



Received: 04 February 2017
Accepted: 08 April 2017
First Published: 19 April 2017

*Corresponding author: Vikas Kumar Mishra, Department of Mathematics, Indian Institute of Technology Patna, Patna, India
E-mail: vikas.mishra45@gmail.com

Reviewing editor:
James Lam, University of Hong Kong, Hong Kong

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SYSTEMS & CONTROL | RESEARCH ARTICLE

Index reduction for rectangular descriptor systems via feedbacks

Vikas Kumar Mishra^{1*}, Nutan Kumar Tomar¹ and Mahendra Kumar Gupta¹

Abstract: Index plays a fundamental role in the study of descriptor systems. For regular descriptor systems, calculation of the index can be performed by calculating the index of the nilpotent matrix obtained by means of the Weierstrass canonical form. Notwithstanding, if the system is not regular, there is no algebraic technique to determine the index of the system. A sufficient algebraic criterion is provided to determine the index of a general linear time-invariant descriptor systems. Thereafter, we provide an alternate but lucid proof of the fact that impulse controllability is necessary and sufficient for the existence of a semistate feedback such that the closed loop system is of the index at most one. Finally, a sufficient test for the existence of a semistate feedback such that the closed loop system is of the index at most two is provided. Examples are given to illustrate the presented theory.

Subjects: Applied Mathematics; Systems & Control Engineering; Control Engineering

Keywords: descriptor systems; controllability; feedback control design; singular value decomposition (SVD); index reduction

ABOUT THE AUTHORS

Vikas Kumar Mishra is currently with the Mechanical Engineering Department of Ben-Gurion University of the Negev, Beer-sheva, Israel as a Post-Doctoral Fellow. He has completed his PhD from Indian Institute of Technology Patna, India in 2016. After completing his PhD, he worked as a faculty member in the Department of Mathematics, National Institute of Technology Jamshedpur, India. He is mainly interested in analysis and design of control systems.

Nutan Kumar Tomar is currently an assistant professor in the Department of Mathematics, Indian Institute of Technology Patna, India. He received his PhD from the Department of Mathematics, Indian Institute of Technology Roorkee, India in 2008. His main research interests include the controllability and optimal control problems in infinite-dimensional spaces.

Mahendra Kumar Gupta was awarded his PhD degree from Department of Mathematics, Indian Institute of Technology Patna, India in 2016. He is currently a Post-Doctoral Fellow in Department of Chemical Engineering, Indian Institute of Technology Madras, India. His main research interests include control and estimation, observer design and descriptor systems.

PUBLIC INTEREST STATEMENT

Descriptor systems play a vital role in the modeling of physical systems where dynamics of the system is constrained. It is well known that the solvability of descriptor systems requires certain smoothness assumptions on its input. Index, loosely speaking, is a measure of the number of differentiations required by the input so that the system has a solution. As higher index problems are difficult to handle, at least, numerically. Therefore, it is always desirable to minimize the index of the system. There are two known techniques to reduce the index: with feedback and without feedback. In the present article, we have developed some techniques to determine the index and to reduce the index by applying feedback.

1. Introduction

Consider an linear time-invariant (LTI) continuous-time descriptor system

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad (1)$$

where $E, A \in \mathbb{R}^{m \times n}$, and $B \in \mathbb{R}^{m \times r}$ are known constant matrices. The vectors $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^r$ are called as semistate vector and control/input vector, respectively. System (1) is termed as regular descriptor system if $m = n$ and there exists a $\lambda \in \mathbb{C}$ such that the matrix pencil $(\lambda E - A)$ is invertible. The study of descriptor systems of the form (1) enables us to model complex dynamical systems with algebraic constraints. There is a wide class of problems which are most appropriately modeled in the form of general descriptor systems (1). Some particular examples are electrical circuits (Riaza, 2008), chemical processes (Kumar & Daoutidis, 1999), biological systems (Zhang, Liu, & Zhang, 2012), constrained mechanical systems (Small, 2011), multi-body dynamics (Eich-Soellner & Führer, 1998), and large-scale systems where various sub-models are linked together (Singh & Liu, 1973).

