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Yilun Shang

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*Corresponding author: Yilun Shang,
Einstein Institute of Mathematics,
Hebrew University of Jerusalem,
Jerusalem 91904, Israel; SUTD-MIT
International Design Center, Singapore
University of Technology and Design,
Singapore 138682, Singapore
E-mail: shylmath@hotmail.com

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Group consensus in generic linear multi-agent systems with inter-group non-identical inputs

Yilun Shang^{1*}

Abstract: This paper studies the group consensus problem for generic linear multi-agent systems under directed information flow. External adapted inputs are introduced to realize the intra-group synchronization as well as the inter-group separation. Without imposing complicated algebraic criteria or restrictive graphic conditions on the interaction topology, we show that the group consensus can be achieved by designing appropriate gains given any magnitude of the coupling strengths among the agents. Numerical examples are presented to illustrate the availability of our results.

Subjects: Automation Control, Linear & Multilinear Algebra, Robotics & Cybernetics

Keywords: network of multiple agents, linear time-invariant system, group consensus, cooperative control

1. Introduction

Consensus problems for multi-agent systems have been attracting many researchers in recent years, due to their broad applications in various areas such as swarming/flocking (Gazi & Passino, 2003; Shang & Bouffanais, 2014; Tanner, Jadbabaie, & Pappas, 2007), distributed sensor networks (Kar & Moura, 2009), and multi-vehicle formation control (Lian & Deshmukh, 2006; Ren & Sorensen, 2008). As a fundamental issue in cooperative control of multiple agents, the main goal of consensus problem is to design appropriate distributed algorithms (referred to as consensus protocols), such

ABOUT THE AUTHOR

Yilun Shang is a researcher at Hebrew University of Jerusalem. He completed his doctoral studies at Shanghai Jiao Tong University in 2010, and held appointments at University of Texas at San Antonio and Singapore University of Technology and Design. His research work involves the development of network science and complex system control. He is particularly interested in random graph, self-organization, and agent-based modeling.

PUBLIC INTEREST STATEMENT

The field of consensus problem has evolved rapidly in the last decades. More complicated consensus patterns are explored including group (or cluster) consensus, where certain components inside the network show isochronous synchronization phenomenon. This paper addresses the group consensus problem of a network of continuous-time agents with generic linear dynamics. Such systems include single-integrator, double-integrator, and higher order integrator dynamics as special cases. By introducing external adapted inputs, group consensus can be realized exponentially fast. Here, the external control inputs contribute to the intra-group synchronization and inter-group separation, without imposing complicated algebraic criteria or restrictive topological constraints. Moreover, only non-negative weights are assigned to the communication links, which have practical applicability.



that the states of all agents converge to a common value with information exchanges between each other. In Olfati-Saber and Murray (2004), a theoretical framework for consensus problems was provided for continuous-time networked dynamical systems under both fixed and switching topologies. In Jadbabaie, Lin, and Morse (2003), asymptotic consensus protocols based on nearest neighbor rules were designed for first-order discrete-time systems. Up to now, numerous papers concerning consensus protocols, including average consensus (Fagnani & Zampieri, 2009; Lin & Jia, 2008), finite-time consensus (Shang, 2012), stochastic consensus (Hatano & Mesbahi, 2005; Huang, Dey, Nair, & Manton, 2010), leader-following consensus (Wen, Duan, Chen, & Yu, 2014; Wen, Li, Duan, & Chen, 2013), and quantized consensus (Kashyap, Başar, & Srikant, 2007), have been published. For details, we refer the readers to survey papers (Cao, Yu, Ren, & Chen, 2013; Olfati-Saber, Fax, & Murray, 2007) and the references therein.

