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## SYSTEMS & CONTROL | RESEARCH ARTICLE

# Frequency interval balanced truncation of discrete-time bilinear systems

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**Abstract:** This paper presents the development of a new model reduction method for discrete-time bilinear systems based on the balanced truncation framework. In many model reduction applications, it is advantageous to analyze the characteristics of the system with emphasis on particular frequency intervals of interest. In order to analyze the degree of controllability and observability of discrete-time bilinear systems with emphasis on particular frequency intervals of interest, new generalized frequency interval controllability and observability gramians are introduced in this paper. These gramians are the solution to a pair of new generalized Lyapunov equations. The conditions for solvability of these new generalized Lyapunov equations are derived and a numerical solution method for solving these generalized Lyapunov equations is presented. Numerical examples which illustrate the usage of the new generalized frequency interval controllability and observability gramians as part of the balanced truncation framework are provided to demonstrate the performance of the proposed method.

**Subjects:** Dynamical Control Systems; Non-Linear Systems; Systems & Control Engineering

**Keywords:** model reduction; bilinear systems; balanced truncation; frequency interval gramians; finite frequency interval



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This research work is applicable to both engineering and mathematical problems which can be formulated as bilinear systems.

### PUBLIC INTEREST STATEMENT

Nonlinear mathematical models are commonly used to describe the processes in many branches of engineering. Bilinear systems are an important class of nonlinear systems which have well-established theories and are applicable to many practical applications. Mathematical models in the form of bilinear systems can be found in a variety of fields such as the mathematical models which describe the processes of electrical networks, hydraulic systems, heat transfer, and chemical processes. Many nonlinear systems can be modeled as bilinear systems with appropriate state feedback or can be approximated as bilinear systems by using the bilinearization process. The mathematical modeling of a large-scale bilinear system may result in a high-order bilinear model. To address the complexity associated with high-order models, we present a new model reduction technique for discrete-time bilinear systems.

## 1. Introduction

Model reduction which is of fundamental importance in many modeling and control applications deals with the approximation of a higher order model by a lower order model such that the input-output behavior of the original system is preserved to a required accuracy. The balanced truncation model reduction technique originally developed by Moore for continuous-time linear systems is one of the most widely applied model reduction techniques (Moore, 1981). In recent years, many variations to this original balanced truncation technique have been developed (Li, Yu, Gao, & Zhang, 2014; Minh, Battle, & Fossas, 2014; Opmeer & Reis, 2015; Zhang, Wu, Shi, & Zhao, 2015).

One of the further developments to the original balanced truncation technique was the work by Gawronski and Juang which involved the development of frequency interval controllability and observability gramians (Gawronski & Juang, 1990). The significance of emphasizing particular frequency intervals of interest in a variety of control engineering problems has led to extensive theoretical developments in robust control techniques which emphasize particular frequency intervals of interest which have been presented in (Ding, Du, & Li, 2015; Ding, Li, Du, & Xie, 2016; Du, Fan, & Ding, 2016; Imran & Ghafoor, 2015; Li & Yang, 2015; Li, Yin, & Gao, 2014; Li, Yu, & Gao, 2015).

In the context of discrete-time systems, digital systems are designed to work with signals with known frequency characteristics, therefore it is essential to have model reduction techniques which generate reduced-order models which function well with signals which have specified frequency characteristics. The works by Horta, Juang, and Longman (1993), Wang and Zilouchian (2000) and more recently by Imran and Ghafoor (2014) described the formulation of frequency interval gramians for discrete-time systems.

Bilinear systems are an important category of non-linear systems which have well-established theories (Al-Baiyat, Bettayeb, & Al-Saggaf, 1994; Dorissen, 1989; D'Alessandro, Isidori, & Ruberti, 1974; Shaker & Tahavori, 2014b, 2015). Many non-linear systems in various branches of engineering can be well represented by bilinear systems. Similar to the case of linear systems, the mathematical modeling process to obtain bilinear system models may result in obtaining high-order models. Fortunately, by formulating a state space model for these bilinear system models, the application of model reduction techniques becomes possible to reduce the order of these bilinear system models. The balanced truncation technique for continuous-time bilinear systems has been presented in Zhang & Lam (2002) whereas the balanced truncation technique for discrete-time bilinear systems has been presented in Zhang, Lam, Huang, and Yang (2003). More recently further developments have been carried out to the original balanced truncation technique for continuous-time bilinear systems in order to reduce the approximation error between the outputs of the original bilinear model and reduced-order bilinear model by incorporating time and frequency interval techniques (Shaker & Tahavori, 2014a, 2014c).

The contributions of this paper are as follows. Firstly, new generalized frequency interval controllability and observability gramians are defined for discrete-time bilinear systems. Secondly, it is shown that these frequency interval controllability and observability gramians are solutions to a pair of new generalized Lyapunov equations. Thirdly, conditions for solvability of these new generalized Lyapunov equation are proposed together with a numerical solution method for solving these new Lyapunov equations. Finally, numerical examples are provided to demonstrate the performance of the proposed method relative to existing techniques.

The notation used in this paper is as follows.  $M^*$  refers to the transpose of the matrix  $M$  if  $M \in \mathbb{R}^{n \times m}$  and complex conjugate transpose if  $M \in \mathbb{C}^{n \times m}$ . The  $\otimes$  symbol denotes a Kronecker product.

