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Boundary controllability of impulsive nonlinear fractional delay integro-differential system

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Abstract: By using the strongly continuous semigroup theory and the Banach contraction principle, we study the boundary controllability of time varying delay impulsive nonlinear fractional integrodifferential system in Banach spaces. An example is provided to illustrate the theory.

Subjects: Differential Equations; Functional Analysis; Mathematical Analysis

Keywords: impulsive fractional nonlinear integrodifferential system; mild solution; boundary controllability; fixed point theorem; semigroup theory

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1. Introduction

Recently, there has been increasing interest in studying the problem of controllability of nonlinear systems. Zhang (2000) obtained results on local exact controllability of semilinear integrodifferential systems in abstract spaces by means of Banach fixed point theorem. Balasubramanian, Dauer, and Loganathan (2002) considered a class of semilinear functional integrodifferential equations in Banach space setting and provided sufficient conditions for the controllability. Balachandran, Balasubramaniam, and Dauer (1996) established sufficient conditions for the local null controllability of nonlinear functional differential systems. Dauer and Balasubramaniam (1997) established sufficient conditions for the null controllability of semilinear integrodifferential systems in Banach space. Kwun, Park, and Ryu (1991) discussed the approximate controllability for delay Volterra systems while Balachandran and Sakthivel (1998) established a set of sufficient conditions for the controllability of delay integrodifferential systems in Banach spaces. Several abstract settings have

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PUBLIC INTEREST STATEMENT

The notion of controllability is of great importance in mathematical control theory. Many fundamental problems of control theory such as pole-assignment, stabilizability and optimal control may be solved under the assumption that the system is controllable. The problem of controllability is to show the existence of control function, which steers the solution of the system from its initial state to final state.

Also, impulsive differential equations, i.e. differential equations involving impulse effect, appear as a natural description of observed evolution phenomena of several real world problems.

Therefore, in this paper, we study the boundary controllability of time varying delay impulsive nonlinear fractional integrodifferential system in Banach spaces using the strongly continuous semigroup theory and the Banach contraction principle.

been developed to describe the distributed control system on a domain in which the control is enacted through the boundary. Fattorini (1968) developed a semigroup approach for boundary control systems. Balakrishnan (1976) showed that the solution of a parabolic boundary control equation with L^2 controls can be expressed as a mild solution to an operator equation using semigroup theory. Barbu (1980) discussed the general theory of boundary control systems and the existence of solutions for boundary control systems governed by parabolic equations with nonlinear boundary conditions. Balachandran and Anandhi (2000, 2001a, 2001b) discussed the boundary controllability of semilinear systems and delay integrodifferential systems in Banach spaces. Hamdy (see Ahmed, 2010, 2012) discussed the boundary controllability of nonlinear fractional integrodifferential systems. In this paper we study the boundary controllability of delay nonlinear fractional integrodifferential system.

Let E and U be a two real Banach spaces with $\|\cdot\|$ and $|\cdot|$, respectively. Let σ be a closed, linear and densely defined operator in E. In addition, let τ be a linear operator (the boundary operator) with domain in E and range in some Banach space X. We consider the following boundary control delay nonlinear fractional integrodifferential system of the form

$$\begin{cases} {}^cD^\alpha x(t) = \sigma x(t) + f(t, x(\gamma_1(t)), \int_0^t h(t,s) g(s, x(\gamma_2(s)))ds); \quad \tau x(t) = B_1 u(t), \quad t \in J = [0,b], \quad t \neq t_k, \\ \Delta x|_{t=t_k} = I_k(x(t_k^-)), \quad k = 1, 2, \ldots, m \\ x(0) = x_0, \end{cases} \tag{1.1}$$

where ${}^cD^\alpha$ is the Caputo fractional derivative of order $0 < \alpha < 1$, the delay.

