3))$ is a ϕ -primary ideal of R , where $(m_1, m_2, m_3) \in M - N$. Define $\phi: S(M) \rightarrow S(M) \cup \{\emptyset\}$ by $\phi(K) = \{(0, 0, 0)\}$ for every submodule K . Notice that $2 \cdot 3(0, 1, 1) \in N - \phi(N)$, but $2(0, 1, 1) \notin N$ and $3^n(0, 1, 1) \notin N$ for all positive integer n . Therefore, N is not a ϕ -classical primary submodule of M .

2. Let $M = \mathbf{Z}_{10}$ be an \mathbf{Z}_{10} -module. Define $\phi: S(M) \rightarrow S(M) \cup \{\emptyset\}$ by $\phi(K) = \{[0]\}$ for every submodule K . Consider the submodule $N = \{[0]\}$ of an R -module M . Clearly, N is a ϕ -classical primary submodule of M . Define $\phi: I(R) \rightarrow I(R) \cup \{\emptyset\}$ by $\phi(I) = \emptyset$ for every ideal I of R . Notice that $[2][5] \in \{[0]\} = (N:[1]) - \phi(N:[1])$, but $[2] \notin (N:[1])$ and $[5]^n \notin (N:[1])$ for all positive integer n . Therefore, $(N:[1])$ is not a ϕ -primary ideal of R .

THEOREM 2.6 Let $(\phi(N):m) = \phi(N:m)$ for all $m \in M - N$. Then N is a ϕ -classical primary submodule of M if and only if $(N:m)$ is a ϕ -primary ideal of R for all $m \in M - N$.

Proof It is clear from Theorem 2.4. □

THEOREM 2.7 If N is a ϕ -classical primary submodule of an R -module M , then $(N:r) = \{m \in M: rm \in N\}$ is a ϕ -classical primary submodule of M for every $r \in R - (N:M)$ with $(\phi(N):r) \subseteq \phi(N:r)$.

Proof Let $a, b \in R$ and $m \in M$ such that $abm \in (N:r) - \phi(N:r)$. Then, $rabm \in N$ and $abm \notin \phi(N:r)$. Since $(\phi(N):r) \subseteq \phi(N:r)$, we have $rabm \in N - \phi(N)$. By assumption, $arm \in N$ or $b^nm \in N$ for some positive integer n . Therefore, $am \in (N:r)$ or $b^nm \in N \subseteq (N:r)$ for some positive integer n . Hence, $(N:r)$ is a ϕ -classical primary submodule of M . □

Remark 2.8 Let N be a ϕ -classical primary submodule of M , and $r_1 \in R - (N:M), r_2 \in R - ((N:r_1):M), \dots$ with $(\phi(N):r_1) \subseteq \phi(N:r_1), ((\phi(N):r_1):r_2) \subseteq \phi((N:r_1):r_2), \dots$. Then $(N:r_1), ((N:r_1):r_2), \dots$ are ϕ -classical primary submodules of M and $N \subseteq (N:r_1) \subseteq ((N:r_1):r_2) \subseteq \dots$

A submodule N of an R -module M is said to be irreducible if N is not the intersection of two submodules of M which properly contain it.

Proposition 2.9 Let N be an irreducible submodule of an R -module M . For every $r \in R$ if $(N:r) = (N:r^2)$, then N is a ϕ -classical primary submodule of M .

Proof Let $a, b \in R$ and $m \in M$ such that $abm \in N - \phi(N)$. Suppose that $am \notin N$ and $b^nm \notin N$ for all positive integer n . Clearly, $N \subseteq (N + Ram) \cap (N + Rb^nm)$ for all positive integer n . Let $m_0 \in (N + Ram) \cap (N + Rb^nm)$. This implies that $m_0 \in N + Ram$ and $m_0 \in N + Rb^nm$. Then, there exist $r_1, r_2 \in R$ and $n_1, n_2 \in N$ such that $n_1 + r_1am = m_0 = n_2 + r_2b^nm$. Since $an_1 + r_1a^2m = am_0 = an_2 + ar_2b^nm$, we have $a^2r_1m \in N$. It follows that $r_1m \in (N:a^2)$. By the assumption, $r_1m \in (N:a)$. So, $r_1am \in N$. Thus, $N = (N + Ram) \cap (N + Rb^nm)$. Now since N is an irreducible of M , we have $am \in N + Ram \subseteq N$ or $b^nm \in N + Rb^nm \subseteq N$, a contradiction. Hence, N is a ϕ -classical primary submodule of M . □

COROLLARY 2.10 Let R be a Boolean ring. If N is an irreducible submodule of M , then N is a ϕ -classical primary submodule of M .

