Symmetric surface complex waves in Goubau Line

E.A. Kuzmina\textsuperscript{1*} and Y.V. Shestopalov\textsuperscript{2}

Abstract: Existence of symmetric surface complex waves in a Goubau line—a perfectly conducting cylinder of circular cross-section covered by a concentric dielectric layer—is proved by constructing perturbation of the spectrum of symmetric real waves with respect to the imaginary part of the permittivity of the dielectric cover. Closed-form iteration procedures for calculating the roots of the dispersion equation (DE) in the complex domain supplied with efficient choice of initial approximation are developed. Numerical modeling is performed with the help of a parameter-differentiation method applied to the analytical and numerical solution of DEs.

Keywords: Goubau line; losses; complex surface wave; dispersion equation; attenuation

1. Introduction
A Goubau line (GL) is a perfectly conducting cylinder of circular cross-section covered by a concentric dielectric layer. GL is a basic type of open metal-dielectric waveguides (see Figure. 1).

ABOUT THE AUTHOR
The research group under informal leadership of Yury Shestopalov, professor of the University of Gävle, Sweden, comprises several tens of researchers working in different countries. Mathematical modeling for electromagnetics constitutes key research activities of the group. One of main topics is the development of modern mathematical and numerical methods for the analysis of electromagnetic wave propagation in open structures and inhomogeneous nonlinear media. A team at the Moscow Technological University MIREA involving three professors and a PhD student Ekaterina Kuzmina studies among all complex waves in open waveguides and lossy dielectrics. The results reported in the paper Symmetric Surface Complex Waves in Goubau Line constitute an important milestone in the development of a general theory of wave propagation, which is a topical issue of the team and lead to a variety of new applications for modern antennas and wireless technology.

PUBLIC INTEREST STATEMENT
Transmission of signals by miniature easy-to-manufacture devices is crucial for providing wireless detection of objects. This technique is incorporated in various modern payment facilities like contactless cards and mobile phones equipped with the corresponding apps. The transmission devices employ artificial lossy dielectric media, creating theoretical background for the analysis of electromagnetic wave propagation in such dielectrics and lines is an urgent task to ensure robust wireless communication at short distances. The paper Symmetric Surface Complex Waves in Goubau Line highlights the discovery of a new type of waves in a circular wire covered by a concentric dielectric layer. The obtained results perfectly fit the purpose of designing simple and efficient tool to be used in wireless technologies.

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The discovery and explanation of the nature of surface waves in a GL date back to classical studies (Goubau, 1950; Harms, 1907; Sommerfeld, 1899). Many works deal with analysis of complex waves in open waveguides (e.g., Barlow & Brown, 1962; Felsen & Marcuvitz, 1973; John, 1977; Marcuse, 1974; Snyder & Love, 1983). Various aspects of the mathematical theory of complex waves are developed in Jablonski (1994); Kartchevski et al. (2005); Rajevsky & Rajevsky (2010); Shestopalov & Kuzmina (2016); Shestopalov, Kuzmina, & Samokhin (2014); Shestopalov & Shestopalov (1996). However, rigorous mathematical proof of the existence of complex waves in a GL has not been completed. Typical approaches developed by many authors searching for complex waves are reduced to numerical solution in the complex domain of dispersion equations (DEs) associated with symmetric waves in a GL or incorrect replacement of the initial DEs by simplified equations employing various approximations or asymptotic expansions. Such numerical-analytical investigations are not supported by mathematically correct proofs of the existence and distribution of the complex wave spectra, although the occurrence of complex waves in GLs have been reported by many authors (Fikioris & Roumeliotis, 1979; Overfelt, Halterman, Feng, & Bowling, 2009; Rao & Hamid, 1979; Sherman & Hennessy, 1983).

The first step toward rigorous analysis of complex waves in a GL using perturbation techniques and based on the theory set forth in Shestopalov et al. (2014) has been made in Shestopalov & Kuzmina (2016). The present study is devoted to the development of this approach, complementing the technique with further necessary mathematical tools and results, both analytical and numerical.

2. Statement
Consider the propagation of symmetric (azimuthally independent) TM eigenwaves described in terms of nontrivial solutions to homogeneous Maxwell’s equations in a GL with a and b, b>a, being the radii of the internal (perfectly conducting) and external (dielectric) cylinders. The eigenwaves have the nonzero electromagnetic field components

\[ H = [0, H_2(r, z), 0], \quad E = [E_1(r, z), 0, E_3(r, z)], \]

\[ E_1 = -\frac{i\beta}{k_0^2} \frac{d\phi}{dr} e^{-i\omega z}, \quad E_3 = \phi(r)e^{-i\omega z}, \quad H_2 = -\frac{i\omega \epsilon}{k_0^2} \frac{d\phi}{dr} e^{-i\omega z}. \]

Here, \( E_1 \) refers to the x-component of E, and so on. Parameter \( \beta \) is the wave propagation constant (spectral parameter), \( \epsilon \gg 1 \) is relative permittivity of the homogeneous dielectric, and \( k_0^2 \) is the freespace wavenumber. Respectively, \( \phi \) satisfies the Bessel equation:

\[ \frac{1}{r} \frac{d}{dr} \left(r \frac{d\phi}{dr}\right) + k_0^2 \phi = 0, \]

and the longitudinal wavenumber
\[ k_1^2 = \begin{cases} k_0^2 - \beta^2, & r > b, \\ \bar{k}_0^2 - \beta^2, & a < r < b. \end{cases} \]