## 2. Preliminaries

### 2.1. Controllability and observability gramians of discrete-time linear systems

Considering the following time-invariant and asymptotically stable discrete-time linear system  $(A, B, C)$ :

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned} \tag{1}$$

where  $u \in \mathbb{R}^p$ ,  $y \in \mathbb{R}^q$ ,  $C \in \mathbb{R}^n$  are the input, output and states respectively.  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times p}$ ,  $C \in \mathbb{R}^{q \times n}$  are matrices with appropriate dimensions.

**Definition 1** The discrete-time domain controllability and observability gramian definitions are given by:

$$P = \sum_{k=0}^{\infty} A^k B B^* (A^*)^k \tag{2}$$

$$Q = \sum_{k=0}^{\infty} (A^*)^k C^* C A^k \tag{3}$$

**Remark 1** It is established that (2) and (3) satisfy the following Lyapunov equations:

$$A P A^* - P + B B^* = 0 \tag{4}$$

$$A^* Q A - Q + C^* C = 0 \tag{5}$$

**Remark 2** By applying a direct application of Parseval's theorem to (2) and (3), the controllability and observability gramians in the frequency domain are given by:

$$P = \frac{1}{2\pi} \int_{-\pi}^{\pi} (e^{j\theta} I - A)^{-1} B B^* (e^{-j\theta} I - A^*)^{-1} d\theta \tag{6}$$

$$Q = \frac{1}{2\pi} \int_{-\pi}^{\pi} (e^{-j\theta} I - A^*)^{-1} C^* C (e^{j\theta} I - A)^{-1} d\theta \tag{7}$$

where  $I$  is an identity matrix.

### 2.2. Frequency interval controllability and observability gramians of discrete-time linear systems

**Definition 2** The frequency interval controllability and observability gramians for discrete-time systems are defined as (Horta et al., 1993):

$$P_{cf} = \frac{1}{2\pi} \int_{\delta\theta} (e^{j\theta} I - A)^{-1} B B^* (e^{-j\theta} I - A^*)^{-1} d\theta \tag{8}$$

$$Q_{cf} = \frac{1}{2\pi} \int_{\delta\theta} (e^{-j\theta} I - A^*)^{-1} C^* C (e^{j\theta} I - A)^{-1} d\theta \tag{9}$$

where  $\delta\theta = [\theta_1, \theta_2]$  is the frequency range of operation and  $0 \leq \theta_1 < \theta_2 \leq \pi$ . Due to the symmetry of the discrete Fourier transform, the integration is carried out throughout the frequency intervals  $[\theta_1, \theta_2]$  and  $[-\theta_2, -\theta_1]$  (Horta et al., 1993). Therefore the gramians  $P_{cf}$  and  $Q_{cf}$  in (8) and (9) will always be real.

Remark 3 It has been shown that the frequency interval controllability and observability gramians defined in (8) and (9) are the solutions to the following Lyapunov equations (Wang et al., 2000):

$$AP_{cf}A^* - P_{cf} + X_{cf} = 0 \tag{10}$$

$$A^*Q_{cf}A - Q_{cf} + Y_{cf} = 0 \tag{11}$$

where

$$X_{cf} = F_{cf}BB^* + BB^*F_{cf}^* \tag{12}$$

$$Y_{cf} = F_{cf}^*C^*C + C^*CF_{cf} \tag{13}$$

and

$$F_{cf} = \frac{-(\theta_2 - \theta_1)}{2\pi}I + \frac{1}{2\pi} \int_{\delta\theta} (I - Ae^{-j\theta})^{-1}d\theta \tag{14}$$

### 2.3. Controllability and observability gramians of discrete-time bilinear systems

Considering the following discrete-time bilinear system represented by:

$$x(k+1) = Ax(k) + \sum_{j=1}^m N_j x(k)u_j(k) + Bu(k) \tag{15}$$

$$y(k) = Cx(k) \tag{16}$$

where  $x(k) \in \mathbb{R}^{n \times n}$  is the state vector,  $u(k) \in \mathbb{R}^{m \times m}$  is the input vector and  $u_j(k)$  is the corresponding  $j$ th element of  $u(k)$ ,  $y(k) \in \mathbb{R}^{q \times q}$  is the output vector and  $A, B, C$  and  $N_j$  are matrices with suitable dimensions. This bilinear system is denoted as  $(A, N_j, B, C)$ .

The controllability gramian for this system is defined as (Zhang et al., 2003):

$$P = \sum_{i=1}^{\infty} \sum_{k_i=0}^{\infty} \dots \sum_{k_1=0}^{\infty} P_i P_i^* \tag{17}$$

where

$$P_1(k_1) = A^{k_1}B$$

$$P_i(k_1, \dots, k_i) = A^{k_i} [ N_1 P_{i-1} \quad N_2 P_{i-1} \dots N_m P_{i-1} ], \quad i \geq 2$$

whereas the observability gramian is defined as (Zhang et al., 2003):

$$Q = \sum_{i=1}^{\infty} \sum_{k_i=0}^{\infty} \dots \sum_{k_1=0}^{\infty} Q_i^* Q_i \tag{18}$$

where

$$Q_1(k_1) = CA^{k_1}$$

$$Q_i(k_1, \dots, k_i) = \begin{bmatrix} Q_{i-1} N_1 \\ Q_{i-1} N_2 \\ \vdots \\ Q_{i-1} N_m \end{bmatrix}$$