Proof It is clear from Proposition 2.9. □

THEOREM 2.11 Let M, M' be two R -modules and let $f: M \rightarrow M'$ be a homomorphism. Suppose that $\phi': S(M') \rightarrow S(M') \cup \{\emptyset\}$ is a function. Then, the following statements hold:

- (1) If N' is a ϕ' -classical primary submodule of M' and $f(\phi(N)) \subseteq \phi'(f(N))$, then $f^{-1}(N')$ is a ϕ -classical primary submodule of M .
- (2) Let f be surjectivity. If N is a ϕ -classical primary submodule of M and $f(\phi(N)) \subseteq \phi'(f(N))$, then $f(N)$ is a ϕ' -classical primary submodule of M' .

Proof 1. Let $a, b \in R$ and $m \in M$ such that $abm \in f^{-1}(N') - \phi(f^{-1}(N'))$. Since f is homomorphism, $abf(m) = f(abm) \in N'$. Clearly, $abf(m) = f(abm) \notin \phi'(N')$ so $abf(m) \in N' - \phi'(N')$. By assumption, $f(am) = af(m) \in N'$ or $f(b^nm) = b^nf(m) \in N'$ for some positive integer n . Thus $am \in f^{-1}(N')$ or $b^nm \in f^{-1}(N')$. Therefore, $f^{-1}(N')$ is a ϕ -classical primary submodule of M .

2. Let $a, b \in R$ and $m' \in M'$ such that $abm' \in f(N) - \phi'(f(N))$. Since f is surjectivity, there exists $m \in M$ such that $m' = f(m)$. Therefore, $f(abm) = abf(m) \in f(N)$. So $abm \in N$. Clearly, $abm \notin \phi(N)$. It implies that $abm \in N - \phi(N)$. By assumption, $am \in N$ or $b^n m \in N$ for some positive integer n . Thus, $am' \in f(N)$ or $b^n m' \in f(N)$. Hence, $f(N)$ is a ϕ' -classical primary submodule of M' . \square

Let N be a submodule of an R -module M and let $\phi: S(M) \rightarrow S(M) \cup \{\emptyset\}$ be a function. Define $\phi_N: S(M/N) \rightarrow S(M/N) \cup \{\emptyset\}$ by

$$\phi_N(K/N) = \begin{cases} (\phi(K) + N)/N & ; \phi(K) \neq \emptyset \\ \emptyset & ; \phi(K) = \emptyset, \end{cases}$$

for every submodule K of M with $N \subseteq K$ (Zamani, 2010). Zamani (2010) gives relations between ϕ -prime submodules of M and ϕ_N -prime submodules of M/N . This leads us to give relations between ϕ -classical primary submodules of M and ϕ_N -classical primary submodules of M/N .

THEOREM 2.12 Let N, K be two submodules of M . If K is a ϕ -classical primary submodule of M , then K/N is a ϕ_N -classical primary submodule of M/N .

Proof Let $a, b \in R$ and $m \in M$ such that $ab(m + N) \in (K/N) - \phi_N(K/N) = (K/N) - (\phi(K) + N)/N = (K - \phi(K))/N$. Clearly, $abm \in K - \phi(K)$. By assumption, $am \in K$ or $b^n m \in K$ for some positive integer n . Therefore, $a(m + N) \in K/N$ or $b^n(m + N) \in K/N$ for some positive integer n . Hence, K/N is a ϕ_N -classical primary submodule of M/N . \square

THEOREM 2.13 Let N, K be two submodules of M . If K/N is a ϕ_N -classical primary submodule of M/N , then K is a ϕ -classical primary submodule of M .

Proof Let $a, b \in R$ and $m \in M$ such that $abm \in K - \phi(K)$. Then, $ab(m + N) = abm + N \in (K - \phi(K))/N = K/N - (\phi(K) + N)/N = (K/N) - \phi_N(K/N)$. By the given hypothesis, $a(m + N) \in K/N$ or $b^n(m + N) \in K/N$ for some positive integer n . Thus, $am \in K$ or $b^n m \in K$ for some positive integer n . Hence, K is a ϕ -classical primary submodule of M . \square

Now, by Theorem 2.12 and Theorem 2.13, we have the following corollary.

COROLLARY 2.14 Let N, K be two submodules of M . Then K is a ϕ -classical primary submodule of M if and only if K/N is a ϕ_N -classical primary submodule of M/N .