Determination of surface waves can be reduced (Shestopalov et al., 2014) to the following singular Sturm–Liouville boundary eigenvalue problems on the half-line with a discontinuous (piecewise constant) coefficient in the differential equation:

\[ \mathcal{L}\phi = 0, \quad r > a, \quad \phi(a) = 0, \phi \in C^1[a, +\infty) \cap C^2(a, b) \cap C^2(b, +\infty), \]
\[ |\phi|_{r=b} = \left[ \int_a^b d\phi dr \right]_{r=b} = 0. \] \hspace{1cm} (4)

To investigate complex symmetric waves complement (4) with a condition at infinity which is a version of the Riechartd–Sveshnikov radiation condition (Shestopalov & Shestopalov, 1996)

\[ \phi(r) = AH_0^{(1)}(r\bar{\nu}), \quad r > b, \] \hspace{1cm} (5)

and represent potential function \( \phi(r) \) accordingly as

\[ \phi(r) = \begin{cases} \frac{g}{\nu(r)} [J_0(\nu x) Y_0(x) - Y_0(\nu x) J_0(x)], & a < r < b. \end{cases} \] \hspace{1cm} (6)

Here, \( J_0(x) \), \( Y_0(x) \), and \( H_0^{(1)} \) denote, respectively, the Bessel and Neumann functions of the order \( k = 0, 1 \) and the zero-order Hankel function of the first kind; \( \nu = \beta k_0, x = \kappa \sqrt{c - \gamma^2}, u = \kappa \sqrt{c - 1}, \) \( \kappa = k_0 \alpha, k_0 = \frac{2\nu^2}{c} \), where \( c \) is the speed of light in vacuum, \( f \) frequency, and \( s = ba > 1, \gamma \) may be complex quantity; and \( \bar{\nu} = \frac{1}{\nu} \sqrt{u^2 - x^2} = k_0 \sqrt{1 - \gamma^2} \) is the transverse wavenumber of the medium outside GL (vacuum).

The resulting boundary value problem (4)–(5) constitutes a non-self-adjoint eigenvalue problem with respect to (w.r.t.) spectral parameter \( \gamma \). Their eigenvalues, real or complex, specify symmetric waves (1), (2) having normalized propagation constant (longitudinal wavenumber) \( \gamma \); these waves are called, respectively, real or complex.

The conditions at infinity in (4)–(5) and the form (2), (6) can be taken as a definition of symmetric (surface) waves in GL. The real surface waves are described in terms of real-valued quantities; in particular, the boundary operator in (4) is defined (as real-valued functions of a real variable \( \gamma \) or \( \lambda = \gamma^2 \)) on a certain interval \( I \). The spectrum of surface waves may be empty or they may consist of several (real) points located on this interval. The results of the classical Sturm–Liouville theory concerning existence and distribution of the (real) spectrum are not applicable here, and we will perform in this work the corresponding analysis from the very beginning by reducing the boundary eigenvalue problems under study to DEs. This will, in turn, enable us to prove all the required statements and determine the eigenvalues numerically or analytically.

In view of the logarithmic singularity of the Hankel function \( H_0^{(1)}(z) \) at \( z = 0 \), eigenvalue problem (4)–(5) is considered, following (Shestopalov & Shestopalov, 1996) at \( \gamma \in \Lambda \), where \( \Lambda \) is the multisheet Riemann surface of the function \( f(\gamma) = \ln \sqrt{1 - \gamma^2} \); \( \Lambda_0 \) is the principal (“proper”) sheet of this Riemann surface specified by the condition \( \Im \gamma \geq 0 \).

Note that the differential operator of boundary eigenvalue problem (4)–(5) is not defined at: (i) \( s = 1 \) when the interval supporting transmission conditions in (4) vanishes and the differential operator degenerates; (ii) the points \( \gamma \in \Gamma_5 = \{ \pm 1, \sqrt{c} \} \) which are branch points on the Riemann surface \( \Lambda \), \( \Gamma_5 \) may be thus classified as singular spectral set and \( s = 1 \) as a singular value for the differential operator of the boundary eigenvalue problem. At the singular spectral points \( \gamma = \pm 1 \), the potential spectral function in (6) is not defined. Below, it will be shown that the points belonging to \( \Gamma_5 \) are singularities in the dependence of eigenvalues of (4)–(5) on the problem parameters \( s, u, \) and \( \kappa \).
3. Real surface waves in GL

3.1. Dispersion equation

For real surface waves in GL, the condition at infinity is exponential decay of the potential function in (4)–(5). By applying the transmission (continuity) condition to (6), we obtain DE (Shestopalov et al., 2014) for symmetric surface waves in GL

\[ F_0(x, u, s) = G_2(x, u, s) - F_0(x, s) = 0, \]  

(7)

where

\[ G_2(x, u, s) = \frac{1}{\Phi_1(x, s)} K_0(sw), \quad F_0(x, s) = \frac{1}{\Phi_2(x, s)} \Phi_1(x, s), \quad w = \sqrt{u^2 - x^2}, \]  

(8)

\[ \Phi_1(x, s) = J_0'(sx, x) = J_0(x)Y_0'(sx) - J_0'(x)Y_0(sx), \]

\[ \Phi_2(x, s) = J_1'(sx, x) = J_0(x)Y_1'(sx) - J_1'(sx)Y_0(x). \]

Here, functions \( J_{m}^{\pi} \) and \( Y_{m}^{\pi} \) are introduced by formulas (23)–(25) in Appendix, and notations \( \Phi_1 \) and \( \Phi_2 \) are used in (Shestopalov et al., 2014).