The present paper is focused on the determination and reduction of index of descriptor systems of the form (1). There are various notions of index for descriptor systems, for example differentiation index (Campbell & Gear, 1995), perturbation index (Hairer & Wanner, 1996), tractability index (März, 1992), and strangeness index (Kunkel & Mehrmann, 1994). In this paper, we focus on the so-called Kronecker index which is defined below for LTI descriptor systems. The Kronecker index of a descriptor system, loosely speaking, provides an idea about the smoothness required by the control variables of the system to have an impulse free solution for consistent initial conditions. It can be seen that the degree of difficulty in the numerical solutions of descriptor systems emanates with the increase in the number of the index of the system. This gives rise a call for the index determination and reduction for descriptor systems. There are two commonly used techniques to reduce the index of an LTI descriptor system: with feedback (Berger & Reis, 2015; Bunse-Gerstner, Mehrmann, & Nichols, 1992; Byers, Geerts, & Mehrmann, 1997; Chu, Mehrmann, & Nichols, 1999) and without feedback (Berger & Van Dooren, 2015; Takamatsu & Iwata, 2008). Many researchers have worked on the index reduction for square descriptor systems (Bunse-Gerstner et al., 1992; Byers et al., 1997; Chu et al., 1999; Takamatsu & Iwata, 2008), however, the papers on the index reduction for rectangular descriptor systems are limited (Berger & Reis, 2015; Berger & Van Dooren, 2015). In articles (Berger & Reis, 2015; Berger & Van Dooren, 2015), the authors have considered the system in behavioral sense and have used a reinterpretation of system variables to reduce the index of the system at most one. It is notable that in the present paper, we reduce the index to at most one and two by feedbacks without using a reinterpretation of systems variables. Berger and Reis (2013) have proved that there exists a semistate feedback matrix $K \in \mathbb{R}^{r \times n}$ such that $[EA + BK] \in \Sigma_{m,n}$ is of index at most one if and only if the system (1) is impulse controllable (I-controllable) by means of feedback canonical form (Loiseau, Özçaldıran, Malabre, & Karcıanias, 1991). We have also provided an alternate but lucid proof of the above result without utilizing the feedback canonical form. Next, it is well known that if the descriptor system is regular, then the index of the system is equal to the index of nilpotency of the nilpotent matrix obtained in the equivalent Weierstrass canonical form (WCF) of the system. However, there is no algebraic technique to find the index of a general LTI descriptor system. In this direction, a sufficient algebraic test to determine the index of an LTI descriptor system is also provided in this article.

The rest of the paper is organized as follows. The coming section is the main section of the paper which deals with the index determination and reduction for general LTI descriptor systems. Section 3 presents the examples which illustrates the presented theory. Finally, conclusions of the paper are provided in Section 4.

2. Index determination and reduction

We first recall the Kronecker canonical form from Gantmacher (1959).

Kronecker Canonical Form (KCF) Corresponding to any matrix pair $[EA] \in \Sigma_{m,n}$, there exists invertible matrices $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ such that the pencil $(\lambda E - A)$ can be brought to the KCF

$$U(\lambda E - A)V = \text{block-diag}(0_{\delta \times \delta'}, \lambda E_\eta - A_\eta, \lambda I_\rho - J_\rho, \lambda N_\sigma - I_\sigma, \lambda E_\epsilon - A_\epsilon), \quad (2)$$

where δ, δ', ρ are nonnegative integers; $J_\rho \in \mathbb{R}^{\rho \times \rho}$ is a Jordan matrix and η, ϵ, σ are unique multi-indices. We use the notation for any multi-index η as $\eta = (\eta_1, \eta_2, \dots, \eta_{l(\eta)})$, $|\eta| = \sum_{i=1}^{l(\eta)} \eta_i$. Moreover, the block matrices have the following properties:

(1) $\lambda E_\eta - A_\eta$ has a block diagonal structure and each block takes the form

$$\lambda E_{\eta_i} - A_{\eta_i} = \lambda \begin{bmatrix} I & \\ & 0^T \end{bmatrix} - \begin{bmatrix} 0^T & \\ & I \end{bmatrix},$$

with order $(\eta_i + 1) \times \eta_i$;

(2) N_σ is a nilpotent matrix in Jordan form;

(3) $\lambda E_\epsilon^T - A_\epsilon^T$ has the same block structure as $\lambda E_\eta - A_\eta$; naturally their dimensions are different.