Proof The proof is similar to Theorems 2.12, 2.13 and so the details are left to the reader. \square

3. Properties of ϕ -classical primary submodules

Let S be a multiplicatively closed set in R and let T be a set of all pairs (x, s) , where $x \in M, s \in S$. Define a relation on T by $(x, s) \sim (x', s')$ if and only if there exists $t \in S$ such that $t(sx' - s'x) = 0$. Then \sim is an equivalence relation on T (Larsen & McCarthy, 1971). For $(a, s) \in M \times S$, denote the equivalence class of \sim which contains (a, s) by $\frac{a}{s}$ and denote a set of all equivalence classes of \sim by $S^{-1}M$. Then $S^{-1}M$ can be given the structure of an $S^{-1}R$ -module under operations for which $\frac{m}{s} + \frac{n}{t} = \frac{tm+sn}{st}, \frac{a}{s} \frac{m}{t} = \frac{am}{st}$ for all $m, n \in M$ and $s, t \in S, a \in R$. The $S^{-1}R$ -module $S^{-1}M$ is called the module of fractions of M with respect to S , its zero element is $\frac{0}{1}$ and this is equal to $\frac{0}{s}$ for all $s \in S$ (Larsen & McCarthy, 1971). We know that every submodule of $S^{-1}M$ is of the form $S^{-1}N$ for some submodule N of M (Sharp, 2000).

Let S be a multiplicatively closed set in R and let $\phi: S(M) \rightarrow S(M) \cup \{\emptyset\}$ be a function. Define $\phi_S: S(S^{-1}M) \rightarrow S(S^{-1}M) \cup \{\emptyset\}$ by

$$\phi_S(S^{-1}N) = \begin{cases} S^{-1}(\phi(N)) & ;\phi(N) \neq \emptyset \\ \emptyset & ;\phi(N) = \emptyset. \end{cases}$$

Zamani (2010) gives relations between ϕ -prime submodules of M and ϕ_S -prime submodules of $S^{-1}M$. This leads us to give relations between ϕ -classical primary submodules of M and ϕ_S -classical primary submodules of $S^{-1}M$.

THEOREM 3.1 *Let S be a multiplicatively closed subset of R . If N is a ϕ -classical primary submodule of M with $S \cap (N:M) = \emptyset$ then $S^{-1}N$ is a ϕ_S -classical primary submodule of $S^{-1}M$.*

Proof Since $S \cap (N:M) = \emptyset$, we have $S^{-1}N$ is a proper submodule of $S^{-1}M$. Let $a_1, a_2 \in R, s_1, s_2, s_3 \in S$ and $m \in M$ such that $\frac{a_1 a_2 m}{s_1 s_2 s_3} \in S^{-1}N - \phi_S(S^{-1}N)$. Then there exists $s \in S$ such that $sa_1a_2m \in N$. If $sa_1a_2m \in \phi(N)$, then $\frac{a_1 a_2 m}{s_1 s_2 s_3} = \frac{sa_1a_2m}{ss_1s_2s_3} \in S^{-1}\phi(N) = \phi_S(S^{-1}N)$, a contradiction. Now if $sa_1a_2m \notin \phi(N)$, then $sa_1a_2m \in N - \phi(N)$. By assumption, $a_1sm \in N$ or $a_2sm \in N$ for some positive integer n . Thus, $\frac{a_1 m}{s_1 s_3} = \frac{sa_1m}{ss_1s_3} \in S^{-1}N$ or $\left(\frac{a_2}{s_2}\right)^n \frac{m}{s_3} = \frac{sa_2^n m}{ss_2^n s_3} \in S^{-1}N$ for some positive integer n . Hence, $S^{-1}N$ is a ϕ_S -classical primary submodule of $S^{-1}M$. □

THEOREM 3.2 *Let S be a multiplicatively closed subset of R . If $S^{-1}N$ is a ϕ_S -classical primary submodule of $S^{-1}M$ such that $S \cap Zd(N/\phi(N)) = \emptyset, S \cap Zd(M/N) = \emptyset$, then N is a ϕ -classical primary submodule of M .*