The controllability and observability gramians defined in (17) and (18) are the solution to the following generalized Lyapunov equations (Zhang et al., 2003):

$$APA^* - P + \sum_{j=1}^{\infty} N_j P N_j^* + BB^* = 0 \tag{19}$$

$$A^*QA - Q + \sum_{j=1}^{\infty} N_j^* Q N_j + C^*C = 0 \tag{20}$$

The generalized Lyapunov equations corresponding to the controllability and observability gramians in (19) and (20) can be solved iteratively. The controllability gramian can be obtained by (Zhang et al., 2003):

$$P = \lim_{i \rightarrow \infty} \hat{P}_i \tag{21}$$

where

$$\begin{aligned} A\hat{P}_1A^* - \hat{P}_1 + BB^* &= 0, \\ A\hat{P}_iA^* - \hat{P}_i + \sum_{j=1}^{\infty} N_j\hat{P}_{i-1}N_j^* + BB^* &= 0, \quad i \geq 2 \end{aligned} \tag{22}$$

whereas the observability gramian can be obtained by (Zhang et al., 2003)

$$Q = \lim_{i \rightarrow \infty} \hat{Q}_i \tag{23}$$

where

$$\begin{aligned} A^*\hat{Q}_1A - \hat{Q}_1 + C^*C &= 0, \\ A^*\hat{Q}_iA - \hat{Q}_i + \sum_{j=1}^{\infty} N_j^*\hat{Q}_{i-1}N_j + C^*C &= 0, \quad i \geq 2 \end{aligned} \tag{24}$$

### 3. Main work

#### 3.1. Frequency interval controllability and observability gramians of discrete-time bilinear systems

For a particular discrete-time frequency interval  $\Omega = [\gamma_1, \gamma_2]$ , we define the frequency interval controllability and observability gramians as follows:

**Definition 3** The generalized frequency interval controllability gramian for discrete-time bilinear systems is defined as:

$$\hat{P}(\theta) = \sum_{i=1}^{\infty} \frac{1}{(2\pi)^i} \int_{\delta\theta} \dots \int_{\delta\theta} \hat{P}_i(\theta_1, \dots, \theta_i) \hat{P}_i^*(\theta_1, \dots, \theta_i) d\theta_1 \dots d\theta_i \tag{25}$$

where  $\delta\theta = [\gamma_1, \gamma_2]$  and

$$\begin{aligned} \hat{P}_1(\theta_1) &= (e^{j\theta_1}I - A)^{-1}B \\ &\vdots \\ \hat{P}_i(\theta_1, \dots, \theta_i) &= (e^{j\theta_i}I - A)^{-1} \begin{bmatrix} N_1\hat{P}_{i-1} & N_2\hat{P}_{i-1} & \dots & N_m\hat{P}_{i-1} \end{bmatrix} \end{aligned}$$

Similarly, the generalized frequency interval observability gramian is defined as:

$$\hat{Q}(\theta) = \sum_{i=1}^{\infty} \frac{1}{(2\pi)^i} \int_{\delta\theta} \dots \int_{\delta\theta} \hat{Q}_i^*(\theta_1, \dots, \theta_i) \hat{Q}_i(\theta_1, \dots, \theta_i) d\theta_1 \dots d\theta_i \tag{26}$$

where  $\delta\theta = [\gamma_1, \gamma_2]$  and

$$\begin{aligned} \hat{Q}_1(\theta_1) &= C(e^{-j\theta_1}I - A^*)^{-1} \\ &\vdots \\ \hat{Q}_i(\theta_1, \dots, \theta_i) &= \begin{bmatrix} N_1 \hat{Q}_{i-1} \\ N_2 \hat{Q}_{i-1} \\ \vdots \\ N_m \hat{Q}_{i-1} \end{bmatrix} (e^{-j\theta_i}I - A^*)^{-1} \end{aligned}$$

These gramians defined in (25) and (26) are the solution to a pair of new generalized Lyapunov equations which is presented in Theorem 1. Lemmas 1, 2 and 3 together with Theorem 1 presented in the following sections are interrelated such that Lemma 1 and Lemma 2 are required as part of proving Lemma 3, whereas Lemma 3 is required for proving Theorem 1.

**LEMMA 1** Let  $A$  be a square matrix which is also stable and let  $M$  be a matrix with the appropriate dimension. If  $X$  satisfies the following equation:

$$X = \sum_{i=0}^{+\infty} A^i M (A^i)^* \tag{27}$$

It follows that  $X$  is the solution to:

$$AXA^* - X + M = 0 \tag{28}$$

*Proof* Since  $X = \sum_{i=0}^{+\infty} A^i M (A^i)^*$  and  $A$  is stable, it follows that:

$$\begin{aligned} AXA^* - X &= A \sum_{i=0}^{+\infty} A^i M (A^i)^* (A^i)^* - \sum_{i=0}^{+\infty} A^i M (A^i)^* \\ &= \sum_{i=0}^{+\infty} A^{i+1} M (A^*)^{i+1} - \sum_{i=0}^{+\infty} A^i M (A^i)^* \end{aligned}$$

Since  $\sum_{i=0}^{+\infty} A^{i+1} M (A^*)^{i+1} = \sum_{i=1}^{+\infty} A^i M (A^*)^i$

$$AXA^* - X = \sum_{i=1}^{+\infty} A^i M (A^*)^i - \sum_{i=0}^{+\infty} A^i M (A^i)^* = -M.$$

**LEMMA 2** Let  $A$  be a square matrix which is also stable and let  $R$  be a matrix with the appropriate dimension. If  $Y$  satisfies the following

$$Y = \sum_{i=0}^{+\infty} (A^i)^* R (A^i) \tag{29}$$

It follows that  $Y$  is the solution to

$$A^* Y A - Y + R = 0 \tag{30}$$

*Proof* Similar to the proof of Lemma 1 and is therefore omitted for brevity.