Note that for any matrix pair $[E A] \in \Sigma_{m,n}$ and $\mu \in \mathbb{N}$, the following matrix

$$E_\mu = \left[\begin{array}{cccc} E & & & \\ A & E & & \\ & A & E & \\ & & \ddots & \ddots \\ & & & A & E \end{array} \right] \left. \vphantom{\begin{array}{cccc} E & & & \\ A & E & & \\ & A & E & \\ & & \ddots & \ddots \\ & & & A & E \end{array}} \right\} \mu \text{ block rows} \in \mathbb{R}^{\mu m \times \mu n}, \quad (3)$$

has rank

$$\text{rank } E_\mu = \mu \text{ rank } E_\eta + \mu \text{ rank } I_\rho + \mu \text{ rank } E_\epsilon + (\mu - 1) \text{ rank } I_\sigma + \text{rank } N_\sigma^\mu. \quad (4)$$

This can be seen by writing the matrix E_μ in the KCF.

Now, to define the index of descriptor system (1), we adopt the following notations

Definition 2.1 Let matrix pencil $(\lambda E - A)$ be given in the KCF (2). Then, the index of the matrix pencil or the system (1) is given by

$$\nu = \max\{\theta_1, \theta_2\}. \quad (5)$$

We now prove the following theorem on the rank of two successive $E_{\mu'}s$.

THEOREM 2.2 The system has no η -blocks and the index of nilpotency of the matrix N_σ in (2) is at most μ if and only if

$$\text{rank } E_{\mu+1} = \text{rank} \begin{bmatrix} E & A \end{bmatrix} + \text{rank } E_\mu. \quad (6)$$

Proof Writing the matrices E and A in the KCF, the LHS of (6) is equal to

$$(\mu + 1) \text{ rank } E_\eta + (\mu + 1) \text{ rank } I_\rho + (\mu + 1) \text{ rank } E_\epsilon + \mu \text{ rank } I_\sigma + \text{rank } N_\sigma^{\mu+1}.$$

Similarly, the right-hand side of (6) is equal to

$$(\mu + 1) \text{ rank } E_\epsilon + (\mu + 1) \text{ rank } I_\rho + \mu \text{ rank } E_\eta + \text{rank} \begin{bmatrix} E_\eta & A_\eta \end{bmatrix} + \mu \text{ rank } I_\sigma + \text{rank } N_\sigma^\mu.$$

The above relationships enable us to write

$$\text{rank} E_\eta + \text{rank} N_\sigma^{\mu+1} = \text{rank} \begin{bmatrix} E_\eta & A_\eta \end{bmatrix} + \text{rank} N_\sigma^\mu + \text{rank} N_\sigma^\mu \tag{7}$$

The η -blocks in the KCF exists if and only if $\text{rank} E_\eta < \text{rank} \begin{bmatrix} E_\eta & A_\eta \end{bmatrix}$. Moreover, $\text{rank} N_\sigma^{\mu+1} \leq \text{rank} N_\sigma^\mu$. Thus, (7) holds if and only if η -blocks are void and $\text{rank} N_\sigma^{\mu+1} = \text{rank} N_\sigma^\mu$. This completes the proof. \square

The following remark provides an insight of the above theorem.

Remark 2.3 The condition (6) guarantees the nonexistence of η -blocks and the index of nilpotency of the matrix N_σ in (2) is at most μ . Therefore, the index of the system is determined by only nilpotent blocks and is at most μ . Moreover, the contribution of the zero rows in the KCF in determining the index is at most one which coincides with the minimum value of $\mu = 1$. Furthermore, this test does not provide any idea about the case when index is zero. Thus, if Equation (6) holds for the least value of $\mu = \nu$ (say) and $\nu > 1$ then ν is the index of the system (1).

For $\mu = 1$, Equation (6) reduces to

$$\text{rank} \begin{bmatrix} E & 0 \\ A & E \end{bmatrix} = \text{rank} \begin{bmatrix} E & A \end{bmatrix} + \text{rank} E. \tag{8}$$

According to the authors' best knowledge, Equation (8) appeared for the first time in the work of Hou (2004) where it has been proved to be equivalent to the fact that the system has no impulsive modes and admits arbitrary initial conditions. Recently, condition (8) has been proved to be equivalent to the fact that the system has index at most one (Berger & Reis, 2013). Now, we present the following theorem on the decomposition of system matrices which will be used in deriving the main results on index reduction of this paper.

