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*Corresponding author: Wenbin Lin, School of Physical Science and Technology, Southwest Jiaotong University, Chengdu 610031, China
E-mail: wl@swjtu.edu.cn

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ASTRONOMY & ASTROPHYSICS | RESEARCH ARTICLE

Does Birkhoff's theorem really hold?

Wenbin Lin^{1*}

Abstract: Birkhoff's theorem states that the external gravitational field of any spherically-symmetric system is static and is described by Schwarzschild metric. In this paper, we demonstrate that it is inconsistent with the direct post-Newtonian solution to Einstein field equations in the weak field limit. Moreover, we identify the flaw in the derivation of Birkhoff's theorem—the consequences of making the coordinate transformations for solving Einstein field equations are ignored, which is due to the misuse of the covariance in general relativity.

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1. Introduction

In the theory of general relativity, a non-static system usually produces a time-dependent gravitational field. However, Birkhoff's theorem (Birkhoff, 1923; Weinberg, 1972) states that any spherically-symmetric system must generate the static exterior gravitational field which is characterized by Schwarzschild metric. It follows from this theorem that the radial motion of a spherically-symmetric system does not have any gravitational effect.

In this work, we first study the exterior gravitational field of a system, in which the mass distribution is spherically-symmetric and there exist radial and rotational motions. This system will reduce



Wenbin Lin

ABOUT THE AUTHORS

Wenbin Lin graduated from Xiamen University in 1992, and received two PhD degrees, one in Astrophysics from University of Science & Technology of China in 1999 and the other in Electrical & Computer Engineering from Duke University in 2005. His research interests include general relativity and gravitation, cosmology, electromagnetic field and microwave technology, and scientific computing.

The research reported in this paper demonstrates that Birkhoff's theorem is not correct, which is resulted from the misuse of the covariance of general relativity. It is pointed out that the coordinate transformations taken in solving Einstein field equations may make the final solution not directly correspond to the original problem. This conclusion is very important, since the coordinate transformations have often been made for solving Einstein field equations in many literatures.

PUBLIC INTEREST STATEMENT

Before solving Einstein field equations, the energy and momentum distributions of gravitational source have to be specified. This implies that there exists a reference frame and the solution corresponds to this reference frame by default. It is usually believed that based on the general covariance one can make any coordinate transformation in solving Einstein field equations. However, the covariance in general relativity only means that the fundamental equations, e.g. Einstein field equations, take the same form for any space-time coordinates. It does not mean that the solution of Einstein field equations will be kept unchanged after making coordinate transformations, since in relativity the coordinate transformations include changes of reference frame. Therefore, the consequences of making any coordinate transformation should be taken into account when the resultant solution is related to the original problem, otherwise wrong conclusions may be reached. Here we take Birkhoff's theorem as an example to illustrate this important issue.

to a dynamic spherically-symmetric system when the rotational motion does not exist. The reason for including the rotational motion here is that it can produce a gravitational vector potential which has physical effects on the post-Newtonian dynamics (Weinberg, 1972), and we will show that the radial motion of the system, similar to the rotational one, can also cause a gravitational vector potential. Basing on the fact that the post-Newtonian solution to Einstein's field equations is unique once the harmonic-coordinate conditions are imposed, we conclude that Birkhoff's theorem is not consistent with the theory of post-Newtonian approximations, which has been well established (Will, 2011). Finally, we re-visit the derivation of Birkhoff's theorem and identify the flaw leading to the inconsistency.

This paper is organized as follows: Section 2 gives a brief review on the theory of post-Newtonian approximations. Section 3 presents the direct post-Newtonian solution for the spherically-symmetric mass system with radial and rotational motions, which reduces to the one for the spherically-symmetric system when the rotational motion is dropped. In Section 4 we give another derivation for the solution of the same problem. In Section 5 we discuss the inconsistency between Birkhoff's theorem and the post-Newtonian solution. The conclusions are given in Section 6.

2. Theory of post-Newtonian approximations

We start with Einstein field equations, which read

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi T_{\mu\nu}, \tag{1}$$

where $R_{\mu\nu}$ and R are Ricci tensor and Ricci scalar, $g_{\mu\nu}$ denotes metric tensor and $T_{\mu\nu}$ is matter energy-momentum tensor. Throughout this paper, we use the geometrized units, in which the speed of light in vacuum and the gravitational constant are unity. Greek indices run from 0 to 3 and Latin indices run from 1 to 3.