For real surface waves in GL, all quantities in (7), (8) are real.

Functions \( \Phi_k(x, s), k = 1, 2 \), entering DE (7) are sums of products of cylindrical functions. The existence of real zeros of such functions and distribution of zeros on the real line are reported, e.g. in [20]. A more detailed analysis of the dependence of zeros on parameter \( s \) is performed in (Shestopalov et al., 2014). According to these results, \( \Phi_k(x, s) \) have each as functions of real variable \( x \) for a fixed \( s > 1 \) a countable increasing sequence of positive simple real zeros forming a set \( \lambda_k^s(s) = \{ h_m^k(s), m = 1, 2, \ldots \} \), \( k = 1, 2 \).

Equation (7) with (8) may be conditionally called singular form of DE because \( F_0(x, u, s) \) and \( F_0(x, u, s) \) have singularities (poles) at real zeros of \( \Phi_2(x, s) \) that are points belonging to the set \( \lambda_2^s(s) \).

The equation

\[ F_0(x, u, s) = \epsilon_0w\Phi_2(x, s)K_0(sw) - x\Phi_2(x, s)K_1(sw) = 0 \]

(9)

constitutes regular form of DE. Here, \( F_0(x, u, s) \) is a continuous real-valued function of \( x \in (0, u) \) for every \( s > 1 \) and \( u > 0 \), and has singularities at \( x = 0 \) and \( x = u \). Outside set \( \lambda_2^s(s) \), regular and singular forms of DE are equivalent.

List the known facts (Abramowitz & Stegun, 1972) concerning distribution of the set of (real) zeros of functions \( \Phi_k(x, s), k = 1, 2 \), necessary to investigate the properties of real and complex spectra of symmetric waves.

- The distance \( d_m^k(s) = |h_{m-1}^k(s) - h_m^k(s)|, k = 1, 2 \), between two neighboring zeros tends for a fixed \( s > 1 \) to \( \pi \) as \( m \to \infty \);

- There are numbers \( r_k^s(s) > 0 \) such that \( r_k^s(s) = \min_m d_m^k(s), m = 1, 2, \ldots, k = 1, 2, s > 1; \)

- Zeros \( h_m^1(s) \in \lambda_1^s(s) \) and singularities \( h_m^1(s) \in \lambda_1^s(s) \) (\( m = 1, 2, \ldots \)) of \( \cot \theta(z) = \frac{\Phi_2(x, s)}{\Phi_1(x, s)} \) alternate: \( h_m^1(s) \in (h_{m}^1(s), h_{m-1}^1(s)), \) and \( \min_{m=m-1} |h_{m}^1(s) - h_{m-1}^1(s)| < 0.5r_0(s), m = 1, 2, \ldots, s > 1; \)

- \( h_m^k(s), m = 1, 2, \ldots, k = 1, 2 \) are decreasing functions of \( s > 1 \), as illustrated by Figure. 2.
Conditions

\[ u(13) \Phi_s(1, \frac{1}{2}) \]

is the first (minimal) zero of function \( \Phi(x,s) \) and \( \Phi_s(x,s) \)
(red) vs \( s \in [2, 6] \).

For a given \( u > 0 \), (real) zeros of \( \Phi_1(x,s) \) and \( \Phi_2(x,s) \) may coincide at certain \( s = s' \):

\[ x_{m,m'}^* = h^1_m(s') = h^2_m(s') \in (0, u), \]

so that

\[ \Phi_1(x_{m,m'}^*, s') = \Phi_2(x_{m,m'}^*, s') = 0. \]

In this case, \( x_{m,m'}^* \) is a root of a (“regular”) DE,

\[ F_G(x_{m,m'}^*, u, s') = 0. \]

Such roots of DE (7) form, for a given \( u > 0 \), a special (maybe empty) set of real points \( D'(u) \subset (0, u) \). Figure 2 shows plots of the first three roots \( h^k_m = h^k_m(s), \) \( m = 1, 2, 3, k = 1, 2 \) of \( \Phi_1(x, 2) \) and \( \Phi_2(x, 2) \) vs \( s \). This result suggests that set \( D'(u) \subset (0, u) \) is empty. It means that DE (7) describes the whole spectrum of symmetric surface waves in GL.

3.2. Singular points and “cutoff” conditions

Function \( G_3(x, u, s) \) in (8) considered w.r.t. \( x \) or \( \gamma \) has a removable singularity at \( x = u \) or \( \gamma = \pm 1 \), so that for an \( s > 1 \),

\[ \lim_{x \to u} G_3(x, u, s) = 0. \]

On the other hand, outside set \( \mathcal{H}^2(s), \) \( F_G(x, s = 1) = 0, \) hence

\[ \lim_{(x,s) \to (u, 1)} F_G(x, u, s) = 0 \]

(more detailed analysis is performed in (Shestopalov et al., 2014)), so that DE (7) is formally valid at the singular spectral points \( \gamma = \pm 1 \), or \( x = u \), and singular value \( s = 1 \) for the differential operator of boundary eigenvalue problem (4)–(5). For \( s > 1 \) in a close proximity of \( s = 1 \), positive roots of DE (7) virtually merge with \( \gamma = 1 \) or \( x = u > 0 \), as shown in Figures. 10–13.