THEOREM 2.4 Assume $[E \ A \ B] \in \Sigma_{m,n,r}$. Then, there exist orthogonal matrices $M \in \mathbb{R}^{m \times m}$, $N \in \mathbb{R}^{n \times n}$, and $U \in \mathbb{R}^{r \times r}$ such that

$$MEN = \begin{bmatrix} \Sigma_E & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} n_0 \\ p_0 \\ q_0 \\ s \end{matrix}, \quad MAN = \begin{bmatrix} A_{11} & A_{121} & A_{122} \\ A_{211} & \Sigma_{A_{22}} & 0 \\ A_{2121} & 0 & 0 \\ A_{2122} & 0 & 0 \end{bmatrix} \begin{matrix} n_0 \\ p_0 \\ q_0 \\ s \end{matrix}, \tag{9a}$$

$$\text{and } MBU = \begin{bmatrix} B_{11} & B_{12} \\ B_{211} & B_{212} \\ \Sigma_{B_{22}} & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} n_0 \\ p_0 \\ q_0 \\ s \end{matrix}, \tag{9b}$$

where Σ_E , $\Sigma_{A_{22}}$, and $\Sigma_{B_{22}}$ are diagonal positive definite matrices. Moreover, $n_0 + p_0 + l = n$ and $n_0 + p_0 + q_0 + s = m$.

Proof Let $\text{rank} E = n_0$. Then, there exist orthogonal matrices $M_1 \in \mathbb{R}^{m \times m}$ and $N_1 \in \mathbb{R}^{n \times n}$ such that

$$M_1 E N_1 = \begin{bmatrix} \Sigma_E & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} n_0 \\ p \end{matrix}, \quad M_1 A N_1 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{matrix} n_0 \\ p \end{matrix}, \quad \text{and } M_1 B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \begin{matrix} n_0 \\ p \end{matrix},$$

where $n_0 + k = n$ and $n_0 + p = m$. Let $\text{rank} A_{22} = p_0$. Then, there exist orthogonal matrices $M_2 \in \mathbb{R}^{p \times p}$ and $N_2 \in \mathbb{R}^{k \times k}$ such that

$$M_2 A_{22} N_2 = \begin{bmatrix} \Sigma_{A_{22}} & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} p_0 \\ q \end{matrix}.$$

This implies that for

$$\tilde{M}_2 = \begin{bmatrix} I_{n_0} & 0 \\ 0 & M_2 \end{bmatrix} M_1 \text{ and } \tilde{N}_2 = N_1 \begin{bmatrix} I_{n_0} & 0 \\ 0 & N_2 \end{bmatrix},$$

the following hold:

$$\tilde{M}_2 E \tilde{N}_2 = \begin{bmatrix} \Sigma_E & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} n_0 \\ p_0 \\ q \end{matrix}, \tilde{M}_2 B = \begin{bmatrix} B_1 \\ B_{21} \\ B_{22} \end{bmatrix} \begin{matrix} n_0 \\ p_0 \\ q \end{matrix}, \text{ and}$$

$$\tilde{M}_2 A \tilde{N}_2 = \begin{bmatrix} A_{11} & A_{121} & A_{122} \\ A_{211} & \Sigma_{A_{22}} & 0 \\ A_{212} & 0 & 0 \end{bmatrix} \begin{matrix} n_0 \\ p_0 \\ q \end{matrix},$$

where $M_2 A_{21} = \begin{bmatrix} A_{211} \\ A_{212} \end{bmatrix}$, $A_{12} N_2 = \begin{bmatrix} A_{121} & A_{122} \end{bmatrix}$, $M_2 B_2 = \begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix}$, $n_0 + p_0 + l = n$, and $n_0 + p_0 + q = m$. Again, let us assume that $\text{rank } B_{22} = q_0$. Then, there exist orthogonal matrices $M_3 \in \mathbb{R}^{q \times q}$ and $N_3 \in \mathbb{R}^{r \times r}$ such that

$$M_3 B_{22} N_3 = \begin{bmatrix} \Sigma_{B_{22}} & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} q_0 \\ (r - q_0) \\ s \end{matrix}.$$