For a non-relativistic system, we can solve Einstein field equations via the method of post-Newtonian approximations, which does not rely on any assumption except that its energy-momentum tensor and the corresponding metric can be expanded with some small parameters (Weinberg, 1972). Let \bar{M} , \bar{v} , and \bar{r} represent the typical values of mass, velocity, and distance in a non-relativistic system. In the post-Newtonian approximations, the metric is expanded in the powers of \bar{v}^2 , which is roughly of the same order of the typical Newtonian potential $\bar{\phi} = -\bar{M}/\bar{r}$, as follows (Weinberg, 1972)

$$g_{00} = -1 + \overset{2}{g}_{00} + \overset{4}{g}_{00} \dots, \tag{2}$$

$$g_{0i} = \overset{3}{g}_{0i} + \overset{5}{g}_{0i} + \dots, \tag{3}$$

$$g_{ij} = \delta_{ij} + \overset{2}{g}_{ij} + \overset{4}{g}_{ij} + \dots, \tag{4}$$

where $i = 1, 2, 3$. δ_{ij} is Kronecker's delta, and the symbol $\overset{N}{g}_{\mu\nu}$ denotes the terms in $g_{\mu\nu}$ of order \bar{v}^N . The corresponding energy-momentum tensor is expanded as

$$T^{00} = \overset{0}{T}_{00} + \overset{2}{T}_{00} + \dots, \tag{5}$$

$$T^{0i} = \overset{1}{T}_{0i} + \overset{3}{T}_{0i} + \dots, \tag{6}$$

$$T^{ij} = \overset{2}{T}_{ij} + \overset{4}{T}_{ij} + \dots, \tag{7}$$

where the symbol $\overset{N}{T}_{\mu\nu}$ denotes the terms in $T^{\mu\nu}$ of order $(\bar{M}/\bar{r}^3)\bar{v}^N$.

In general, Einstein field equations have only six independent equations due to Bianchi equalities, while the metric tensor has ten independent components. Therefore, we need four additional conditions to fix the metric tensor. In the theory of post-Newtonian approximations, the harmonic-coordinate conditions are used to remove this redundancy. The harmonic-coordinate conditions can be written as

$$g^{\mu\nu}\Gamma_{\mu\nu}^{\lambda} = 0, \tag{8}$$

with $\Gamma_{\mu\nu}^{\lambda}$ being the affine connection.

Substituting Equations (2)–(7) into Equation (1), and making use of the harmonic-coordinate conditions Equation (8), we can obtain the solution to Einstein’s field equations in the weak-field limit as follows (Fock, 1959; Weinberg, 1972)

$$g_{00}^2(\mathbf{x}, t) = -2\phi = 2 \int \frac{T_{00}^0(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} d^3x', \tag{9}$$

$$g_{0i}^3(\mathbf{x}, t) = \zeta_i = -4 \int \frac{T_{0i}^1(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} d^3x', \tag{10}$$

$$g_{ij}^2(\mathbf{x}, t) = -2\phi \delta_{ij}. \tag{11}$$

where ϕ is Newtonian potential, and ζ_i denotes the i -component of a gravitational vector potential ζ , which is generated by mass current and has physical effects (Kopeikin & Fomalont, 2007; Kopeikin & Mashhoon, 2002; Sereno, 2002). It is worth emphasizing that the post-Newtonian solution (Equations (9)–(11)) is unique under the harmonic-coordinate conditions. The gravitational vector potential ζ plays important roles in the post-Newtonian dynamics, for example, it can cause the time delay and deflection for the light propagation. The gravitational effect due to the mass current is also called as gravitomagnetic effect (Kopeikin & Mashhoon, 2002).