In Shestopalov et al. (2014), we have demonstrated that, for arbitrary \( s = b / \alpha > 1 \), there exists a root \( x^1 = x^1(s) \) of DE (7) (more specifically, implicit function \( x^1(s) \) specified by equation (7)). In other words, for an arbitrarily thin dielectric layer, there exists (at least one) higher order surface wave propagating in the GL under the condition

\[ u = k_0 \Delta s - 1 \geq h_1^1(s). \]

where \( h_1^1(s) \) is the first (minimal) zero of function \( \Phi_2(x,s) : \Phi_1(h_1^1(s), s) = 0. \)

Relationship (14) is the necessary condition that provides the existence of at least one such wave. Taking into account the domain of the functions entering DE (7), we have the condition \( x \in (0, \kappa \sqrt{\epsilon}) \).

Figure 2. Plots of the first three roots \( h^k_m = h^k_m(s), m = 1, 2, 3, k = 1, 2, \) of \( \Phi(x,s) \) and \( \Phi_s(x,s) \) (red) vs \( s \in [2, 6] \).
We can determine “cutoff” values of \( u \) according to the condition:

\[
    u = \kappa \sqrt{\epsilon - 1} = h_1'(s)
\]

(15)

by solving numerically w.r.t. \( s \) for a given \( u \) the equation

\[
    \Phi_1(u, s) = 0.
\]

(16)

and finding its first (minimal) root \( s = s_1'(u) \).

Equation (16) defines implicit functions \( s = s_m'(u) \) \((m = 1, 2, \ldots)\) that, for a fixed \( u > 0 \), are sequences of zeros of \( \Phi_1(u, s) \). The implicit function \( s = s_1'(u) \) can be determined, for a given \( u_0 > 0 \), as a solution to the Cauchy problem

\[
    \frac{ds_1'}{du} = T_1(u, s_1'), \quad s_1'(u_0) = s_{10};
\]

\[
    T_1(u, s) = \frac{\partial \Phi_1(u, s)}{\partial s} = \frac{s}{u} + \frac{W_{1001}(u, su)}{W_{0110}(u, su)},
\]

(17)

where \( W_{ij} \) are given by formulas (23) and (24). When deriving (17), it is taken into account that:

\[
    \frac{d}{ds} \Phi_1(x, s) = x \Phi_2(x, s).
\]

(18)

In view of (18), the Cauchy problem (17) is uniquely solvable on a certain interval \( u \in (u_0, u_1) \) outside the set of zeros of \( \Phi_2(u, s) \); on this interval, \( s_1'(u) \) is a one-to-one function. Figure 3 shows \( s_1'(u) \) calculated at the initial condition specified by \( u_0 = 2 \) and \( s_{10} = 2.554 \); function \( s_1'(u) \) is monotonic on an interval \( u \in (2.75) \) and one-to-one mapping of \( u \in (2.75) \) onto \( s_1' \in (1.418, 2.554) \).

Figure 3. Plots of \( s_1'(u) \), the first root of

\[
    \Phi_1(u, s) = J_0(su)Y_2(u) - J_2(u)Y_0(su),
\]

\( s_1'(u) \) is calculated from (17) with the initial condition specified by \( u_0 = 2 \) and \( s_{10} = 2.554 \); function \( s_1'(u) \) is monotonic on an interval \( u \in (2.75) \) and one-to-one mapping of \( u \in (2.75) \) onto \( s_1' \in (1.418, 2.554) \).
Summarize properties of real surface waves in GL proved in Shestopalov et al. (2014) for all $u>0$ and $s>1$:

- there exists (at least) one root $x^0 = x^0(s,u) \in I(s)$ of equation (7); this root corresponds to the fundamental surface wave, which exists for all $u>0$;

- the cross-sectional structure of the $n$th higher order wave is characterized by the number of zeros (oscillations) of potential function $\phi_n(r)$ on $(a,b)$; namely, $\phi_n(r)$ has exactly $n$ zeros on $(a,b)$ (see Figure. 4) and the intervals between two neighboring zeros contained in the interval tend to zero as $n$ tends to $\infty$.

4. Surface complex waves

In this section, we validate the existence and consider the properties of the spectrum of surface complex waves in GL w.r.t. spectral parameters $\gamma$ or $x$ entering DE (7) as (regular) perturbations $\gamma = \gamma(t)$ or $x = x(t)$ with respect to the parameter $t = \omega \varepsilon$.