Now, we obtain the desired decomposition by setting the matrices M , N , and U as follows

$$M = \begin{bmatrix} I_{n_0} & 0 & 0 \\ 0 & I_{p_0} & 0 \\ 0 & 0 & M_3 \end{bmatrix} \tilde{M}_2, N = N_1 \begin{bmatrix} I_{n_0} & 0 \\ 0 & N_2 \end{bmatrix}, \text{ and } U = N_3,$$

with the following decompositions

$$M_3 A_{212} = \begin{bmatrix} A_{2121} \\ A_{2122} \end{bmatrix}, B_1 N_3 = \begin{bmatrix} B_{11} & B_{12} \end{bmatrix}, \text{ and } B_{21} N_3 = \begin{bmatrix} B_{211} & B_{212} \end{bmatrix}. \quad \square$$

Next, we recall the following result from Berger and Reis (2013).

PROPOSITION 2.5 *The system (1) is I-controllable if and only if*

$$\text{rank} \begin{bmatrix} E & 0 & 0 \\ A & E & B \end{bmatrix} = \text{rank} \begin{bmatrix} E & A & B \end{bmatrix} + \text{rank } E. \quad (10)$$

The following theorem provides an equivalent condition to the I-controllability of the system (1).

THEOREM 2.6 *The system (1) is I-controllable if and only if the block matrix A_{2122} is identically zero in the decomposition (9).*

Proof We know that the I-controllability of (1) is equivalent to (10). Now, applying the decomposition (9) to the condition (10), we obtain

$$\begin{aligned} \text{rank} & \begin{bmatrix} \Sigma_E & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_{11} & A_{121} & A_{122} & \Sigma_E & 0 & 0 & B_{11} & B_{12} \\ A_{211} & \Sigma_{A_{22}} & 0 & 0 & 0 & 0 & B_{211} & B_{212} \\ A_{2121} & 0 & 0 & 0 & 0 & 0 & \Sigma_{B_{22}} & 0 \\ A_{2122} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \text{rank} \begin{bmatrix} \Sigma_E & 0 & 0 & A_{11} & A_{121} & A_{122} & B_{11} & B_{12} \\ 0 & 0 & 0 & A_{211} & \Sigma_{A_{22}} & 0 & B_{211} & B_{212} \\ 0 & 0 & 0 & A_{2121} & 0 & 0 & \Sigma_{B_{22}} & 0 \\ 0 & 0 & 0 & A_{2122} & 0 & 0 & 0 & 0 \end{bmatrix} + \text{rank } E, \end{aligned}$$

which is equivalent to the fact that $A_{2122} \equiv 0$. Hence the theorem is proved. \square

The following theorem provides an equivalent criterion for the system to have index at most one.

THEOREM 2.7 *The system (1) has index at most one if and only if the block matrix $\begin{bmatrix} A_{2121} \\ A_{2122} \end{bmatrix}$ is identically zero in the decomposition (9).*

Proof We know that the system (1) has index at most one if and only if (8) holds. Now, applying the decomposition (9) to (8), we obtain

$$\begin{aligned} \text{rank} & \begin{bmatrix} \Sigma_E & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ A_{11} & A_{121} & A_{122} & \Sigma_E & 0 & 0 \\ A_{211} & \Sigma_{A_{22}} & 0 & 0 & 0 & 0 \\ A_{2121} & 0 & 0 & 0 & 0 & 0 \\ A_{2122} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \text{rank} \begin{bmatrix} \Sigma_E & 0 & 0 & A_{11} & A_{121} & A_{122} \\ 0 & 0 & 0 & A_{211} & \Sigma_{A_{22}} & 0 \\ 0 & 0 & 0 & A_{2121} & 0 & 0 \\ 0 & 0 & 0 & A_{2122} & 0 & 0 \end{bmatrix} + \text{rank } E, \end{aligned}$$

which is equivalent to the fact that $\begin{bmatrix} A_{2121} \\ A_{2122} \end{bmatrix} \equiv 0$. Hence the theorem is proved. \square

Note that in the above theorems, by the term “identically zero” we mean that the matrices are either empty or null. The next theorem proves that I-controllability is necessary and sufficient for the existence of a semistate feedback such that the closed loop system is of index at most one.