LEMMA 3 Let  $M$  and  $R$  be matrices with the appropriate dimensions and let  $A$  be stable, if  $\hat{P}_{cf}$  and  $\hat{Q}_{cf}$  satisfy:

$$\hat{P}_{cf} = \frac{1}{2\pi} \int_{\delta\theta} (e^{j\theta}I - A)^{-1}M(e^{-j\theta}I - A^*)^{-1}d\theta \quad (31)$$

$$\hat{Q}_{cf} = \frac{1}{2\pi} \int_{\delta\theta} (e^{-j\theta}I - A^*)^{-1}R(e^{j\theta}I - A)^{-1}d\theta \quad (32)$$

then  $\hat{P}_{cf}$  and  $\hat{Q}_{cf}$  are the solution to the following generalized Lyapunov equations:

$$A\hat{P}_{cf}A^* - \hat{P}_{cf} = X_{cf} \quad (33)$$

$$A^*\hat{Q}_{cf}A - \hat{Q}_{cf} = Y_{cf} \quad (34)$$

where

$$X_{cf} = -FM - MF^* \quad (35)$$

$$Y_{cf} = -F^*R - RF \quad (36)$$

and

$$F = \frac{(\theta_1 - \theta_2)}{2\pi}I + \frac{1}{2\pi} \int_{-\theta_2}^{\theta_2} (I - Ae^{-j\theta})^{-1}d\theta \dots \quad (37)$$

$$- \frac{1}{2\pi} \int_{-\theta_1}^{\theta_1} (I - Ae^{-j\theta})^{-1}d\theta$$

*Proof* In this part we will prove that (31) is the solution to (33). This proof is a further development of the proof of equation 4.1a in Wang and Zilouchian (2000). The proof that (32) is the solution to (34) can then be obtained similarly by using lemma 2 and therefore is omitted for brevity. Firstly (28) can be re-written as follows:

$$- (e^{j\theta}I - A)X(e^{-j\theta}I - A^*) + X(e^{-j\theta}I - A^*)e^{j\theta}I \dots \quad (38)$$

$$+ (e^{j\theta}I - A)Xe^{-j\theta}I = M$$

Multiplying (38) from the left by  $(e^{j\theta}I - A)^{-1}$  and from the right by  $(e^{-j\theta}I - A^*)^{-1}$  followed by integrating both sides by  $\frac{1}{2\pi} \int_{\delta\theta} d\theta$  yields:

$$\frac{1}{2\pi} \int_{\delta\theta} (e^{j\theta}I - A)^{-1}M(e^{-j\theta}I - A^*)^{-1}d\theta$$

$$= -\frac{1}{2\pi} \int_{\delta\theta} Xd\theta + \frac{1}{2\pi} \int_{\delta\theta} (e^{j\theta}I - A)^{-1}Xe^{j\theta}d\theta + \dots$$

$$\frac{1}{2\pi} \int_{\delta\theta} Xe^{-j\theta}(e^{-j\theta}I - A^*)^{-1}d\theta \quad (39)$$

$$= -\frac{1}{2\pi} \int_{\delta\theta} Xd\theta + \left( \frac{1}{2\pi} \int_{\delta\theta} (I - e^{-j\theta}IA)^{-1}d\theta \right) X + \dots$$

$$X \left( \frac{1}{2\pi} \int_{\delta\theta} (I - e^{-j\theta}IA)^{-1}d\theta \right)^*$$

Denoting  $K_1 = \frac{1}{2\pi} \int_{\delta\theta} (I - e^{-j\theta}IA)^{-1}d\theta$ , (39) can be re-written as:

$$\hat{P}_{cf} = -\frac{1}{2\pi} \int_{\delta\theta} Xd\theta + K_1X + XK_1^* \tag{40}$$

Substituting (40) into the left-hand side of (33) yields:

$$\begin{aligned} & A \left[ -\frac{1}{2\pi} \int_{\delta\theta} Xd\theta + K_1X + XK_1^* \right] A^* \dots \\ & - \left[ -\frac{1}{2\pi} \int_{\delta\theta} Xd\theta + K_1X + XK_1^* \right] = X_{cf} \end{aligned} \tag{41}$$

It has been shown in Wang and Zilouchian (2000) that the property  $AK_1 = K_1A$  and  $AK_1^* = K_1^*A$  holds true. As a result (41) can be re-written as

$$\begin{aligned} X_{cf} &= -\frac{1}{2\pi} \int_{\delta\theta} 1d\theta I [AXA^* - X] + K_1 [AXA^* - X] + [AXA^* - X] K_1^* \\ &= \left( \frac{1}{2\pi} \int_{\delta\theta} 1d\theta I \right) (M) - K_1 M - M K_1^* \\ &= -\left( -\frac{1}{4\pi} \int_{\delta\theta} 1d\theta I + K_1 \right) (M) - (M) \left( -\frac{1}{4\pi} \int_{\delta\theta} 1d\theta I + K_1 \right)^* \end{aligned} \tag{42}$$