3. Post-Newtonian solution for a spherically-symmetric mass system with radial and rotational motions

Now we apply the theory of post-Newtonian approximations to calculate the far field of a spherically-symmetric mass, with radial and rotational motions. The system’s center is at the origin of coordinates. The radius and density of the system are denoted by a and ρ . The radial velocity can be written as $\mathbf{v} = v\mathbf{x}'/|\mathbf{x}'|$, and the rotational angular velocity is denoted by $\boldsymbol{\omega}$. For simplicity, the rotation is assumed to be around with z -axis. ρ , v and $\boldsymbol{\omega}$ may be functions of the radial variable $r' \equiv |\mathbf{x}'|$ and the time t , and a may be dependent on t . The corresponding energy-momentum tensor of this system can be written as

$$T_{00}^0 = \begin{cases} \rho, & \text{for } r' \leq a, \\ 0, & \text{for } r' > a, \end{cases} \tag{12}$$

$$T_{0i}^1 = T_{00}^0 \left[\frac{\mathbf{x}'}{|\mathbf{x}'|} v + \boldsymbol{\omega} \times \mathbf{x}' \right]_i. \tag{13}$$

For the field point \mathbf{x} far away from the gravitational source point \mathbf{x}' , the denominator $|\mathbf{x} - \mathbf{x}'|$ in Equations (9)–(10) can be expanded as follow

$$|\mathbf{x} - \mathbf{x}'|^{-1} = \frac{1}{r} + \frac{\mathbf{x} \cdot \mathbf{x}'}{r^3} + \dots, \tag{14}$$

where $r \equiv |\mathbf{x}|$ is the distance from the coordinate origin to the field point.

Substituting Equations (12)–(13) into Equations (9)–(10), and making use of Equation (14), we can obtain the Newtonian and gravitational vector potentials for the spherically-symmetric mass system as follows

$$\phi = -\frac{1}{r} \underbrace{\left(4\pi \int_0^a \rho r'^2 dr' \right)}_M, \tag{15}$$

$$\zeta \approx -\underbrace{\frac{\mathbf{x}}{r^3} \left(\frac{16\pi}{3} \int_0^a v \rho r'^3 dr' \right)}_{\zeta_{\text{rad}}} + \underbrace{\frac{\mathbf{x}}{r^3} \times \left(\frac{16\pi}{3} \int_0^a \omega \rho r'^4 dr' \right)}_{\zeta_{\text{rot}}}, \tag{16}$$

where M denotes the non-relativistic mass of the system. The gravitational vector potential ζ includes two parts. In Equation (16), the first term is caused by the radial motion and denoted as ζ_{rad} . The second term is produced by the rotational motion and denoted as ζ_{rot} . Both ζ_{rad} and ζ_{rot} may be time-dependent. Notice that ζ_{rot} has a more familiar form of $\frac{2\mathbf{x} \times \mathbf{J}}{r^3}$ with \mathbf{J} being the angular momentum of system (Weinberg, 1972).

The gravitational vector potentials ζ_{rad} and ζ_{rot} are both predicted by the direct post-Newtonian solution. Moreover, Their magnitudes are also comparable, if the radial velocity and the rotational velocity have same order of magnitudes. When there does not exist the rotational motion, Equation (16) represents the gravitational vector potential due to the radial motion of a spherically-symmetric system. The physical effects of ζ_{rot} have been studied in detail (Kopeikin & Mashhoon, 2002; Sereno, 2002, 2005; Weinberg, 1972), however, due to Birkhoff’s theorem, the fact that the radial motion of a spherically-symmetric system can produce the gravitational vector potential ζ_{rad} has hitherto been overlooked.

4. Another derivation

In this section we give a more intuitive derivation for Equations (15) and (16).

The far fields of a moving particle with mass m , position \mathbf{x}' and velocity \mathbf{u} , can be characterized by the multipole fields as follows (He & Lin, 2014; Kopeikin & Fomalont, 2007; Weinberg, 1972; Wucknitz & Spherhake, 2004)

$$\phi^{(\text{particle})} = -\frac{m}{|\mathbf{x} - \mathbf{x}'|}, \tag{17}$$

$$\zeta^{(\text{particle})} = -\frac{4 m \mathbf{u}}{|\mathbf{x} - \mathbf{x}'|}, \tag{18}$$

where $\phi^{(\text{particle})}$ and $\zeta^{(\text{particle})}$ denote the Newtonian and gravitational vector potentials caused by the moving particle, respectively. The gravitomagnetic effect produced by the particle’s motion has been explored extensively (He & Lin, 2014; Klioner, 2003; Kopeikin, Polnarev, Schäfer, & Vlasov, 2007; Kopeikin & Schäfer, 1999; Wucknitz & Spherhake, 2004).