$\gamma_i(t): J \to J, i = 1, 2,$ are continuous functions, the state $x(\cdot)$ takes values in the Banach space E, $B_1: U \to X$ is a linear continuous operator, the control function $u \in L^2(J, U)$, a Banach space of admissible control functions, $h: J \times J \to R$ is a continuous function, $\Delta x|_{t=t_k} = I_k(x(t_k^-))$, where $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively and the nonlinear operators $f: J \times E \times E \to E, g: J \times E \to E$ are given. Let $A: E \to E$ be the linear operator defined by

$$D(A) = \{x \in D(\sigma); \tau x = 0\}, Ax = \{\sigma x, \ \text{for}\ x \in D(A)\}.$$

The operator A is the infinitesimal generator of an analytic semigroup $T(t)$ on E and there exists a constant $M > 0$ such that $\|T(t)\| \leq M$. We assume without loss of generality that $0 \in \rho(A)$. This allows us to define the fractional power $(-A)^\gamma$, for $0 < \gamma \leq 1$, as a closed linear operator on its domain $D((-A)^\gamma)$ with inverse $(-A)^{-\gamma}$.

We will introduce the following basic properties of $(-A)^\gamma$.

THEOREM 1.1 (see Pazy, 1983).

(1) $E_\gamma = D((-A)^\gamma)$ is a Banach space with the norm $\| x \|_\gamma = \| (-A)^\gamma x \|, \quad x \in E_\gamma$.
(2) $S(t): E \to E_\gamma$ for each $t > 0$ and $(-A)^\gamma S(t)x = S(t)(-A)^\gamma x$ for each $x \in E_\gamma$ and $t \geq 0$.
(3) For every $t > 0$, $(-A)^\gamma S(t)$ is bounded on E and there exists a positive constant C_γ such that $\| (-A)^\gamma S(t) \| \leq \frac{C_\gamma}{t^\gamma}$.
(4) If $0 < \beta < \gamma \leq 1$, then $E_\gamma \hookrightarrow E_\beta$ and the embedding is compact whenever the resolvent operator of A is compact.

2. Preliminaries

Let us recall the following known definitions.

Definition 2.1 (see Podlubny, & EI-Sayed, 1996; Podlubny, 1999; Miller & Ross, 1993; Samko, Kilbas, & Marichev, 1993). The fractional integral of order α with the lower limit zero for a function f can be defined as

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(s)}{(t-s)^{1-\alpha}}ds, \quad t > 0, \quad \alpha > 0,$$

provided the right-hand side is pointwise defined on $[0, \infty)$, where Γ is the Gamma function.

Definition 2.2 (see Miller & Ross, 1993; Podlubny & EI-Sayed, 1996; Podlubny, 1999; Samko et al., 1993). The Caputo derivative of order α with the lower limit zero for a function f can be written as

$$^{c}D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}}ds = I^{n-\alpha}f^{(n)}(t), \quad t > 0, \quad 0 \le n-1 < \alpha < n.$$

If f is an abstract function with values in X, then the integrals appearing in the above definitions are taken in Bochner's sense.

Let $Y = C(J, B_r)$ and $B_r = \{y \in Y: \|y\| \le r\}$ for some $r > 0$.

We assume the following hypotheses to prove the controllability of the system (1.1):

(H1) $D(\sigma) \subset D(\tau)$ and the restriction of τ to $D(\sigma)$ is continuous relative to graph norm of $D(\sigma)$.

(H2) There exists a linear continuous operator $B:U \to E$ such that $\sigma B \in L(U, E)$; $\tau(Bu) = B_1 u$ for all $u \in U$. Also $Bu(t)$ is continuously differentiable and $\|(-A)^{\alpha} Bu\| \le C\|B_1 u\|$ for all $u \in U$, where C is some positive constant.

(H3) (i) $f:J \times E \times E \to E$ is continuous and there exist constants $N_1 > 0$ and $N_2 > 0$ such that for all $v_1, v_2 \in B_r$ and $w_1, w_2 \in E$ we have

$$\|f(t, v_1, w_1) - f(t, v_2, w_2)\| \le N_1[\|v_1 - v_2\| + \|w_1 - w_2\|], \quad N_2 = \max_{t \in J}\|f(t, 0, 0)\|.$$

(ii) $g:J \times E \to E$ is continuous and there exist constants $L_1 > 0$ and $L_2 > 0$ such that for all $v_1, v_2 \in B_r$ we have

$$\|g(t, v_1) - g(t, v_2)\| \le L_1[\|v_1 - v_2\|], \quad L_2 = \max_{t \in J}\|g(t, 0)\|.$$

(iii) The functions $I_k:E \to E$ are continuous and there exist constants $L_3 > 0$, $L_4 > 0$, such that for all $v_1, v_2 \in B_r$ we have

$$\|I_k(v_1) - I_k(v_2)\| \le L_3[\|v_1 - v_2\|], \quad L_4 = \max_{t \in J}\|I_k(0)\|.$$

(H4) There exists a constant L such that $|h(t, s)| \le L$ for $(t, s) \in J \times J$.