(42) is equivalent to the right-hand side of (35). By comparing both expressions we have

$$F = -\frac{1}{4\pi} \int_{\delta\theta} 1d\theta I + K_1 \tag{43}$$

Due to the symmetry of the discrete Fourier transform, the integrations are carried out throughout the frequency intervals  $[\theta_1, \theta_2]$  and  $[-\theta_2, -\theta_1]$  (Horta et al., 1993). Therefore we have

$$\begin{aligned} F &= \frac{(\theta_1 - \theta_2)}{2\pi} I + \frac{1}{2\pi} \int_{-\theta_2}^{\theta_2} (I - Ae^{-j\theta})^{-1} d\theta \dots \\ &\quad - \frac{1}{2\pi} \int_{-\theta_1}^{\theta_1} (I - Ae^{-j\theta})^{-1} d\theta \end{aligned}$$

Lemma 3 derived in the previous section is now applied as part of the proof of Theorem 1 as follows.

**THEOREM 1** The frequency interval controllability and observability gramians  $\hat{P}(\theta)$  and  $\hat{Q}(\theta)$  defined in (25) and (26) are the solutions to the following generalized Lyapunov equations:

$$\begin{aligned} & A\hat{P}(\theta)A^* - \hat{P}(\theta) + F \left( \sum_{j=1}^m N_j \hat{P}(\theta) N_j^* \right) + \dots \\ & \left( \sum_{j=1}^m N_j \hat{P}(\theta) N_j^* \right) F^* + FBB^* + BB^*F^* = 0 \end{aligned} \tag{44}$$

$$\begin{aligned} & A^* \hat{Q}(\theta) A - \hat{Q}(\theta) + F^* \left( \sum_{j=1}^m N_j^* \hat{Q}(\theta) N_j \right) + \dots \\ & \left( \sum_{j=1}^m N_j^* \hat{Q}(\theta) N_j \right) F + F^* C^* C + C^* C F = 0 \end{aligned} \tag{45}$$



*Proof* The proof that the frequency interval controllability gramian  $\hat{P}(\theta)$  defined in (25) is the solution to the generalized Lyapunov equation in (44) is presented in this section. The proof that the frequency interval observability gramian  $\hat{Q}(\theta)$  defined in (26) is the solution to the generalized Lyapunov equation in (45) can be obtained in a similar manner and therefore is omitted for brevity. Firstly let:

$$\tilde{P}_1(\theta) = \frac{1}{2\pi} \int_{\delta\theta} \hat{P}_1(\theta) \hat{P}_1^*(\theta) d\theta_1 \tag{46}$$

⋮

$$\tilde{P}_i(\theta) = \frac{1}{2\pi} \int_{\delta\theta} \dots \int_{\delta\theta} \hat{P}_i(\theta_1, \theta_2, \dots, \theta_i) \hat{P}_i^*(\theta_1, \theta_2, \dots, \theta_i) d\theta_1 \dots d\theta_i \tag{47}$$

we have

$$\hat{P}(\theta) = \sum_{i=1}^{\infty} \tilde{P}_i(\theta) \tag{48}$$

Using Lemma 3 with  $M = BB^*$ , it is observed that  $\tilde{P}_1(\theta)$  is the solution to

$$A\tilde{P}_1(\theta)A^* - \tilde{P}_1(\theta) + FBB^* + BB^*F^* = 0 \tag{49}$$

For  $\tilde{P}_2(\theta)$  we have

$$\begin{aligned} \tilde{P}_2(\theta) &= \frac{1}{(2\pi)^i} \int_{\delta\theta} \int_{\delta\theta} \hat{P}_2(\theta_1, \theta_2) \hat{P}_2^*(\theta_1, \theta_2) d\theta_1 d\theta_2 \\ &= \frac{1}{(2\pi)^i} \int_{\delta\theta} \int_{\delta\theta} (e^{j\theta}I - A) \left[ N_1 \hat{P}_1 \dots N_m \hat{P}_1 \right] \times \dots \\ &\quad \begin{bmatrix} \hat{P}_1^* N_1^* \\ \vdots \\ \hat{P}_1^* N_m^* \end{bmatrix} (e^{-j\theta}I - A^*) d\theta_1 d\theta_2 = \frac{1}{2\pi} \int_{\delta\theta} (e^{j\theta}I - A)^{-1} \times \dots \\ &\quad \left( \sum_{j=1}^m N_j \left( \frac{1}{2\pi} \int_{\delta\theta} \hat{P}_1(\theta) \hat{P}_1^*(\theta) d\theta_1 \right) N_j^* \right) (e^{-j\theta}I - A^*)^{-1} d\theta_2 \\ &= \frac{1}{2\pi} \int_{\delta\theta} (e^{j\theta}I - A)^{-1} \left( \sum_{j=1}^m N_j \tilde{P}_1(\theta) N_j^* \right) (e^{-j\theta}I - A^*)^{-1} d\theta_2 \end{aligned}$$

Denoting  $M = \sum_{j=1}^m N_j \tilde{P}_1(\theta) N_j^*$ , Lemma 3 applies and as a result  $\tilde{P}_2(\theta)$  will be the solution to:

$$A\tilde{P}_2(\theta)A^* - \tilde{P}_2(\theta) + F \left( \sum_{j=1}^m N_j \tilde{P}_1(\theta) N_j^* \right) + \left( \sum_{j=1}^m N_j \tilde{P}_1(\theta) N_j^* \right) F^* = 0 \tag{50}$$