The spherically-symmetric mass system with radial and rotational motions can be thought to be composed of infinite particles with the mass of $m = \rho d^3 x'$ and the velocity of $\mathbf{u} = \frac{\mathbf{x}'}{|\mathbf{x}'|} v + \boldsymbol{\omega} \times \mathbf{x}'$. The total Newtonian and gravitational vector potentials of the system can be obtained by the summation of the corresponding contributions from all particles as follows

$$\phi = \sum_{\text{particle}} \phi^{(\text{particle})} = \int -\frac{\rho d^3x'}{|\mathbf{x} - \mathbf{x}'|} = -\frac{1}{r} \left(4\pi \int_0^a \rho r'^2 dr' \right), \quad (19)$$

$$\zeta = \sum_{\text{particle}} \zeta^{(\text{particle})} = \int -\frac{4\mathbf{u} \rho d^3x'}{|\mathbf{x} - \mathbf{x}'|} \approx -\frac{\mathbf{x}}{r^3} \left(\frac{16\pi}{3} \int_0^a v \rho r'^3 dr' \right) + \frac{\mathbf{x}}{r^3} \times \left(\frac{16\pi}{3} \int_0^a \omega \rho r'^4 dr' \right), \quad (20)$$

which are exactly the same as Equations (15) and (16). Here we have also made use of Equation (14). We can see that ζ_{rot} and ζ_{rad} originate from the same velocity \mathbf{u} of particles, and both are indispensable elements to characterize the far fields of the system.

Specifically, when the system has only radial motion, each particle has velocity $\mathbf{u} = \frac{x'}{|\mathbf{x}'|} \mathbf{v}$ and thus can produce a gravitational vector potential. The summation for the particles' gravitational vector potentials is *nonzero*, and it is these vector potentials that constitute the gravitational vector potential ζ_{rad} of the dynamic spherically-symmetric system.

5. Discussions

In the above we have achieved the post-Newtonian solution for the spherically-symmetric system in the harmonic coordinates, which in the metric form can be written as

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 - \zeta_{\text{rad}} \cdot d\mathbf{x} dt + \left(1 + \frac{2M}{r}\right) d\mathbf{x}^2. \quad (21)$$

Now we re-visit the derivation for Birkhoff's theorem. The most general metric for a time-dependent spherically-symmetric system should have the form (Weinberg, 1972)

$$ds^2 = -C(r, t) dt^2 + D(r, t) dr^2 + 2E(r, t) dr dt + r^2 F(r, t) (d\theta^2 + \sin^2 \theta d\varphi^2). \quad (22)$$

In the derivation for Birkhoff's theorem, before solving Einstein's field equations, some coordinate transformations have been applied on this metric to eliminate the time-spatial component and the to-be-determined metric is written in the following form (Weinberg, 1972)

$$ds^2 = -B(r, t) dt^2 + A(r, t) dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2). \quad (23)$$

Solving Einstein's field equations for this metric form, one obtains that the solution of the spherically-symmetric system is Schwarzschild metric. In the harmonic coordinates, the weak-field limit for Schwarzschild metric can be written as (Weinberg, 1972)

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 + \frac{2M}{r}\right) d\mathbf{x}^2, \quad (24)$$

which is different from Equation (21).

Someone might argue that the time-spatial component in Equation (21) can be eliminated by some coordinate transformations based on the general covariance of relativity, and the resultant metric will reduce to Equation (24). This viewpoint is misleading and not correct. The arguments are as follows:

- The post-Newtonian solution to Einstein field equations (Equations (9)–(11)) is unique when the harmonic-coordinate conditions are imposed, and this can be seen clearly from the derivations in page 212–219 of Weinberg's textbook (Weinberg, 1972). Equation (21) is the direct result from Equations (9)–(11), thus it is *the one and only one* solution to Einstein field equations with the harmonic-coordinate conditions in the weak-field limit for the spherically-symmetric system.

- General covariance means that the fundamental equations of general relativity, e.g. Einstein field equations, take the same form for any space-time coordinates. The covariance does not mean that the predictions of general relativity (the solutions to the Einstein field equations) will keep unchanged after making coordinate transformations, since in relativity the coordinate transformations include changes of reference frame.
- More specifically, according to the covariance of Einstein field equations (see Equation (1)), if we perform a coordinate transformation on the to-be-determined metric tensor $g_{\mu\nu}$ to eliminate its time-spatial component, we must perform the same coordinate transformation on the matter energy-momentum tensor $T_{\mu\nu}$. In this case, the components of the matter energy-momentum tensor will be changed, then if we solve Einstein field equations for this new matter energy-momentum tensor, the solution we obtained will not correspond to the original matter energy-momentum distribution anymore.