(H5) There exists a constant q such that for all $x_1, x_2 \in E\|x_1(\gamma_i(t)) - x_2(\gamma_i(t))\| \le q\|x_1(t) - x_2(t)\|$, for $i = 1, 2$.

(H6) The linear operator W from $L^2(J, U)$ into E is defined by

$$Wu = \int_0^b (b-s)^{\alpha-1}[\sigma T_{\alpha}(b-s) - AT_{\alpha}(b-s)]Bu(s)ds$$

has an induced inverse operator W^{-1} which takes values in $L^2(J, U)/kerW$ and there exists a positive constant K, $K_1 > 0$ and $K_2 > 0$ such that $\|(-A)^{-\beta}\| \le K$, $0 < \beta \le 1$, $\|B_1\| \le K_1$ and $\|W^{-1}\| \le K_2$ (see Quinn & Carmichael 1984, 1988).

(H7) There exists a constant $r > 0$ such that

$$M\|x_0\| + K_1K_2\left[\frac{M\|\sigma\|b^\alpha}{\Gamma(\alpha+1)} + \frac{KC_{1-\beta}\Gamma(1+\beta)b^{\alpha\beta}}{\beta\Gamma(1+\alpha\beta)}\right][\|x_1\| + M\|x_0\| + \frac{b^\alpha M}{\Gamma(\alpha+1)}\left(N_1(r + bL(L_1r + L_2)) + N_2\right) + \frac{\alpha Mm}{\Gamma(\alpha+1)}(L_3r + L_4)] + \frac{b^\alpha M}{\Gamma(\alpha+1)}[N_1(r + bL(L_1r + L_2)) + N_2] + \frac{\alpha Mm}{\Gamma(\alpha+1)}(L_3r + L_4) \le r.$$

Let $x(t)$ be the solution of the system (1.1). Then we can define a function $z(t) = x(t) - Bu(t)$ and from our assumption we see that $z(t) \in D(A)$. Hence (1.1) can be written in terms of A and B as

$$\begin{cases} {}^cD^\alpha z(t) &= Az(t) + \sigma Bu(t) - B\,({}^cD^\alpha u(t)) + f(t, x(\gamma_1(t)), \int_0^t h(t,s)g(s, x(\gamma_2(s))))ds, \quad t \in J, \quad t \neq t_k, \\ \Delta z|_{t=t_k} &= \Delta x|_{t=t_k}, \quad k = 1, 2, \dots, m \\ z(0) &= x_0 - Bu(0). \end{cases} \tag{2.1}$$

From (2.1) and fractional calculus, the integral form of the system (1.1) can be written in the form

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{Ax(s)}{(t-s)^{1-\alpha}}ds - \frac{1}{\Gamma(\alpha)}\int_0^t \frac{ABu(s)}{(t-s)^{1-\alpha}}ds + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\sigma Bu(s)}{(t-s)^{1-\alpha}}ds + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(s, x(\gamma_1(s)), \int_0^s h(s,\tau)g(\tau, x(\gamma_2(\tau)))d\tau)}{(t-s)^{1-\alpha}}ds, \quad t \in J, \quad t \neq t_k,$$

$$\Delta x|_{t=t_k} = I_k(x(t_k^-)), \quad k = 1, 2, \dots, m.$$

and hence, the mild solution of the system (1.1) is given by

$$x(t) = S_\alpha(t)x_0 + \int_0^t (t-s)^{\alpha-1}[\sigma T_\alpha(t-s) - AT_\alpha(t-s)]Bu(s)ds + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s, x(\gamma_1(s)), \int_0^s h(s,\tau)g(\tau, x(\gamma_2(\tau)))d\tau)ds + \sum_{0<t_k<t} T_\alpha(t-t_k)I_k(x(t_k^-)), t \in J$$