Similarly, according to Lemma 3,  $\tilde{P}_i(\theta)$  will be the solution to

$$A\tilde{P}_i(\theta)A^* - \tilde{P}_i(\theta) + F \left( \sum_{j=1}^m N_j \tilde{P}_{i-1}(\theta) N_j^* \right) + \left( \sum_{j=1}^m N_j \tilde{P}_{i-1}(\theta) N_j^* \right) F^* = 0 \tag{51}$$

Adding (51) to (49) and applying a summation to infinity as in the right-hand side of (48) yields

$$A \sum_{i=1}^{\infty} \tilde{P}_i(\theta) A^* - \sum_{i=1}^{\infty} \tilde{P}_i(\theta) + F \left( \sum_{j=1}^m N_j \sum_{i=2}^{\infty} \tilde{P}_{i-1}(\theta) N_j^* \right) + \dots$$

$$\left( \sum_{j=1}^m N_j \sum_{i=2}^{\infty} \tilde{P}_{i-1}(\theta) N_j^* \right) F^* + FBB^* + BB^*F^* = 0 \tag{52}$$

Equivalently, we have

$$A \sum_{i=1}^{\infty} \tilde{P}_i(\theta) A^* - \sum_{i=1}^{\infty} \tilde{P}_i(\theta) + F \left( \sum_{j=1}^m N_j \sum_{i=1}^{\infty} \tilde{P}_i(\theta) N_j^* \right) + \dots$$

$$\left( \sum_{j=1}^m N_j \sum_{i=1}^{\infty} \tilde{P}_i(\theta) N_j^* \right) F^* + FBB^* + BB^*F^* = 0 \tag{53}$$

Finally applying the property in (48) to (53)

$$A \hat{P}(\theta) A^* - \hat{P}(\theta) + F \left( \sum_{j=1}^m N_j \hat{P}(\theta) N_j^* \right) + \left( \sum_{j=1}^m N_j \hat{P}(\theta) N_j^* \right) F^* + FBB^* + BB^*F^* = 0$$

**3.2. Conditions for solvability of the Lyapunov equations corresponding to frequency interval controllability and observability gramians**

In this section, the condition for solvability of the generalized Lyapunov equation in (44) which corresponds to the frequency interval controllability gramian defined in (25) is presented herewith in Theorem 2. The condition for solvability of the generalized Lyapunov equation in (45) which corresponds to the frequency interval observability gramian defined in (26) can be derived in a similar manner and is therefore omitted for brevity.

**THEOREM 2** The generalized Lyapunov equation in (44) is solvable and has a unique solution if and only if

$$W = (A \otimes A) - (I \otimes I) + (N_j \otimes FN_j) + (FN_j \otimes N_j) \tag{54}$$

is non-singular.

*Proof* Let  $\text{vec}(\cdot)$  be an operator which converts a matrix into a vector by stacking the columns of the matrix on top of each other. This operator has the following useful property [20]

$$\text{vec}(M_1, M_2, M_3) = (M_3^* \otimes M_1) \text{vec}(M_2) \tag{55}$$

Applying  $\text{vec}(\cdot)$  on both sides of (44) together with the property in (54) yields

$$\left\{ (A \otimes A) - (I \otimes I) + \left( \sum_{j=1}^m N_j \otimes FN_j \right) + \left( \sum_{j=1}^m FN_j \otimes N_j \right) \right\} \text{vec}(\hat{P}(\theta)) = -\text{vec}(FBB^* + BB^*F) \tag{56}$$

The generalized Lyapunov equation in (44) is solvable provided that this equation presented in (56) is solvable and has a unique solution. It follows that (56) is solvable and has a unique solution if and only if

$$W = (A \otimes A) - (I \otimes I) + (N_j \otimes FN_j) + (FN_j \otimes N_j)$$

is non-singular. □

### 3.3. Numerical solution method for the Lyapunov equations corresponding to the frequency interval controllability and observability gramians

The iterative procedure for solving bilinear Lyapunov equations in previous studies can also be applied to obtain the solution to the generalized Lyapunov equation in (44) -  $\hat{P}(\theta)$  as follows (Shaker et al., 2014a, 2014c; Zhang et al., 2003; Zhang & Lam, 2002):

$$\hat{P}(\theta) = \lim_{i \rightarrow +\infty} \tilde{P}_i(\theta) \tag{57}$$

where

$$A\tilde{P}_1(\theta)A^* - \tilde{P}_1(\theta) + FBB^* + BB^*F^* = 0$$

$$A\tilde{P}_i(\theta)A^* - \tilde{P}_i(\theta) + F\left(\sum_{j=1}^m N_j\tilde{P}_{i-1}(\theta)N_j^*\right) + \dots \tag{58}$$

$$\left(\sum_{j=1}^m N_j\tilde{P}_{i-1}(\theta)N_j^*\right)F^* + FBB^* + BB^*F^* = 0, \quad i \geq 2 \tag{59}$$

This iterative procedure can also be applied to solve the generalized Lyapunov equation corresponding to the frequency interval observability gramian in (45).

### 3.4. Model reduction algorithm

The procedure for obtaining the reduced-order model is described as follows

Step 1: The frequency interval controllability and observability gramians are calculated by solving (44) and (45), respectively.