Proof Since $S \cap Zd(N/\phi(N)) = \emptyset$, we have N as a proper submodule of M . Let $a, b \in R$ and $m \in M$ such that $abm \in N - \phi(N)$. Then, $\frac{a b m}{1 1 1} = \frac{abm}{1} \in S^{-1}N$. If $\frac{a b m}{1 1 1} \in \phi_S(S^{-1}N) = S^{-1}\phi(N)$, then there exists $s \in S$ such that $sabm \in \phi(N)$ which is a contradiction. If $\frac{a b m}{1 1 1} \notin \phi_S(S^{-1}N)$, then $\frac{a b m}{1 1 1} \in S^{-1}N - \phi_S(S^{-1}N)$. By assumption, $\frac{a m}{1 1 1} \in S^{-1}N$ or $\left(\frac{b}{1}\right)^n \frac{m}{1} \in S^{-1}N$ for some positive integer n . If $\frac{a m}{1 1 1} \in S^{-1}N$, there exists $s \in S$ such that $sam \in N$. Thus, $s(am + N) = sam + N = N$. Since $S \cap Zd(M/N) = \emptyset$, we have $am \in N$. Now if $\left(\frac{b}{1}\right)^n \frac{m}{1} \in S^{-1}N$, then it is clear that $b^n m \in N$. Hence, N is a ϕ -classical primary submodule of M . □

In view of Theorem 3.1 and Theorem 3.2, we have the following result.

COROLLARY 3.3 *Let S be a multiplicatively closed subset of a commutative ring R and let $S \cap (N:M) = \emptyset, S \cap Zd(N/\phi(N)) = \emptyset, S \cap Zd(M/N) = \emptyset$ for $N \in S(M)$. Then, N is a ϕ -classical primary submodule of M if and only if $S^{-1}N$ is a ϕ_S -classical primary submodule of $S^{-1}M$.*

Proof The proof is similar to Theorem 3.1 and Theorem 3.2. □
 In the following result, we give an equivalent definition of ϕ -classical primary submodules.

THEOREM 3.4 *Let N be a proper submodule of M . The following conditions are equivalent:*

- (1) N is a ϕ -classical primary submodule of M .
- (2) For every $a, b \in R, (N:ab) \subseteq (\phi(N):ab) \cup (N:a) \cup (N:b^n)$ for some positive integer n .

Proof (1 \Rightarrow 2) Let $m \in (N:ab)$. Then $abm \in N$. If $abm \in \phi(N)$, then $m \in (\phi(N):ab) \subseteq (\phi(N):ab) \cup (N:a) \cup (N:b^n)$. If $abm \notin \phi(N)$, then $abm \in N - \phi(N)$. By assumption, $am \in N$ or $b^n m \in N$ for some positive integer n . Therefore, $m \in (N:a)$ or $m \in (N:b^n)$ for some positive integer n . Hence, $(N:ab) \subseteq (\phi(N):ab) \cup (N:a) \cup (N:b^n)$ for some positive integer n .

(2 \Rightarrow 1) Let $a, b \in R$ and $m \in M$ such that $abm \in N - \phi(N)$. Then, $m \in (N:ab)$ and $m \notin (\phi(N):ab)$. Since $(N:ab) \subseteq (\phi(N):ab) \cup (N:a) \cup (N:b^n)$ for some positive integer n , we have $m \in (N:a) \cup (N:b^n)$ for some positive integer n . Clearly, $am \in N$ or $b^n m \in N$ for some positive integer n . Hence, N is a ϕ -classical primary submodule of M . □

THEOREM 3.5. *Let N be a proper submodule of M . The following conditions are equivalent:*

- (1) N is a ϕ -classical primary submodule of M .
- (2) For every $a \in R$ and $m \in M$ if $a^n m \notin N$ for all positive integer n , then $(N:am) = (\phi(N):am) \cup (N:m)$.
- (3) For every $a \in R$ and $m \in M$ if $a^n m \notin N$ for all positive integer n , then $(N:am) = (\phi(N):am)$ or $(N:am) = (N:m)$.

Proof (1 \Rightarrow 2) Clearly, $(\phi(N):am) \cup (N:m) \subseteq (N:am)$. On the other hand, let $r \in (N:am)$. Then $m \in (N:ar)$. Thus by Theorem 3.4, $m \in (\phi(N):ar)$ or $m \in (N:r)$ or $m \in (N:a^{n_1})$ for some positive integer n_1 . This implies that $r \in (\phi(N):am)$ or $a^{n_1} m \in N$ or $r \in (N:m)$. Since $a^n m \notin N$, we have $r \in (\phi(N):am) \cup (N:m)$. Therefore, $(N:am) \subseteq (\phi(N):am) \cup (N:m)$ and hence $(N:am) = (\phi(N):am) \cup (N:m)$.

(2 \Rightarrow 3) By the fact that if an ideal (a subgroup) is the union of two ideals (two subgroups), then it is equal to one of them.