(see El-Borai, 2002, 2006; Zhou, Jiao, & Li, 2010) where ξ_α is a probability density function defined on $(0, \infty)$ and

$$S_\alpha(t)x = \int_0^\infty \xi_\alpha(\theta)T(t^\alpha\theta)xd\theta, \quad T_\alpha(t)x = \alpha\int_0^\infty \theta\xi_\alpha(\theta)T(t^\alpha\theta)xd\theta.$$

Remark 2.1 $\xi_\alpha(\theta) \ge 0, \theta \in (0, \infty), \int_0^\infty \xi_\alpha(\theta)d\theta = 1$ and $\int_0^\infty \theta\xi_\alpha(\theta)d\theta = \frac{1}{\Gamma(1+\alpha)}$ (see Zhou et al. (2010)).

Definition 2.3 The system (1.1) is said to be controllable on the interval J if for every

$x_0, x_1 \in E$, there exists a control $u \in L^2(J, U)$ such that the solution $x(\cdot)$ of the system (1.1) satisfies $x(b) = x_1$.

LEMMA 2.1 *(see El-Borai, 2006). The operators $S_\alpha(t)$ and $T_\alpha(t)$ have the following properties:*

(I) for any fixed $x \in E$, $\| S_\alpha(t)x \| \le M \| x \|$, $\| T_\alpha(t)x \| \le \frac{\alpha M}{\Gamma(\alpha+1)} \| x \|$;
(II) $\{S_\alpha(t), t \ge 0\}$ and $\{T_\alpha(t), t \ge 0\}$ are strongly continuous;
(III) for every $t > 0$, $S_\alpha(t)$ and $T_\alpha(t)$ are also compact operators;
(IV) for any $x \in E$, $\beta \in (0,1)$ and $\delta \in (0,1)$, we have $-AT_\alpha(t)x = (-A)^{1-\beta}T_\alpha(t)(-A)^\beta x$ and $\| (-A)^\delta T_\alpha(t) \| \le \frac{\alpha C_\delta\Gamma(2-\delta)}{t^{\alpha\delta}\Gamma(1+\alpha(1-\delta))}$, $t \in (0, b]$.

3. Main result

THEOREM 3.1 *If the hypotheses (H1)–(H7) are satisfied, then the problem (1.1) is controllable on J provided that*

$$[b^\alpha N_1 + b^{\alpha+1} N_1 L L_1 + \alpha m]\left[\frac{M^2 b^\alpha K_1 K_2 \|\sigma\|}{\Gamma^2(\alpha+1)} + \frac{MKK_1K_2C_{1-\beta}\Gamma(1+\beta)b^{\alpha\beta}}{\beta\Gamma(1+\alpha)\Gamma(1+\alpha\beta)} + \frac{M}{\Gamma(\alpha+1)}\right] \le \Lambda, \quad 0 \le \Lambda < 1.$$

Proof Using the hypothesis (H6), for an arbitrary function $x(\cdot)$ define the control

$$\begin{aligned} u(t) = W^{-1}[x_1 - S_\alpha(b)x_0 - \int_0^b (b-s)^{\alpha-1} T_\alpha(b-s) f(s, x(\gamma_1(s)), \int_0^s h(s,\tau) g(\tau, x(\gamma_2(\tau)))d\tau)ds \\ + \sum_{0<t_k<b} T_\alpha(b-t_k) I_k(x(t_k^-))](t). \end{aligned}$$

We shall show that the operator Φ defined by

$$\begin{aligned} \Phi x(t) = S_\alpha(t)x_0 + \int_0^t (t-s)^{\alpha-1}[\sigma T_\alpha(t-s) - AT_\alpha(t-s)]BW^{-1}[x_1 - S_\alpha(b)x_0 \\ - \int_0^b (b-\tau)^{\alpha-1} T_\alpha(b-\tau) f(\tau, x(\gamma_1(\tau)), \int_0^\tau h(\tau,\eta) g(\eta, x(\gamma_2(\eta)))d\eta)d\tau \\ + \sum_{0<t_k<b} T_\alpha(b-t_k) I_k(x(t_k^-))](s)ds \\ + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s, x(\gamma_1(s)), \int_0^s h(s,\tau) g(\tau, x(\gamma_2(\tau)))d\tau)ds \\ + \sum_{0<t_k<t} T_\alpha(t-t_k) I_k(x(t_k^-)) \end{aligned}$$

has a fixed point. This fixed point is then a solution of (1.1). Clearly $\Phi x(b) = x_1$, which means that the control u steers the impulsive fractional delay integrodifferential system (1.1) from the initial state x_0 to final state x_1 in time b provided we can obtain a fixed point of the nonlinear operator Φ.