Step 2: Both of these frequency interval controllability and observability gramians obtained by solving (44) and (45) are simultaneously diagonalized by using a suitable transformation matrix denoted by  $T$  such that

$$T\hat{P}(\theta)T^T = T^{-T}\hat{Q}(\theta)T^{-1}$$

Step 3: Transform and partition to get the realization

$$\bar{A} = T^{-1}AT = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \bar{N} = T^{-1}NT = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}$$

$$\bar{B} = T^{-1}B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad \bar{C} = CT = [C_1 \quad C_2]$$

Step 4: The reduced order model is given by  $A_r = A_{11}, N_r = N_{11}, B_r = B_1, C_r = C_1$

## 4. Results and discussion

### 4.1. Numerical example and results

Considering the following fifth-order discrete-time bilinear system originally presented by Hinamoto and Maekawa (1984) which has also been used by Zhang et al. (2003).

**Table 1. Exact values of  $y(k)$  from Figure 1**

k	y(k)			E1	E2
	Original	Proposed	Zhang et al. (2003)		
20	99.35	<b>100.5</b>	110.4	<b>1.150</b>	11.050
21	103.1	<b>104.2</b>	114.6	<b>1.1</b>	11.5
22	106.6	<b>107.2</b>	117.7	<b>0.6</b>	11.1
23	109.7	<b>109.4</b>	119.9	<b>0.3</b>	10.2
24	112.6	<b>110.9</b>	120.8	<b>1.7</b>	8.2
25	115.2	<b>111.6</b>	120.6	<b>3.6</b>	5.4

$$x(k+1) = Ax(k) + \sum_{j=1}^m N_j x(k) u_j(k) + Bu(k) \tag{60}$$

$$y(k) = Cx(k)$$

where

$$A = \begin{bmatrix} 0 & 0 & 0.024 & 0 & 0 \\ 1 & 0 & -0.26 & 0 & 0 \\ 0 & 1 & 0.9 & 0 & 0 \\ 0 & 0 & 0.2 & 0 & -0.06 \\ 0 & 0 & 0.15 & 1 & 0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 0.8 \\ 0.6 \\ 0.4 \\ 0.2 \\ 0.5 \end{bmatrix}, \quad N = \begin{bmatrix} 0.1 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0 & 0 & 0.5 \end{bmatrix}$$

$$C = [ 0.2 \quad 0.4 \quad 0.6 \quad 0.8 \quad 1.0 ]$$

The proposed technique involves firstly obtaining the frequency interval controllability and observability gramians defined in Theorem 1 and subsequently using these gramians as part of the balanced truncation-based technique described in Section 3.4 (Moore, 1981; Zhang & Lam, 2002; Zhang et al., 2003). This fifth-order model  $\{A, N, B, C\}$  is reduced to the following second-order model  $\{A_{r1}, N_{r1}, B_{r1}, C_{r1}\}$  in the form of (60) by using the proposed technique for the frequency interval  $\Omega = [0.04\pi, 0.3\pi]$

Similarly, by applying the proposed technique for the frequency interval  $\Omega = [0, 0.1\pi]$  to this fifth-order model  $\{A, N, B, C\}$ , a third-order discrete-time bilinear system with the following system matrices  $\{A_{r2}, N_{r2}, B_{r2}, C_{r2}\}$  in the form of (60) is obtained

$$A_{r1} = \begin{bmatrix} 0.8247 & -0.2446 \\ 0.2394 & 0.5495 \end{bmatrix}, \quad B_{r1} = \begin{bmatrix} 1.2986 \\ -0.7881 \end{bmatrix}, \quad N_{r1} = \begin{bmatrix} 0.3214 & 0.1276 \\ 0.0878 & 0.2862 \end{bmatrix}$$

$$C_{r1} = [ 1.2974 \quad 0.7789 ]$$

**Table 2. Exact values of  $y(k)$  from Figure 2**

k	y(k)			E1	E2
	Original	Proposed	Zhang et al. (2003)		
20	99.35	<b>100.8</b>	96.49	<b>1.45</b>	2.86
21	103.1	<b>104.5</b>	99.1	<b>1.4</b>	4
22	106.6	<b>107.7</b>	101.2	<b>1.1</b>	5.4
23	109.7	<b>110.5</b>	102.9	<b>0.8</b>	6.8
24	112.6	<b>112.9</b>	104.3	<b>0.3</b>	8.3
25	115.2	<b>114.9</b>	105.3	<b>0.3</b>	9.9

$$A_{r2} = \begin{bmatrix} 0.7588 & -0.1963 & 0.0881 \\ 0.1932 & 0.4929 & 0.2810 \\ 0.0947 & -0.2836 & 0.2436 \end{bmatrix}, \quad B_{r2} = \begin{bmatrix} 1.4647 \\ -1.0558 \\ -0.6156 \end{bmatrix}, \quad N_{r2} = \begin{bmatrix} 0.3260 & 0.1387 & 0.0108 \\ 0.0998 & 0.3008 & -0.1865 \\ -0.0108 & -0.0910 & 0.3204 \end{bmatrix}$$

$$C_{r2} = [ 1.4583 \quad 1.0710 \quad -0.5848 ]$$

For comparison, we apply the method by Zhang et al. (2003) which yields the following second- and third-order discrete-time bilinear systems with the following system matrices  $\{A_{r3}, N_{r3}, B_{r3}, C_{r3}\}$  and  $\{A_{r4}, N_{r4}, B_{r4}, C_{r4}\}$  in the form of (60)

$$A_{r3} = \begin{bmatrix} 0.8032 & -0.2445 \\ 0.1692 & 0.6050 \end{bmatrix}, \quad B_{r3} = \begin{bmatrix} 1.3335 \\ -0.8717 \end{bmatrix},$$

$$N_{r3} = \begin{bmatrix} 0.3387 & 0.1301 \\ 0.1075 & 0.2851 \end{bmatrix}, \quad C_{r3} = [ 1.3615 \quad 0.7053 ]$$

and

$$A_{r4} = \begin{bmatrix} 0.8032 & -0.2445 & 0.0373 \\ 0.1692 & 0.6050 & 0.3500 \\ 0.0732 & -0.3154 & 0.3499 \end{bmatrix}, \quad B_{r4} = \begin{bmatrix} 1.3335 \\ -0.8717 \\ -0.4052 \end{bmatrix}$$