(3 \Rightarrow 1) It is obvious. □

THEOREM 3.6 Let N be a proper submodule of M . The following conditions are equivalent:

- (1) N is a ϕ -classical primary submodule of M .
- (2) For every $a \in R$, $m \in M$ and every ideal I of R if $aIm \subseteq N - \phi(N)$, then $Im \subseteq N$ or $a^n m \in N$ for some positive integer n .
- (3) For every $m \in M$ and every ideal I of R if $Im \not\subseteq N$, then $(N:Im) = (\phi(N):Im)$ or $\sqrt{(N:Im)} = \sqrt{(N:m)}$.
- (4) For every ideals I, J of R if $IJm \subseteq N - \phi(N)$, then $Im \subseteq N$ or $J \subseteq \sqrt{(N:m)}$.

Proof (1 \Rightarrow 2) Let $a \in R$, $m \in M$ and let I be an ideal of R such that $aIm \subseteq N - \phi(N)$. Then $I \subseteq (N:am)$ and $I \not\subseteq (\phi(N):am)$. If $a^n m \in N$ for some positive integer n , then we are done. Let $a^n m \notin N$ for all positive integer n . Therefore by Theorem 3.5, we have $I \subseteq (N:am) = (N:m)$, i.e. $Im \subseteq N$.

(2 \Rightarrow 3) Let I be an ideal of R and $m \in M$ such that $Im \not\subseteq N$. It is easy to see that $(\phi(N):Im) \subseteq (N:Im)$. Assume that $r \in (N:Im)$. Then $rIm \subseteq N$. If $rIm \subseteq \phi(N)$, then $r \in (\phi(N):Im)$ so $(N:Im) = (\phi(N):Im)$. Now if $rIm \not\subseteq \phi(N)$, then $rIm \subseteq N - \phi(N)$. By assumption, $Im \subseteq N$ or $r^n m \in N$ for some positive integer n . Since $Im \not\subseteq N$, we have $r^n m \in N$. Therefore, $r \in \sqrt{(N:m)}$. Clearly, $(N:Im) \subseteq \sqrt{(N:m)}$. Hence $\sqrt{(N:Im)} = \sqrt{(N:m)}$.

(3 \Rightarrow 4) Suppose that $IJm \subseteq N - \phi(N)$, where I, J are ideals of R . Let $Im \not\subseteq N$. By assumption, i.e. $(N:Im) = (\phi(N):Im)$ or $\sqrt{(N:Im)} = \sqrt{(N:m)}$. Since $IJm \subseteq N - \phi(N)$, we have $J \subseteq (N:Im)$ and $J \not\subseteq (\phi(N):Im)$. This implies that $J \subseteq (N:Im) \subseteq \sqrt{(N:Im)} = \sqrt{(N:m)}$.

(4 \Rightarrow 1) It is obvious. □

Let M be an R -module. The M is called a multiplication module if every submodule N of M has the form IM for some ideal I of R (El-Bast & Smith, 1988). Note that, since $I \subseteq (N:M)$ then $N = IM \subseteq (N:M)M \subseteq N$. Thus $N = (N:M)M$. Let N_1 and N_2 be two submodules of M with $N = I_1M$ and $N_2 = I_2M$ for some ideals I_1 and I_2 of R . A product of N_1 and N_2 denoted by N_1N_2 is defined by $N_1N_2 = I_1I_2M$. The following theorem offers a characterization of ϕ -classical primary submodules.

THEOREM 3.7 Let R be a noetherian ring and let N be a proper submodule of a multiplication R -module M . Then the following conditions are equivalent:

- (1) N is a ϕ -classical primary submodule of M .
- (2) If $K_1K_2 \subseteq N - \phi(N)$ for some submodules K_1, K_2 of M , then $K_1 \subseteq N$ or $K_2^n \subseteq N$ for some positive integer n .

Proof (1 \Rightarrow 2) Suppose that K_1, K_2 are submodules of M . Since M is multiplication, there are ideals I_1, I_2 of R such that $K_1 = I_1M$ and $K_2 = I_2M$. Let $m \in M$. Then $I_1I_2m \subseteq I_1I_2M = K_1K_2 \subseteq N - \phi(N)$. By Theorem 3.6, i.e. $I_1m \subseteq N$ or $I_2 \subseteq \sqrt{(N:m)}$. Therefore $I_1M \subseteq N$ or $I_2^nM \subseteq N$ for some positive integer n . Hence, $K_1 \subseteq N$ or $K_2^n \subseteq N$ for some positive integer n .

(2 \Rightarrow 1) Let $m \in M$ and $I_1mI_2m = I_1I_2m \subseteq N - \phi(N)$ for some ideals I_1, I_2 of R . Thus by part 2, i.e. $I_1m \subseteq N$ or $I_2^nM \subseteq N$ for some positive integer n . Thus $I_1M \subseteq N$ or $I_2 \subseteq \sqrt{(N:M)}$. By Theorem 3.6, N is ϕ -classical primary submodule of M . \square

We are finding additional condition to show that a classical primary submodule is a ϕ -classical primary submodule of an R -module M .