First we show that Φ maps Y into itself. For $x \in Y$,

$$\begin{aligned} \|\Phi x(t)\| \le \|S_\alpha(t)x_0\| + \int_0^t (t-s)^{\alpha-1} \|[\sigma T_\alpha(t-s) - AT_\alpha(t-s)]BW^{-1}[x_1 - S_\alpha(b)x_0 \\ - \int_0^b (b-\tau)^{\alpha-1} T_\alpha(b-\tau) f(\tau, x(\gamma_1(\tau)), \int_0^\tau h(\tau,\eta) g(\eta, x(\gamma_2(\eta)))d\eta)d\tau + \sum_{0<t_k<b} T_\alpha(b-t_k) I_k(x(t_k^-))](s)\|ds \\ + \int_0^t (t-s)^{\alpha-1} \|T_\alpha(t-s) f(s, x(\gamma_1(s)), \int_0^s h(s,\tau) g(\tau, x(\gamma_2(\tau)))d\tau)\|ds + \sum_{0<t_k<t} \|T_\alpha(t-t_k) I_k(x(t_k^-))\| \\ \le \|S_\alpha(t)x_0\| + \int_0^t (t-s)^{\alpha-1}[\|T_\alpha(t-s)\|\|\sigma\| + \|(-A)^{1-\beta}T_\alpha(t-s)\|\|(-A)^\beta\|]\|B_1\|\|W^{-1}\|[\|x_1\| + \|S_\alpha(b)x_0\| \\ + \int_0^b (b-\tau)^{\alpha-1}\|T_\alpha(b-\tau)\|[\|f(\tau, x(\gamma_1(\tau)), \int_0^\tau h(\tau,\eta) g(\eta, x(\gamma_2(\eta)))d\eta) - f(\tau,0,0)\| + \|f(\tau,0,0)\|]d\tau \\ + \sum_{0<t_k<b} \|T_\alpha(b-t_k)\|[\|I_k(x(t_k^-)) - I_k(x(0))\| + \|I_k(x(0))\|]]ds \\ + \int_0^t (t-s)^{\alpha-1}\|T_\alpha(t-s)\|[\|f(s, x(\gamma_1(s)), \int_0^s h(s,\tau) g(\tau, x(\gamma_2(\tau)))d\tau) - f(s,0,0)\| + \|f(s,0,0)\|]ds \\ + \sum_{0<t_k<t} \|T_\alpha(t-t_k)\|[\|I_k(x(t_k^-)) - I_k(x(0))\| + \|I_k(x(0))\|] \\ \le M\|x_0\| + K_1K_2\left[\frac{M\|\sigma\|b^\alpha}{\Gamma(\alpha+1)} + \frac{KC_{1-\beta}\Gamma(1+\beta)b^{\alpha\beta}}{\beta\Gamma(1+\alpha\beta)}\right]\left[\|x_1\| + M\|x_0\| + \frac{b^\alpha M}{\Gamma(\alpha+1)}\left(N_1(r + bL(L_1 r + L_2)) + N_2\right)\right. \\ \left. + \frac{\alpha M m}{\Gamma(\alpha+1)}(L_3 r + L_4)\right] + \frac{b^\alpha M}{\Gamma(\alpha+1)}[N_1(r + bL(L_1 r + L_2)) + N_2] + \frac{\alpha M m}{\Gamma(\alpha+1)}(L_3 r + L_4) \le r. \end{aligned}$$

Thus Φ maps Y into itself.