4.1. Linearized expressions for longitudinal wavenumbers of complex waves

We prove the existence of radially symmetric complex surface waves in a GL using the parameter-continuation method and obtaining explicitly linearized expressions for longitudinal wavenumbers of complex waves. Namely, we determine the propagation constant $\gamma$ of a radially symmetric complex surface wave as an implicit function specified by the DE (7) obtained for lossless GL, consider the DE in the complex domain by constructing analytical continuation of the functions involved in the DE, and then build up a continuation of $\gamma$ with respect to the imaginary part $t = \omega \varepsilon$ of the permittivity of the dielectric cover using reduction to the Cauchy problem

$$\frac{d\gamma}{dt} = F_\gamma(\gamma,t), \quad \gamma(0) = \gamma_n,$$

where $\gamma = \gamma(t)$ and $\gamma_n$ are the longitudinal wavenumbers of, respectively, a symmetric complex surface wave considered w.r.t. real parameter $t$ and a real TM surface wave of index $n$,

$$F_\gamma = \frac{\partial F_G/\partial t}{\partial F_G/\partial \gamma} = -i \frac{\frac{1}{2} F_{g1} - \frac{i}{4} G_9}{\gamma [F_{g1} + \frac{1 + \sqrt{\gamma^2 - 1}}{(\sqrt{\gamma^2 - 1})^2}] G_9 - 2},$$

$$F_{g1} = \kappa (\Phi_1(x,s) + \nu \Phi_{11}(x,s) - \nu \Phi_1(x,s) \Phi_{21}(x,s)),$$

is expressed explicitly as a rational function of the mixed products of cylindrical functions, $v = \sqrt{\omega^2 - \gamma^2}$, and

Figure 4. Plots of potential functions $\phi_n(r)$ for a fundamental, $n = 1$, and higher order, $n = 3$ mode.
\( \Phi_1(x, s) = \frac{K}{V} (\nu_{011}^0 (sx, x) + \nu_{011}^0 (x, sx)) \),

\( \Phi_2(x, s) = \frac{K}{V} (\nu_{11}^0 (sx, x) + \frac{s}{2} (\nu_{022}^0 (sx, x) - \nu_{022}^0 (sx, x))) \),

where \( \nu_{ab}^c \) and \( \nu_{ab}^d \) are given by formulas (23) and (24).

Using the technique developed in Shestopalov et al. (2014) and Shestopalov & Kuzmina (2016), we prove that Cauchy problem (19), (20) is uniquely solvable.

In fact, \( F_G \) is regular at \( \gamma = \gamma_n \) and \( t = 0 \) because \( \partial F_G / \partial \gamma \) does not vanish at this point which can be verified using analytical expressions (20) derived in this paper. This implies that complex zeros of \( F_G = F_G(\gamma, t) \) constitute regular perturbations of its real zeros \( \gamma_n \). Roots of DE become complex due to the presence of small imaginary part in parameter which causes small perturbation.

The (local) existence of the solution to Cauchy problem (19), (20) in the vicinities of the initial values that are propagation constants of real symmetric surface waves in a GL with a lossless dielectric cover proves the existence of complex symmetric surface waves in a lossy GL. The latter waves are regular perturbations of the former when considered with respect to the imaginary part of the permittivity.

The dependence \( \gamma = \gamma(t), t \in \Omega \), is determined explicitly using implicit differentiation in the form of a segment of the Taylor series

\[ \gamma_n(t) = \gamma_n + tD_1^n + O(t^2), \quad D_1^n = \frac{d\gamma_n}{dt} \bigg|_{t=0} = F_G(\gamma, t) \bigg|_{t=0, \gamma=\gamma_n} \]  

This fact enables us to perform complete mathematical analysis of the coefficients entering the segment of the Taylor series in (22) and finally the complex surface wave propagation constants themselves.

### 4.2. Specific propagation regimes

The coefficient \( D_0^\gamma \) which specifies, for each higher order complex GL mode with the index \( n = 2, 3, \ldots \), the main contribution caused by the introduction of lossy dielectric, is a function of the problem parameters, \( D_0^\gamma = D_0^\gamma(\bar{v}, \bar{v} = (a, b, \omega, \Re \gamma)) \) and is a purely imaginary quantity (for \( x \in (0, u) \) or \( \gamma_n \in (1, \sqrt{v}) \)). Using explicit linearized expressions (20), we can thus investigate analytically the properties of surface complex waves. We show that for \( n > 1 \), the imaginary part of the longitudinal wavenumber of a symmetric complex surface wave (that is, contribution caused by the lossy dielectric which gives rise to the wave attenuation) is small and its magnitude is bounded uniformly w.r.t. the parameter vector \( \bar{v} \). In fact, \( \gamma_n \) is real and attenuation \( \Re \gamma \) is governed by \( D_0^\gamma \). Performing analytical investigations and numerical simulations described briefly in the next section, we show that the imaginary part of the propagation constants of higher order complex GL modes are small in a broad range if the problem parameters, as illustrated in Figure. 5.

The propagation constants of complex waves \( \gamma = \gamma_n(t) \) depicted in Figure. 5 on the complex \( \gamma \)-plane are determined (for different \( n \)) as the unique solution of the Cauchy problem (19). The existence and uniqueness of solution to this problem can be proved, following the classical theory of ordinary differential equations and using explicit representation of the right-hand side of (19), on a \( t \)-interval \( (0, T_n) \) for sufficiently large \( T_n \). Thus, \( \gamma = \gamma_n(t) \) exist as well-defined functions of parameter \( t \) on \((0, T_n)\) and can be calculated solely by numerical solution of (19) using the Runge–Kutta method for rather large values of \( t \) exceeding unity; practical calculations can be performed up to \( t = 10 \) and even for larger values. Note that \( \gamma_n(t) \) are calculated using the linearized formula and closed-form expressions for \( D_1^n \) up to approximately \( t = 0.001 \). For larger values of \( t \), \( \gamma_n(t) \) are determined numerically using the procedures described above.
We see that the proposed method enables both the proof of existence and computation of the propagation constants \( \gamma = \gamma_n(t) \) of complex waves for small and large values of \( t = \Im(\varepsilon) \) by efficiently combining closed-form approximations and numerical solution by different methods which constitutes an extra power of the developed approach.