THEOREM 3.8 Let $\phi(N)$ be a classical primary submodule of M . Then N is a ϕ -classical primary submodule of M if and only if N is a classical primary submodule of M .

Proof Suppose that N is a classical primary submodule of M . Clearly, N is a ϕ -classical primary submodule of M . Conversely, assume that N is a ϕ -classical primary submodule of M . Let $a, b \in R$ and $m \in M$ such that $abm \in N$. If $abm \notin \phi(N)$, then $abm \in N - \phi(N)$. By assumption, $am \in N$ or $b^n m \in N$ for some positive integer n . Now if $abm \in \phi(N)$, then $am \in \phi(N) \subseteq N$ or $b^n m \in \phi(N) \subseteq N$ for some positive integer n . Hence, N is a classical primary submodule of M . \square

Definition 3.9 (Bataineh & Khuhail, 2011) Let M be an R -module and let $\phi: S(M) \rightarrow S(M) \cup \{\emptyset\}$ be a function where $S(M)$ be a set of all submodules of M . A proper submodule N of M is called a ϕ -primary submodule, if for each $m \in M, a \in R$ with $am \in N - \phi(N)$, then $m \in N$ or $a \in \sqrt{(N:M)}$.

Remark 3.10 It is easy to see that every ϕ -primary submodule is ϕ -classical primary.

The following example shows that the converse of Remark 3.10 is not true.

Example 3.11 Let $R = \mathbf{Z}$ and $M = \mathbf{Z} \times \mathbf{Z}$. Consider the submodule $N = \{0\} \times 4\mathbf{Z}$ of M . Define $\phi: S(M) \rightarrow S(M) \cup \{\emptyset\}$ by $\phi(K) = \{(0, 0)\}$ for every submodule K of M . It is easy to see that N is a ϕ -classical primary submodule of M . Notice that $4(0, 1) \in \{0\} \times 4\mathbf{Z}$, but $(0, 1) \notin \{0\} \times 4\mathbf{Z}$ and $4 \notin \sqrt{(\{0\} \times 4\mathbf{Z} : \mathbf{Z} \times \mathbf{Z})}$. Therefore N is not a ϕ -primary submodule of M .

We provide some relationships between ϕ -classical primary submodules of an R -module M and ϕ -primary submodule of M . However, these results require that M be a cyclic R -module.

THEOREM 3.12 Let M be a cyclic R -module. If N is a ϕ -classical primary submodule of M , then N is a ϕ -primary submodule of M .

Proof Let $r \in R$ and $m_0 \in M = Rm$ for some $m \in M$ such that $rm_0 \in N - \phi(N)$. Then there exists $s \in R$ such that $m_0 = sm$. Therefore, $rs m = rm_0 \in N - \phi(N)$. By assumption, $sm \in N$ or $r^n m \in N$ for some positive integer n . Thus, $m_0 \in N$ or $r^n \in (N:M)$. Hence, N is a ϕ -primary submodule of M . \square

Now, the following result follows immediately from Theorem 3.12.

COROLLARY 3.13 Let M be a cyclic R -module. Then, N is a ϕ -primary submodule of M if and only if N is a ϕ -classical primary submodule of M .

Proof This makes the same assertion as Theorem 3.12. \square

Now, we are finding additional condition to show that a ϕ -primary submodule is a ϕ -classical primary submodule of an R -module M .

LEMMA 3.14 Let N be a ϕ -classical primary submodule of M and let $m_1 \in M$ and $m_2 \in M - N$ such that $\phi(N:m_2)$ is a primary ideal of R and $(\phi(N):m_2) \subseteq \phi(N:m_2)$. Then, $(N:m_2) = (N:rm_2)$ for all $r \in (N:m_1) - \sqrt{(N:m_2)}$.