Next for x_1, $x_2 \in Y$ we obtain

$$
\begin{aligned}
&\|\Phi x_1(t) - \Phi x_2(t)\| \le \\
&\int_0^t (t-s)^{\alpha-1}[\|T_\alpha(t-s)\|\|\sigma\| + \|(-A)^{1-\beta}T_\alpha(t-s)\|\|(-A)^\beta\|]\|B_1\|\|W^{-1}\|[\int_0^b (b-\tau)^{\alpha-1}\|T_\alpha(b-\tau)\| \\
&\times\{\|f(\tau, x_1(\gamma_1(\tau)), \int_0^\tau h(\tau,\eta)g(\eta, x_1(\gamma_2(\eta)))d\eta) - f(\tau, x_2(\gamma_1(\tau)), \int_0^\tau h(\tau,\eta)g(\eta, x_2(\gamma_2(\eta)))d\eta\|\}d\tau \\
&+\sum_{0<t_k<b}\|T_\alpha(b-t_k)\|\|I_k(x_1(t_k^-)) - I_k(x_2(t_k^-))\|]ds + \int_0^t (t-s)^{\alpha-1}\|T_\alpha(t-s)\| \\
&\times\{\|f(s, x_1(\gamma_1(s)), \int_0^s h(s,\tau)g(\tau, x_1(\gamma_2(\tau)))d\tau) - f(s, x_2(\gamma_1(s)), \int_0^s h(s,\tau)g(\tau, x_2(\gamma_2(\tau)))d\tau)\|\}ds \\
&+\sum_{0<t_k<t}\|T_\alpha(t-t_k)\|\|I_k(x_1(t_k^-)) - I_k(x_2(t_k^-))\| \\
&\le K_1K_2\left[\frac{M\|\sigma\|b^\alpha}{\Gamma(\alpha+1)} + \frac{KC_{1-\beta}\Gamma(1+\beta)b^{\alpha\beta}}{\beta\Gamma(1+\alpha\beta)}\right]\left\{\frac{b^\alpha MN_1}{\Gamma(\alpha+1)}[\|x_1(\gamma_1(\tau)) - x_2(\gamma_1(\tau))\| + bLL_1\|x_1(\gamma_2(\eta)) - x_2(\gamma_2(\eta))\|]\right. \\
&\left.+\frac{\alpha M}{\Gamma(\alpha+1)}[\sum_{0<t_k<b}\|x_1(t_k^-) - x_2(t_k^-)\|]\right\} + \frac{b^\alpha MN_1}{\Gamma(\alpha+1)}[\|x_1(\gamma_1(s)) - x_2(\gamma_1(s))\| + bLL_1\|x_1(\gamma_2(\tau)) - x_2(\gamma_2(\tau))\|] \\
&+\frac{\alpha M}{\Gamma(\alpha+1)}[\sum_{0<t_k<t}\|x_1(t_k^-) - x_2(t_k^-)\|] \\
&\le \left[\frac{M^2b^\alpha K_1K_2\|\sigma\|}{\Gamma^2(\alpha+1)} + \frac{MKK_1K_2C_{1-\beta}\Gamma(1+\beta)b^{\alpha\beta}}{\beta\Gamma(1+\alpha)\Gamma(1+\alpha\beta)}\right]\left\{\sup_{t\in J}\|x_1(\gamma_1(t)) - x_2(\gamma_1(t))\|N_1b^\alpha(1+bLL_1)\right. \\
&\left.+\alpha\, m\sup_{t\in J}\|x_1(t) - x_2(t)\|\right\} + \frac{b^\alpha MN_1}{\Gamma(\alpha+1)}\sup_{t\in J}\|x_1(\gamma_1(t)) - x_2(\gamma_1(t))\|[1+bLL_1] + \frac{\alpha M\, m}{\Gamma(\alpha+1)}\sup_{t\in J}\|x_1(t) - x_2(t)\| \\
&\le [b^\alpha N_1 + b^{\alpha+1}N_1LL_1 + \alpha\, m][\frac{M^2b^\alpha K_1K_2\|\sigma\|}{\Gamma^2(\alpha+1)} + \frac{MKK_1K_2C_{1-\beta}\Gamma(1+\beta)b^{\alpha\beta}}{\beta\Gamma(1+\alpha)\Gamma(1+\alpha\beta)} + \frac{M}{\Gamma(\alpha+1)}]\|x_1(t) - x_2(t)\| \\
&\le \Lambda\|x_1(t) - x_2(t)\|.
\end{aligned}
$$

Since $0 \le \Lambda < 1$ then, Φ is a contraction mapping and hence there exists unique fixed point $x \in Y$ such that $\Phi x(t) = x(t)$. Any fixed point of Φ is a mild solution of (1.1) on J which satisfies $x(b) = x_1$. Thus the system (1.1) is controllable on J. □

4. Application

Let Ω be a bounded, open subset of R^n, and let Γ be a sufficiently smooth boundary of Ω.