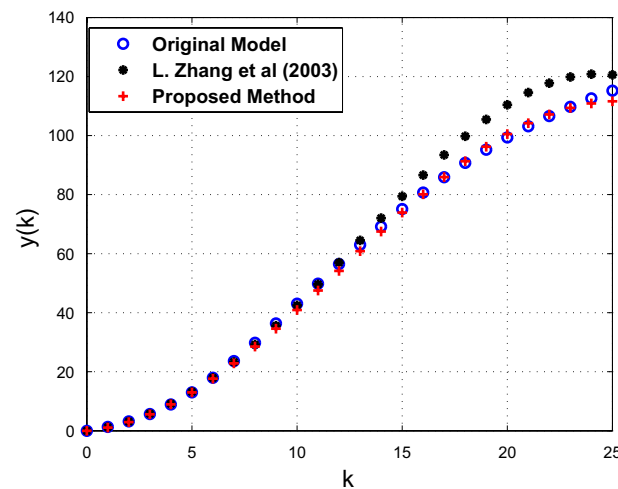
$$N_{r4} = \begin{bmatrix} 0.3387 & 0.1301 & 0.0221 \\ 0.1075 & 0.2851 & -0.1982 \\ -0.0315 & -0.0857 & 0.3059 \end{bmatrix}, \quad C_{r4} = [ 1.3615 \quad 0.7053 \quad -0.2884 ]$$

#### 4.2. Discussion of results

Figure 1 shows the step responses of the original fifth-order model, second-order model obtained using the proposed method for a frequency interval  $\Omega = [0.04\pi, 0.3\pi]$  and a second-order model obtained using the method by Zhang et al. (2003). Table 1 shows the exact values for  $y(k)$  for the discrete times  $k = 20, 21, 22, 23, 24, 25$  from Figure 1. E1 and E2 denote the absolute error between the value of  $y(k)$  of the original model and the value of  $y(k)$  obtained using the proposed method and the method by Zhang et al. (2003), respectively.

On the other hand Figure 2 shows the step responses of the original fifth-order model, third-order model obtained using the proposed method for a frequency interval  $\Omega = [0, 0.1\pi]$  and a third-order model obtained using the method by Zhang et al. (2003). Table 2 shows the exact values for  $y(k)$  for the discrete times  $k = 20, 21, 22, 23, 24, 25$  from Figure 2. E1 and E2 denote the absolute error between the value of  $y(k)$  of the original model and the value of  $y(k)$  obtained using the proposed method and the method by Zhang et al. (2003), respectively. From both

**Figure 1. Step responses of the original model, second-order reduced model obtained using the method by Zhang et al. (2003) and second-order reduced model obtained using the proposed technique for a frequency  $\Omega = [0.04\pi, 0.3\pi]$ .**



**Figure 2. Step responses of the original model, third-order reduced model obtained using the method by Zhang et al. (2003) and third-order reduced model obtained using the proposed technique for a frequency  $\Omega = [0, 0.1\pi]$ .**

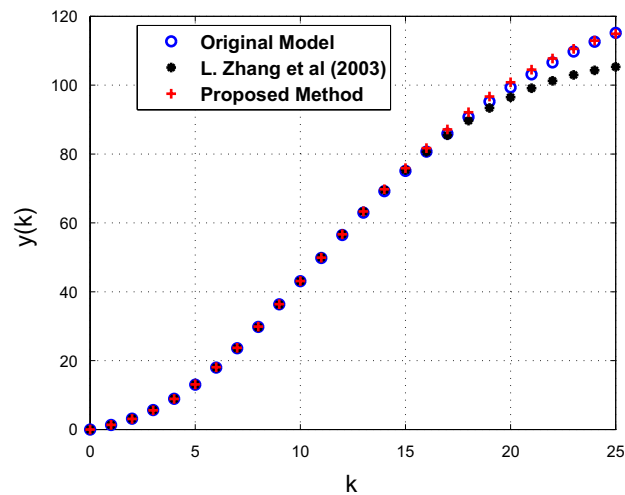


Figure 1, Figure 2, Table 1, and Table 2 it is shown that applying the proposed technique yields a reduced-order model which is a closer approximation to the original model compared to the method by Zhang et al. (2003).

## 5. Conclusion

In conclusion, a new model reduction method for discrete time bilinear systems based on balanced truncation has been developed. The frequency interval controllability and observability gramians for discrete time bilinear systems are introduced and are shown to be solutions to a pair of new generalized Lyapunov equations. The conditions for solvability of these new Lyapunov equations are provided and the numerical solution method used to solve these equations is explained. Numerical results show that the proposed method yields reduced-order models which is a closer approximation to the original model as compared to existing techniques. The technique proposed in this paper is applicable to a variety of non-linear systems which can be formulated as bilinear systems.

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