Proof Let $a \in (N:m_2)$. Then, $a(rm_2) = r(am_2) \in N$ for all $r \in R$. This implies that $(N:m_2) \subseteq (N:rm_2)$. On the other hand, let $a \in (N:rm_2)$. Then $(ar)m_2 = a(rm_2) \in N$. Clearly, $ar \in (N:m_2)$. If $ar \notin \phi(N:m_2)$, then $ar \in (N:m_2) - \phi(N:m_2)$. By assumption, $a \in (N:m_2)$ or $r^n \in (N:m_2)$ for some positive integer n . Now by our hypothesis, $a \in (N:m_2)$. Thus, $(N:rm_2) \subseteq (N:m_2)$. Now if $ar \in \phi(N:m_2)$, then $a \in (N:m_2)$ or $r^n \in (N:m_2)$ for some positive integer n . Again, by the assumption, $a \in (N:m_2)$. Therefore $(N:rm_2) \subseteq (N:m_2)$ and hence $(N:m_2) = (N:rm_2)$. \square

THEOREM 3.15 Let N be a ϕ -classical primary submodule of an R -module M . For any $m_1 \in M$ and $m_2 \in M - N$ such that $\phi(N:m_2)$ is a primary ideal of R and $(N:m_1) - \sqrt{(N:m_2)} \neq \emptyset$, we have $N = (N + Rm_1) \cap (N + Rm_2)$.

Proof Let $m_1 \in M$ and $m_2 \in M - N$. Clearly, $N \subseteq (N + Rm_1) \cap (N + Rm_2)$. For another direction, we show that $(N + Rm_1) \cap (N + Rm_2) \subseteq N$. Let $m \in (N + Rm_1) \cap (N + Rm_2)$. This implies that $m \in N + Rm_1$ and $m \in N + Rm_2$. Then there exist $r_1, r_2 \in R$ and $n_1, n_2 \in N$ such that $n_1 + r_1m_1 = m = n_2 + r_2m_2$. Since $(N:m_1) - \sqrt{(N:m_2)} \neq \emptyset$, there exists $r \in (N:m_1) - \sqrt{(N:m_2)}$. Thus, $rm_1 \in N$. Clearly, $rr_1m_1 \in r_1N \subseteq N$. Now since $rm = rn_1 + rr_1m_1 = rn_2 + rr_2m_2$, we have $rr_2m_2 \in N$. It follows that $r_2 \in (N:rm_2)$. By Lemma 3.14, $r_2 \in (N:m_2)$ so that $r_2m_2 \in N$. Therefore, $m = n_2 + r_2m_2 \in N$ and hence $N = (N + Rm_1) \cap (N + Rm_2)$. \square

THEOREM 3.16 Let N be a submodule of M such that $\phi(N:m) \subseteq (\phi(N):m)$ for all $m \in M - N$. For any $m_1 \in M$ and $m_2 \in M - N$ such that $(N:m_1) - \sqrt{(N:m_2)} \neq \emptyset$, we have $N = (N + Rm_1) \cap (N + Rm_2)$. Then, N is a ϕ -classical primary submodule of M .

Proof Let $m \in M - N$. To show that $(N:m)$ is a ϕ -primary ideal of R . Let $a, b \in R$ such that $ab \in (N:m) - \phi(N:m)$. Assume that $b^n \in R - (N:m)$ for all positive integer n . Since $ab \in (N:m)$, we have $b \in (N:am)$. Clearly, $b \in (N:am) - \sqrt{(N:m)}$. This implies that $(N:am) - \sqrt{(N:m)} \neq \emptyset$. By the assumption, $N = (N + Ram) \cap (N + Rm)$. Now since $am \in (N + Ram) \cap (N + Rm)$, we have $am \in N$. Therefore $a \in (N:m)$. Thus, $(N:a)$ is a ϕ -primary ideal of R . By Theorem 2.4(2), N is a ϕ -classical primary submodule of M . \square

LEMMA 3.17 Let N be a ϕ -classical primary submodule of M and $m_0 \in M - N$ such that $\phi(N:m_0)$ is a primary ideal of R . For each $r \in R$, $m \in M$ and $m_0 \in M - N$ if $rm \in N - \phi(N)$, then $N = (N + Rm) \cap (N + Rr^nm_0)$ for some positive integer n .

Proof Let $r \in R$, $m \in M$ and $m_0 \in M - N$ such that $rm \in N - \phi(N)$. It is clear that $N \subseteq (N + Rm) \cap (N + Rr^nm_0)$ for any positive integer n . On the other hand, we show that $(N + Rm) \cap (N + Rr^nm_0) \subseteq N$ for some positive integer n . We divide our proof into two cases.

Case 1. If $r^nm_0 \in N$ for some positive integer n , then $Rr^nm_0 \subseteq N$. Therefore, $(N + Rm) \cap (N + Rr^nm_0) \subseteq N + Rr^nm_0 \subseteq N$.