Consider the following fractional delay integro-partial differential equation,

$$
\begin{cases}
\frac{\partial^\alpha z(t,y)}{\partial t^\alpha} - \Delta z(t,y) = z(t-\rho, y) + \int_0^t \sin z(s-\rho, y)ds, \quad t\in J, \quad y\in\Omega, \quad t\ne t_k, \\
z(t,0) = u(t,0), \quad \text{on} \quad \Sigma = (0,b)\times\Gamma, \quad t\in J, \quad t\ne t_k, \\
z(t_k^+) - z(t_k^-) = I_k(z(t_k^-)), \quad k = 1,2,\ldots,m, \\
z(t,y) = 0, \quad z(0,y) = z_0(y), \quad y\in\Omega,
\end{cases}
\tag{4.1}
$$

where $\alpha \in (0,1)$, $z_0 \in L^2(\Omega)$ and $u \in L^2(\Sigma)$. Take $E = L^2(\Omega)$, $X = H^{-1/2}(\Gamma)$, $U = L^2(\Gamma)$ $B_1 = I$, the identity operator and $\sigma z = \Delta z$ with domain $D(\sigma) = \{z \in L^2(\Omega): \Delta z \in L^2(\Omega)\}$.

The operator τ is the trace operator such that $\tau z = z|_\Gamma$ is well defined and belongs to $H^{-1/2}(\Gamma)$ for each $z \in D(\sigma)$.

Define the operator $A: D(A) \subset E \to E$ is given by $Az = \Delta z$ with domain $D(A) = H_0^1(\Omega) \cup H^2(\Omega)$ where $H^k(\Omega)$, $H^\beta(\Gamma)$ are usual Sobolev space on Ω, Γ.

It is well known that A generates an analytic semigroup $T(\cdot)$. The spectrum of A consists of the eigenvalues n^2 with corresponding normalized eigenvectors $z_n(y) = \sqrt{2}\sin ny,\ n = 1, 2, 3, \ldots$

In addition, the following properties hold

(a) $\{z_n, \quad n = 1, 2, 3, \ldots\}$ is an orthonormal basis of E,

(b) If $z \in D(A)$ then $Az = \sum_{n=1}^{\infty} n^2 (z, z_n) z_n$,

(c) $T(t)z = \sum_{n=1}^{\infty} e^{\frac{n^2}{1+n^2}t}(z, z_n)z_n$, for every $z \in E$.

We define the linear operator $B{:}L^2(\Gamma) \to L^2(\Omega)$ by $Bu = v_u$, where $v_u \in L^2(\Omega)$ is the unique solution to the Dirichlet boundary value problem,

$$\Delta v_u = 0 \quad \text{in} \quad \Omega,$$

$$v_u = u \quad \text{in} \quad \Gamma.$$

We introduce the following functions:

$$\int_0^t h(t,s)g(s, z(s-\rho))(y)ds = \int_0^t \sin z(s-\rho, y)ds,$$

$$f\left(t, z(t-\rho), \int_0^t h(t,s)g(s, z(s-\rho))ds\right)(y) = z(t-\rho, y) + \int_0^t \sin z(s-\rho, y)ds,$$

where $h(t,s) = 1$. Obviously

$$\begin{aligned}\|[z(t-\rho, y) + \int_0^t \sin z(s-\rho, y)ds] - [(x(t-\rho, y)\\ + \int_0^t \sin x(s-\rho, y)ds]\| \leq (1+b)\|z(s-\rho, y) - x(s-\rho, y)\|.\end{aligned}$$

Choose b and other constants such that the conditions (H1)–(H7) are satisfied. Consequently Theorem 3.1 can be applied for the system (4.1), so the system (4.1) is controllable on [0, b].

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