THEOREM 2.8 *The system (1) is I-controllable if and only if there exists a semistate feedback matrix $K \in \mathbb{R}^{l \times n}$ such that the closed loop system is of index at most one.*

Proof Let the system (1) be I-controllable. Then, respecting Theorem, I-controllability is equivalent to the fact that the matrix A_{2122} is identically zero. Assume the semistate feedback matrix

$$K = U \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \end{bmatrix} N^T,$$

where K_{ij} are to be designed; matrices U and N are same as in the decomposition (9). If we take $\bar{A} = M(A + BK)N$, then we obtain

$$\bar{A} = \begin{bmatrix} A_{11} + B_{11}K_{11} + B_{12}K_{21} & A_{121} + B_{11}K_{12} + B_{12}K_{22} & A_{122} + B_{11}K_{13} + B_{12}K_{23} \\ A_{211} + B_{211}K_{11} + B_{212}K_{21} & \Sigma_{A_{22}} + B_{211}K_{12} + B_{212}K_{22} & B_{211}K_{13} + B_{212}K_{23} \\ A_{2121} + \Sigma_{B_{22}}K_{11} & \Sigma_{B_{22}}K_{12} & \Sigma_{B_{22}}K_{13} \\ A_{2122} & 0 & 0 \end{bmatrix}$$

If we choose $K_{11} = -\Sigma_{B_{22}}^{-1}A_{2121}$, $K_{12} = 0$, $K_{13} = 0$, K_{21} is an arbitrary matrix, $K_{22} = 0$, and $K_{23} = 0$, then Theorems 2.5 and Theorem 2.6 imply that the closed loop system is of index at most one. Conversely, suppose there exists a feedback matrix $K \in \mathbb{R}^{r \times n}$ given as $K = U \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \end{bmatrix} N^T$ such that $[EA + BK] \in \Sigma_{m,n}$ is of index at most one. Then, we need to prove that the system (1) is I-controllable. For, substituting E and $A + BK$ in (8) in decomposed form and simplifying, we obtain

$$\begin{aligned} & \text{rank} \begin{bmatrix} \Sigma_{A_{22}} + B_{211}K_{12} + B_{212}K_{22} & B_{211}K_{13} + B_{212}K_{23} \\ \Sigma_{B_{22}}K_{12} & \Sigma_{B_{22}}K_{13} \\ 0 & 0 \end{bmatrix} \\ &= \text{rank} \begin{bmatrix} A_{211} + B_{211}K_{11} + B_{212}K_{21} & \Sigma_{A_{22}} + B_{211}K_{12} + B_{212}K_{22} & B_{211}K_{13} + B_{212}K_{23} \\ A_{2121} + \Sigma_{B_{22}}K_{11} & \Sigma_{B_{22}}K_{12} & \Sigma_{B_{22}}K_{13} \\ A_{2122} & 0 & 0 \end{bmatrix} \end{aligned}$$

The last equality holds provided the first column block matrices on the right-hand side must be the linear combinations of the other two column block matrices. This ensures that the matrix A_{2122} is identically zero which implies the I-controllability of system (1) by Theorem 2.5. Hence the theorem is proved. \square

Note that the above theorem in its equivalent generality can also be found in the work of Berger and Reis (2013) where the authors have applied the feedback canonical form (Loiseau et al., 1991) to carry out the result. The beauty of the above theorem is that we have not only provided an independent simpler proof but also the feedback matrix is easily computable. Our approach to design the matrix K is inspired by the work of Hou (2004) where the author has designed an admissible K such that the pair $(EA + BK)$ has no impulsive modes and admits arbitrary initial condition if and only if the system is I-controllable. The proof provided by Hou (2004) uses a decomposition that can be viewed as a particular case of the decomposition (9). Here, the decomposition (9) has also been utilized in the following in designing a semistate feedback such that the closed loop system is of index at most two. For this, we write the following condition that is weaker than the I-controllability of system (1).

$$\text{rank} \begin{bmatrix} E & 0 & 0 & 0 & 0 \\ A & B & E & 0 & 0 \\ 0 & 0 & A & B & E \end{bmatrix} = \text{rank}[E \ A \ B] + \text{rank} \begin{bmatrix} E & 0 & 0 \\ A & B & E \end{bmatrix}. \tag{11}$$

For $\mu = 2$, the Equation (6) reduces to

$$\text{rank} \begin{bmatrix} E & 0 & 0 \\ A & E & 0 \\ 0 & A & E \end{bmatrix} = \text{rank}[E \ A] + \text{rank} \begin{bmatrix} E & 0 \\ A & E \end{bmatrix}. \tag{12}$$

Similar to Theorems 2.5 and 2.6, we now write the following theorems for equivalent conditions to (11) and (12), respectively.