Case 2. If $r^nm_0 \notin N$ for all positive integer n , then $r \notin \sqrt{(N:m_0)}$. Since $rm \in N$, we have $r \in (N:m)$. Clearly, $r \in (N:m) - \sqrt{(N:m_0)} \neq \emptyset$. By Theorem 3.15, $N = (N + Rm) \cap (N + Rm_0)$. Now since $(N + Rr^nm_0) \subseteq (N + Rm_0)$, we have $(N + Rm) \cap (N + Rr^nm_0) \subseteq (N + Rm) \cap (N + Rm_0) = N$. Therefore, $(N + Rm) \cap (N + Rr^nm_0) \subseteq N$. Hence, $N = (N + Rm) \cap (N + Rr^nm_0)$ for some positive integer n . \square

Proposition 3.18 Let N be a ϕ -classical primary submodule of M such that $(\phi(N):m)$ is a primary ideal of R for all $m \in M - N$. Let $m \in M - N$ be such that $(N:m) - \bigcup_{m_i \in M-N} \sqrt{(N:m_i)} \neq \emptyset$, then $N = (N + rm) \cap \bigcup_{m_i \in M-N} (N + Rm_i)$.

Proof The inclusion $N \subseteq (N + rm) \cap \bigcup_{m_i \in M-N} (N + Rm_i)$ is clear. For the other inclusion, let $x \in (N + rm) \cap \bigcup_{m_i \in M-N} (N + Rm_i)$. Then there exist $m' \in M - N$, $r_1, r_2 \in R$ and $n_1, n_2 \in N$ such that $n_1 + r_1 m' = x = n_2 + r_2 m'$. By hypothesis, there exists $a \in (N:m)$ such that $a \notin \sqrt{(N:m')}$. Since $an_1 + ar_1 m' = ax = an_2 + ar_2 m'$, we have $ar_2 m' \in N$. If $ar_2 m' \notin \phi(N)$, then $ar_2 m' \in N - \phi(N)$ and as N is a ϕ -classical primary submodule of M , we have $r_2 m' \in N$ or $a^n m' \in N$ for some positive integer n . Since $a \notin \sqrt{(I:m')}$, it follows that $r_2 m_i \in N$. On the other hand, if $ar_2 m_i \in \phi(N)$, then $ar_2 \in (\phi(N):m_i)$. Since $(\phi(N):m)$ is a primary ideal of R and $a \notin \sqrt{(N:m')}$, it follows that $r_2 \in (\phi(N):m')$ so

$r_2 m_i \in \phi(N) \subseteq N$. In either case, we have $x = n_2 + r_2 m' \in N$. This shows that $N = (N + rm) \cap \bigcup_{m_i \in M-N} (N + Rm_i)$. □

Now, the following result follows immediately from Proposition 3.18

COROLLARY 3.19 Let N is a ϕ -classical primary submodule of M such that $(\phi(N):m)$ is a ϕ -primary ideal of R where $m \in M - N$. If $(N:m) - \bigcup_{m_i \in M-N} \sqrt{(N:m_i)} \neq \emptyset$, then N is not irreducible.

Proof By Proposition 3.17, $N = (N + rm) \cap \bigcup_{m_i \in M-N} (N + Rm_i)$. Since $m \in M - N$, we have $N \subseteq N + rm$ and $N \subseteq \bigcap_{m_i \in M-N} (N + Rm_i)$. Hence N is not irreducible. □

We are finding additional condition to show that a ϕ -primary submodule is a ϕ -classical primary submodule of an R -module. Theorem 3.20 Let N be an irreducible submodule of an R -module M . The followings are equivalent.

- (1) For $m_2 \in M - N$ if $(\phi(N):m_2)$ is a primary ideal of R , then N is a ϕ -primary submodule of M .
- (2) N is a ϕ -classical primary submodule of M .

Proof (1 \Rightarrow 2) It is obvious.

(2 \Rightarrow 1) Let $r \in R$ and $m \in M$ such that $rm \in N - \phi(N)$. Assume that $r^n \notin (N:M)$ for all positive integer n . By Lemma 3.16, i.e. $N = (N + Rm) \cap (N + Rr^n m_0)$ for some positive integer n . Since N is an irreducible submodule of M , we have $N = N + Rm$ or $N = N + Rr^n m_0$. Now since $r^n \notin (N:M)$, we have $N + Rr^n m_0 \not\subseteq N$. Therefore, $m \in Rm \subseteq N + Rm \subseteq N$ and hence N is a ϕ -primary submodule of M . □

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Author details

P. Yiarayong¹
 E-mail: pairote0027@hotmail.com
 ORCID ID: <http://orcid.org/0000-0003-1525-9536>
 M. Siripitukdet¹
 E-mail: manoj@snu.ac.th

¹ Faculty of Science, Department of Mathematics, Naresuan University, Phitsanuloke 65000, Thailand.

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