In summary, the key point leading to Birkhoff's theorem is to solve Einstein field equations for the metric form given in Equation (23), not for the general spherically-symmetric metric form given in Equation (22). The solution Birkhoff obtained—Schwarzschild metric—is not for the general time-dependent spherically-symmetric system, but for the static one.

6. Conclusions

We have shown that Birkhoff's theorem is inconsistent with the post-Newtonian solution to Einstein field equations for a dynamic spherically-symmetric system in the weak-field limit. The time-spatial component of metric, which is missed by Birkhoff's theorem, is indispensable for characterizing the external gravitational field of the dynamic spherically-symmetric system. The radial motion of the spherically-symmetric system can produce gravitomagnetic effects, which may play important roles in the post-Newtonian dynamics.

More important statements and conclusions are as follows. When we solve Einstein field equations for a given gravitational source, the (initial) energy and momentum distributions have to be specified. This implies that there is a reference frame and the solution is corresponding to this reference frame by default. For different reference frames, the matter energy-momentum tensors are different, and thus the corresponding solutions to Einstein field equations are also different, though they can be related via coordinate transformations.

In the derivation of Birkhoff's theorem, before solving Einstein field equations, some coordinate transformations have been applied on the general spherically-symmetric metric tensor to eliminate the metric's time-spatial component. According to the covariance of Einstein field equations, when introducing a coordinate transformation, both the components of the metric tensor and the matter energy-momentum tensor will simultaneously change according to the rules of tensor transformations, and this may change the problem we are solving, since in relativity the coordinate transformations include the changes of reference frame. This important issue is overlooked not only in the derivation of Birkhoff's theorem, but also in many literatures, due to the misinterpretation and misuse of the covariance of general relativity.

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Author details

Wenbin Lin¹

E-mail: wlin@swjtu.edu.cn

ORCID ID: <http://orcid.org/0000-0002-4282-066X>

¹ School of Physical Science and Technology, Southwest Jiaotong University, Chengdu 610031, China.

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References

- Birkhoff, G. (1923). *Relativity and modern physics*. Cambridge, MA: Harvard University Press.
- Fock, V. (1959). *The theory of space, time and gravitation*. (N. Kemmel, Trans.). New York, NY: Pergamon Press.
- He, G., & Lin, W. (2014). Exact harmonic metric for a uniformly moving Schwarzschild black hole. *Communications in Theoretical Physics*, 61, 270.

- Klioner, S. (2003). Light propagation in the gravitational field of moving bodies by means of Lorentz transformation I. Mass monopoles moving with constant velocities. *Astronomy & Astrophysics*, 404, 783–787.
- Kopeikin, S., & Fomalont, E. (2007). Gravimagnetism, causality, and aberration of gravity in the gravitational light-ray deflection experiments. *General Relativity and Gravitation*, 39, 1583–1624.
- Kopeikin, S., & Mashhoon, B. (2002). Gravitomagnetic effects in the propagation of electromagnetic waves in variable gravitational fields of arbitrary-moving and spinning bodies. *Physical Review D*, 65, 064025.
- Kopeikin, S., Polnarev, A., Schäfer, G., & Vlasov, I. (2007). Gravimagnetic effect of the barycentric motion of the Sun and determination of the post-Newtonian parameter γ in the Cassini experiment. *Physics Letters A*, 367, 276280.
- Kopeikin, S., & Schäfer, G. (1999). Lorentz covariant theory of light propagation in gravitational fields of arbitrary-moving bodies. *Physical Review D*, 60, 124002.
- Sereno, M. (2002). Gravitational lensing by spinning and radially moving lenses. *Physics Letters A*, 305, 7–11.
- Sereno, M. (2005). On gravitational lensing by deflectors in motion. *Monthly Notices of the Royal Astronomical Society: Letters*, 359, L19–L22.
- Weinberg, S. (1972). *Gravitation and cosmology: Principles and applications of the general theory of relativity*. New York, NY: Wiley.
- Will, C. M. (2011). On the unreasonable effectiveness of the post-Newtonian approximation in gravitational physics. *Proceedings of the National Academy of Sciences*, 108, 5938–5945.
- Wucknitz, O., & Spherhake, U. (2004). Deflection of light and particles by moving gravitational lenses. *Physical Review D*, 69, 063001.



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