THEOREM 2.9 *In view of the decomposition (9), the condition (11) is equivalent to*

$$\text{rank} \begin{bmatrix} \mathcal{A} & \mathcal{B} \end{bmatrix} = \text{rank} A_{2122}, \tag{13}$$

where $\mathcal{A} = A_{2122} \Sigma_E^{-1} A_{122}$ and $\mathcal{B} = A_{2122} \Sigma_E^{-1} B_{12} - A_{2122} \Sigma_E^{-1} A_{121} \Sigma_{A_{22}}^{-1} B_{212}$.

Proof Applying the decomposition (9) to the left-hand side of (11) and simplifying, it becomes

$$2 \text{rank} \Sigma_E + 2 \text{rank} \Sigma_{B_{22}} + \text{rank} \Sigma_{A_{22}} + \begin{bmatrix} A_{121} & A_{122} & B_{12} & \Sigma_E \\ \Sigma_{A_{22}} & 0 & B_{212} & 0 \\ 0 & 0 & 0 & A_{2122} \end{bmatrix},$$

which is further equivalent to

$$3 \text{rank} \Sigma_E + 2 \text{rank} \Sigma_{B_{22}} + \text{rank} \Sigma_{A_{22}} + \begin{bmatrix} \Sigma_{A_{22}} & 0 & B_{212} \\ -A_{2122} \Sigma_E^{-1} A_{121} & -A_{2122} \Sigma_E^{-1} A_{122} & -A_{2122} \Sigma_E^{-1} B_{12} \end{bmatrix},$$

which then reduces to

$$3 \text{rank} \Sigma_E + 2 \text{rank} \Sigma_{B_{22}} + 2 \text{rank} \Sigma_{A_{22}} + \begin{bmatrix} \mathcal{A} & \mathcal{B} \end{bmatrix}. \tag{14}$$

Again, applying the decomposition (9) to the right-hand side of (11) and simplifying, it becomes

$$3 \text{rank} \Sigma_E + 2 \text{rank} \Sigma_{B_{22}} + 2 \text{rank} \Sigma_{A_{22}} + \text{rank} A_{2122}. \tag{15}$$

The theorem follows in view of (14) and (15). □

THEOREM 2.10 *In view of the decomposition (9), the condition (12) is equivalent to*

$$\text{rank} \begin{bmatrix} A_{2121} \Sigma_E^{-1} A_{122} \\ A_{2122} \Sigma_E^{-1} A_{122} \end{bmatrix} = \text{rank} \begin{bmatrix} A_{2121} \\ A_{2122} \end{bmatrix}. \tag{16}$$

Proof The proof follows by proceeding in a similar fashion as in the proof of Theorem 2.8. □

We now provide a sufficient condition for the existence of a semistate feedback such that the closed loop system has index at most two in the following theorem.

THEOREM 2.11 *If the system (1) satisfies condition (11), $\text{Image } B_{12} \subset \text{Image } A_{122}$ and the matrix A_{121} is identically zero, then there exists a feedback matrix $K \in \mathbb{R}^{r \times n}$ such that $[E + BK] \in \Sigma_{m,n}$ is of index at most two.*