Figures 6 and 7 demonstrate the dependence of \( D_n(x) \) on \( s \) for different values of permittivity \( \varepsilon \) giving evidence that \( D_n(x) \) is uniformly bounded and vanish at certain values of \( s \) and \( \varepsilon \).

We conclude that for every \( \varepsilon > 1 \), there exist several values of \( s \) distributed almost periodically as can be seen in Figures. 6 and 7 at which the imaginary part of the longitudinal wavenumbers of the principal and higher order modes virtually vanish. This property gives rise to propagation regimes in a lossy GL characterized by extremely low attenuation of the principal and higher order complex surface waves.

When losses of a GL dielectric cover are moderate (practically, the imaginary part of the permittivity does not exceed unity), we study the properties of complex surface waves in a GL using an approach based on the analysis of the Taylor series (22) for the wave propagation constants. This method can be justified by the fact that we consider an analytical solution to the Cauchy problem (19) determined for the constructed well-defined analytical continuation of the right-hand side of the differential equation in (1). Remarkably, the coefficient \( D_n(\varepsilon) \) multiplying \( t \) in

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**Figure 5. Values of \( \gamma = \gamma(t) \) on the complex plane \( \gamma \); circles denote points \( \gamma = \gamma(t), t = \Im(\varepsilon) = 0.0001, \ldots, 1 \). The index \( n = 1, 2, \ldots \) indicates the order of the surface complex wave.**

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**Figure 6. Modulus of coefficient \( D_n(x) \) vs \( s \) for different values of permittivity \( \varepsilon \).**

(a) \( \varepsilon = 2, 2.1, \ldots, 3 \)  
(b) \( \varepsilon = 2, 2.4, \ldots, 6 \)
the segment of the Taylor series (22) for $\gamma_n(t)$ that solves Cauchy problem (1) can be determined explicitly which validates the efficiency of the method.

Let us present a summary of the results of our analytical-numerical investigations: for small losses, the attenuation of the GL complex waves (i) may be very low and (ii) affected by the relative thickness $s$ of the cover in an oscillatory manner: it may virtually vanish, and attains a distinct maximum at almost periodically alternating values of $s$.

The propagation constant $\gamma_n$ of real surface waves (the leading term in (22) and initial value in the Cauchy problem (19)) is real and attenuation $\gamma_n$ is governed by coefficient $D_1^n$ which is a function of all four problem parameters forming a vector $\mathbf{v}$ defined above. $D_1^n(\mathbf{v})$ can be determined explicitly. Using the obtained explicit expressions, we show that in a broad range of parameters $D_1^n(\mathbf{v})$ (i) is uniformly bounded, (ii) virtually periodic w.r.t. $s$, and (iii) attains zero values as exemplified by Figures. 6 and 7. The obtained closed-form expressions are very bulky and therefore a complete comprehensive analysis of $D_1^n(\mathbf{v})$ as a function of five variables stretches beyond the scope of this paper. We limit this analysis to a short resume (i)–(iii) of the properties of $D_1^n$ and illustrations in the form of Figures. 6 and 7. However, such analysis may and should be performed as a separate study. Definitely, new propagation regimes will be discovered as a result of this analysis and included to this new work.

4.3. Results of computations of surface complex waves

The steps of calculation of the data presented in Figure. 5 can be characterized as follows: the curves in Figure. 5 plot the propagation constants of higher order complex waves on the complex $\gamma$-plane. They are calculated using the linearized formula for very small $t$-values, up to approximately $t = 0.001$. For larger values of $t$, the calculations are performed stepwise, where the $i$th step is associated with a $t$-interval $(t_i, t_{i+1})$; the complex $\gamma$-values are calculated either using the Newton method in the complex domain (with the possibility of efficient choice of the initial value on each step using closed-form approximation for $\gamma(t)$ and the explicit formulas derived in Appendix for the Newton iterations), or by numerical solution of the Cauchy problem (19) using a version of the Runge–Kutta method. Such calculations yield complex $\gamma = \gamma(t)$ with very small variation in the real part of $\gamma$ and therefore curves in the complex plane shown in Figure. 5 that deviate slightly from vertical lines. Note that the values calculated in this manner by the Newton and Runge–Kutta method virtually coincide. A particular shape of the curves in Figure. 5 is governed by the chosen scale enabling to compare the attenuation of the different wave types.
Thus, the proposed method enables computations of the propagation constants of complex waves by efficiently combining closed-form approximations and numerical solution by different methods which constitutes an extra power of the developed approach.

Graphs of the real and imaginary parts of complex propagation constants $\gamma$ (roots of DE (7)) vs frequency $f = 1 \div 40 \cdot 10^9$ Hz = (1, 40) GHz are shown in Figures. 8 and 9.

Results of analytical-numerical investigation of the properties of the longitudinal wavenumber as functions of the problem parameters demonstrate that in a GL with a lossy dielectric cover, the principal and higher order real surface waves are all transformed to complex waves. Next, for small losses, the attenuation of the GL complex waves (i) may be very low and (ii) (strongly) depends on relative thickness $s$ of the dielectric cover.

The real and imaginary parts of the longitudinal wavenumber as functions of frequency virtually do not change in a broad frequency range as shown by Figure. 8.

Introduction of a small imaginary part of the permittivity of the GL dielectric cover gives rise to complex surface waves with small attenuation characterized by a distinct anomalous behavior (see Figure. 9).

Figure 8. Plots of real (a) and imaginary (b) parts of the longitudinal wavenumber $\gamma$, of the principal surface wave for $\epsilon = 5 \div 0.001i, s = 2.63, \kappa = 2$ vs $f \in f = 1 \div 40$ GHz.

(a) Real part  
(b) Imaginary part

Figure 9. Plots of real (a) and imaginary (b) parts of the longitudinal wavenumber $\gamma$, of the principal surface wave for $\epsilon = 10 \div 0.001i, s = 4.4, \kappa = 2$ vs $f \in f = 1 \div 30$ GHz.

(a) Real part  
(b) Imaginary part
Figure 10. Plots of the longitudinal wavenumber \( \gamma_1(s) \) of the principal complex surface wave vs \( s \in (1.01, 1.5) \) on the complex \( \gamma \)-plane at \( \Re \epsilon = 2 \) and \( \Im \epsilon = 0.01 \) (lower curve) to 1 (upper curve). The curves merge at approximately \( s = 1 \) when the width of the dielectric cover becomes infinitely small.

Figure 11. Plots of the longitudinal wavenumber \( \gamma_1(s), \gamma_2(s) \) and \( \gamma_3(s) \) of the first, second, and third complex surface waves vs \( s \in (1.04, 2.63), s \in (1.79, 2.63) \) and \( s \in (2.33, 2.63) \) on the complex \( \gamma \)-plane at \( \Re \epsilon = 5 \) and \( \Im \epsilon = 0.01 \) (lower curve) to 1 (upper curve). The curves merge at approximately \( s = 1 \) when the width of the dielectric cover becomes infinitely small. The right curve with stars is taken from Figure 5.

Figure 12. Plots of the longitudinal wavenumber \( \gamma_1(s), \gamma_2(s) \) and \( \gamma_3(s) \) of the first, second, and third complex surface wave vs \( s \in (1.61, 4.44), s \in (2.9, 4.44) \) and \( s \in (2.6, 4.44) \) on the complex \( \gamma \)-plane at \( \Re \epsilon = 10 \) and \( \Im \epsilon = 0.01 \) (lower curve) to 1 (upper curve). The curves merge at approximately \( s = 1 \) when the width of the dielectric cover becomes infinitely small.
Figures 10–13 confirm that for $s > 1$ located in a vicinity of the singular value $s = 2$ (and different parameter $\epsilon$ in the interval $[0.01, 1]$ and fixed parameter $\kappa$), positive complex roots of DE (7) virtually merge with $\gamma = 1$ or $x = \omega > 0$.

Another, and maybe the most important, peculiarity of the longitudinal wavenumbers of complex waves as functions of $s$ is occurrence of “tree”-shaped curves on the complex $\gamma$-plane: starting, for larger $s$, at a certain value of $t = \Im \epsilon$ on the $\gamma$-curve parametrized by $t$, the $\gamma(t)$-curves merge then around the spectral singularity $\gamma = 1$ as $s$ decreases approaching the value $s = 1$. Such a behavior, which is clearly seen in all Figures. 10–13, follows from the properties of the differential operator of boundary eigenvalue problem (4)–(5) which is not defined at $s = 1$ when the interval supporting transmission conditions in (4) vanishes and the differential operator degenerates. Correspondingly, $\gamma = 1$ is a singularity in the dependence of eigenvalues of (4)–(5) on parameter $s$.

This factor is valid for fundamental and higher order complex surface waves. In this work, we have proved these statements analytically by considering the dependence of the functions in DEs and operator of the corresponding Sturm–Liouville problems on the line w.r.t. the problem parameters in the complex domain.

Surface complex waves constitute a part of the set of complex waves in GL; this set contains also complex waves corresponding to un-infinitely many complex roots of DE which are not connected with any real surface wave (and are not perturbation of real roots).

The analysis of such “pure” complex waves requires elaboration of specific mathematical technique employing functions of several complex variables and will be performed in a separate study.

5. Conclusion
We have demonstrated the existence of symmetric surface complex waves in a GL as perturbations of symmetric real surface waves w.r.t. the imaginary part of the permittivity of the dielectric cover. We have performed comprehensive analysis of DE in the complex domain revealing cutoff conditions and singularities in the dependence of the complex surface wave spectra on the problem parameters. Closed-form iteration procedures and the codes have been developed for efficient calculation of complex roots of DE based on a mathematically justified method providing efficient choice of initial approximations. Numerical modeling has been performed using a parameter-differentiation method applied to the analytical and numerical solution of DEs.