Proof We assume that the system (1) satisfies conditions (11), $\text{Image } B_{12} \subset \text{Image } A_{122}$, and the matrix A_{121} is identically zero. We already know from Theorem 2.8 that the condition (11) is equivalent to (13). Now, if we choose matrix $K = U \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \end{bmatrix} N^T$, with $K_{11} = -\Sigma_{B_{22}}^{-1} A_{2121}$, $K_{12} = 0$, $K_{13} = 0$, K_{21} is an arbitrary matrix, $K_{22} = 0$, and $K_{23} = 0$. Then we obtain

$$M(A + BK)N = \begin{bmatrix} A_{11} + B_{11}K_{11} + B_{12}K_{21} & A_{121} & A_{122} \\ A_{211} + B_{211}K_{11} + B_{212}K_{21} & \Sigma_{A_{22}} & 0 \\ A_{2121} + \Sigma_{B_{22}} K_{11} & 0 & 0 \\ A_{2122} & 0 & 0 \end{bmatrix}.$$

Using the decomposition (9) and Theorem 2.9, the condition (12) for the matrix pair $(E + BK)$ is equivalent to

$$\text{rank} A_{2122} \Sigma_E^{-1} A_{122} = \text{rank} A_{2122}. \tag{17}$$

The condition (17) is implied by (13), $\text{Image } B_{12} \subset \text{Image } A_{122}$, and the matrix A_{121} is zero. This completes the proof of the theorem. \square

3. Illustrating examples

Example 3.1 Let system (1) be represented by the following matrices

$$E = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, A = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (18)$$

It can be checked that the system (18) is not of index at most one and is I-controllable by (8) and (10), respectively. Applying Theorem 2.7, we obtain a desired feedback matrix K as

$$K = -1.$$

Now, it can be checked that the closed loop system $(E, A + BK)$ is of index at most one.

Example 3.2 Let system (1) be represented by the following matrices

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 2 & 3 & 1 \\ 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 2 \\ 1 & 2 & 3 & 1 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}. \quad (19)$$

It can be checked that the system (19) is not of index at most one and is I-controllable by (8) and (10), respectively. Applying Theorem 2.3 to the system (19), we can get all the decomposed matrices. To design a desired feedback matrix K , we need only the following two matrices

$$A_{2121} = \begin{bmatrix} 2.0000 & 0.0000 \\ 0.8944 & 2.6833 \end{bmatrix} \text{ and } \Sigma_{B_{22}} = \begin{bmatrix} 1.0000 & 0 \\ 0 & 0.4472 \end{bmatrix}.$$

Applying Theorem 2.7, we obtain

$$K = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 2 & 0 & 6 & 0 \end{bmatrix}.$$

Now, it can be checked that the closed loop system $(E, A + BK)$ is of index at most one.

Example 3.3 Let system (1) be represented by the following matrices

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 3 & 4 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 2 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}. \quad (20)$$

It can be checked that the system (20) does not satisfy (6) for $\mu = 1, 2, 3$. So, either it has η -blocks or the index of nilpotency of the matrix N_σ is more than three. Further, it can be checked that the system satisfies (11), $\text{Image } B_{12} \subset \text{Image } A_{122}$, and the matrix A_{121} is zero. Hence, we can apply our Theorem, to design a feedback matrix K such that the closed loop system is of index at most two, to obtain

$$K = \begin{bmatrix} -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now, it can be checked that the closed loop system $(E, A + BK)$ is of index at most two.

4. Concluding remarks

The paper was devoted to the index determination and reduction for general LTI descriptor systems. First, we proved a sufficient condition to calculate the index of a general LTI descriptor system. Then, it has been proved that the I-controllability is necessary and sufficient for the existence of a proportional semistate feedback such that the resulted closed loop system has index at most one. Further, a sufficient condition has been provided for the existence of a proportional semistate feedback such that the closed loop system is of index at most two. Finally, the presented theory has been demonstrated by some examples. In future, we would like to study the index concepts of nonlinear descriptor systems and delay descriptor systems.

Funding

This work was supported by the CSIR, New Delhi under [grant number 25(0195)11/EMR-II].

Author details

Vikas Kumar Mishra¹
E-mail: vikas.mishra45@gmail.com
Nutan Kumar Tomar¹
E-mail: nktomar@iitp.ac.in
Mahendra Kumar Gupta¹
E-mail: mahendra14389@gmail.com

¹ Department of Mathematics, Indian Institute of Technology Patna, Patna, India.

Citation information

Cite this article as: Index reduction for rectangular descriptor systems via feedbacks, Vikas Kumar Mishra, Nutan Kumar Tomar & Mahendra Kumar Gupta, *Cogent Engineering* (2017), 4: 1319786.

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