The proposed method enables one both to complete the proof of the existence and perform computation of complex waves for small and large values of the imaginary part of the permittivity by efficiently combining closed-form approximations and numerical solution by different methods. The latter constitutes an extra power of the developed approach.
Appendix

A family of special functions used in the analysis of DE

Introduce a family \( \mathcal{W}_{mn} \) of special functions used in the analysis of DEs that have the form of sums of products of the Bessel, \( J_m \), and Neumann, \( Y_n \), cylindrical functions of the indices \( m, n = 0, 1, \ldots \)

\[
\mathcal{W}_{mn}^{\pm} (x, y) = J_m(x)Y_n(y) \pm J_p(y)Y_q(x), \tag{23}
\]

\[
J_m^\pm (x,y) = \mathcal{W}_{mn}^{\pm} (x, y), \quad m, n, p, q = 0, 1, \ldots \tag{24}
\]

One can write the DEs in all their different forms and iterative schemes applied below for numerical solution of DEs using functions \( \mathcal{W}_{mn}^{\pm} (x, y) \) and \( J_m^\pm (x, y) \).

Let us remind that in this article, we keep the notation \( \Phi_{1,2} \) for the functions

\[
\Phi_1 (x, s) = J_0^-(sx, x), \quad \Phi_2 (x, s) = \mathcal{W}_{0110}^-(x, sx)
\]

given by (9) to be in line with the formulas used in (Shestopalov et al., 2014).

The functions

\[
\mathcal{W}_{mn}^{\pm} (x, sx) = J_m(x)Y_n(sx) \pm J_p(sx)Y_q(x) \tag{25}
\]
and the corresponding functions $\mathcal{Y}^{0}_{mn}(x,sx)$ or $\mathcal{W}^{0}_{mn}(sx,x)$ and $\mathcal{Y}^{s}_{mn}(sx,x)$ considered w.r.t. one variable $x$ or $s$ constitute an important subfamily of $\mathcal{W}^{0}_{mn}$. Among their important properties, note that each has an infinite sequence of positive simple zeros that alternate for different index sets.

**Explicit iteration procedures**

In this section, we show that roots of all equations and DEs involving sums of products of cylindrical functions of the family $\mathcal{W}^{0}_{mn}$ can be calculated with arbitrary given accuracy using explicit iteration procedures. This circumstance is important as far as efficient implementation of the proposed analytical–numerical technique is concerned because it enables one to complete calculations without the use of solvers, controlling the computational accuracy. Equation (16) can be solved numerically using a compact form of the Newton iterations

$$s_{n+1} = s_n - \frac{1}{u^2} F_g(u, s_n)$$

obtained taking into account (9) and (18).

The equations

$$\Phi_k(x, s) = 0, \quad k = 1, 2,$$

(27)

can be solved numerically w.r.t. $x$ for a given $s > 1$ using the Newton iterations which can be written in a compact form

$$x_{n+1}^{k,m}(s) = x_n^{k,m}(s) - T_k(x_n^{k,m}(s), s), n = 0, 1, 2, \ldots,$$

$$x_0^{k,m}(s) = q_0^{k,m}(s), \quad m = 1, 2, \ldots, \quad k = 1, 2,$$

(28)

$$T_1(p, s) = \frac{\mathcal{Y}_0(p, p)}{s \mathcal{W}_{0110}^{-1}(p, sp) + \mathcal{W}_{1001}^{-1}(p, sp)},$$

$$T_2(p, s) = \frac{\mathcal{W}_{0110}^{-1}(p, sp)}{\mathcal{Y}_1(p, sp) - s \mathcal{W}_{0110}^{-1}(p, sp) + \frac{1}{2} \mathcal{W}_{0110}^{-1}(p, sp)}.$$

(29)

Initial approximation $q_0^{k,m}(s)$ is specified for each particular index $m = 1, 2, \ldots$ and a reference value $s = s^* > 1$ using the properties of functions $\Phi_1(x, s)$ or $\Phi_2(x, s)$ and estimating intervals of location of zeros using the known distribution of zeros of these functions summarized above, particularly, that the distance between neighboring zeros are virtually constant and is close to $x$.

For example, Figure. 3 which presents the graphs of $\Phi_1(x, s)$ and $\Phi_2(x, s)$ for $s = s^* = 2$ suggests the following choice of initial approximations

$$q_0^{1,m}(s^*) = 3 + (m - 1)\pi,$$

$$q_0^{2,m}(s^*) = 5 + (m - 1)\pi, \quad m = 1, 2, \ldots,$$

(30)

which provides up to 12 correct decimals in the calculated values of $x_n^{k,m}(s) \approx h_n^{k,m}(s^*) \approx h_n^{k,m}$ after about 10 iterations (28). To calculate zeros for the values of $s$ differing from $s^*$, i.e. to construct an extension of $h_n^{k,m}(s)$ to an interval $I_{k,m} = (s_1^{k,m}, s_2^{k,m})$ containing $s^*$, one can reduce determination of $y = h_n^{k,m}(s)$ to the Cauchy problem.
\[
\frac{dy}{ds} = Q_m^k(y, s), \quad s \in I_{x_m}, \quad y(s^*) = h_m^{k^*},
\]

\[
Q_m^k(y, s) = -\frac{\partial \Phi_2(y, s)}{\partial s} = -\frac{y \Phi_2(y, s)}{s \Phi_2(y, s) + \mathcal{W}_{1001}(y, sy)}
\]

This extension is correct because, in view of the properties of functions \(\Phi_1(x, s)\) and \(\Phi_2(x, s)\), Cauchy problem (31) is uniquely solvable.