The aforementioned results focus attention on the complete consensus of all agents in a network. However, when carrying out a cooperative task, a group of agents should be capable of coping with unanticipated situations, and may evolve into several subgroups with the changes of environments, situations, or even time. This phenomenon widely exists in engineering and biological systems, from military reconnaissance to heterogeneous robots sorting (Kumar, Garg, & Kumar, 2010), from predator-evasion behaviors of a herd of animals (Schellinck & White, 2011) to opinion formation in social networks (Shang, 2013a, 2014). Suitable protocols have been designed recently to ensure *group consensus*, i.e. the states of all agents within the same subgroup asymptotically converge to a consistent value, while there is no agreement between different subgroups. It is clear that the complete consensus is a special case of group consensus. Group consensus problems for continuous-time single-integrator agents under switching topologies were explored in Yu and Wang (2010) using the Lyapunov direct method and double-tree-form transformations. In Xia and Cao (2011), sufficient and necessary conditions for group consensus were provided for continuous-time multi-agent systems under a couple of different mechanisms concerning whether or not the agents' self-dynamics are identical. Second-order group consensus was addressed in Ma, Wang, and Miao (2014) and Feng, Xu, and Zhang (2014). Algebraic criteria for group consensus were reported in Han, Lu, and Chen (2013) and Shang (2013b) for discrete-time single-integrator dynamics. In addition, a group of continuous-time agents with non-linear self-dynamics (Sun, Bai, Jia, Xiong, & Chen, 2011), time delay (Shang, 2013c), linear time-invariant dynamics (Qin & Yu, 2013; Tan, Liu, & Duan, 2011), and choice-based protocols (Liu & Wong, 2013) can also reach group consensus under some conditions.

It is widely known that the weak coupling strength among agents may lead to instability and inhibits the convergence of the state trajectories. In most realistic systems, however, it is literally impossible to make the coupling strengths arbitrarily large. Thus, it is inevitable to study the sufficient/necessary conditions for group consensus concerning the coupling strengths. In the previously mentioned work, some algebraic criteria were proposed to ensure group consensus. For example, linear matrix inequality conditions were introduced in Yu and Wang (2010) and Xia and Cao (2011), and conditions involving eigenvalues of interaction topologies were proposed in Sun et al. (2011, Shang (2013c), and Tan et al. (2011). The feasibility of these algebraic conditions turns out to be very difficult to check. Alternatively, the authors in Qin and Yu (2013) imposed a graphic constraint on the interaction topology. It is shown that the group consensus can be achieved via pinning control irrespective of how weak or strong the couplings among agents are, if the underlying network has an acyclic partition.

In this paper, continuing with previous works, we study the group consensus of a network of agents with continuous-time generic linear dynamics. By introducing inter-group non-identical inputs, we show that the system can achieve group consensus by designing suitable consensus gains given any magnitude of the coupling strengths. Our result differs from the existing literature in that we impose neither complicated algebraic criteria (Shang, 2013c; Sun et al., 2011; Tan et al., 2011; Xia & Cao, 2011; Yu & Wang, 2010) nor graphic constraints (Liu & Wong, 2013; Ma et al., 2014; Qin & Yu, 2013) on the coupling. Moreover, in the aforementioned works (Qin & Yu, 2013; Shang, 2013c; Sun et al., 2011; Tan et al., 2011; Xia & Cao, 2011; Yu & Wang, 2010), the couplings among

agents in different groups may be negatively weighted to desynchronize the states of agents in different groups. Nevertheless, negative weights are difficult to find practical applications. In our framework, the external adapted inputs help realize inter-group separation. Thanks to that, we only require non-negative weights. Convergence rate and ultimate consensus state can be specified as well. It is worthwhile to mention that similar external inputs mechanisms were dealt with in Han et al. (2013) and Shang (2013b) for discrete-time single-integrator agents, where the coupling strengths need to be sufficiently strong. The approaches used are totally different.

The rest of the paper is organized as follows. The problem to be investigated is formulated in Section 2. Main results are given in Section 3. Simulations are performed in Section 4 to illustrate theoretical results. Conclusions are drawn in Section 5.

The following notations will be used throughout the paper. $\mathbf{1}_n \in \mathbb{R}^n$ (resp. $\mathbf{0}_n \in \mathbb{R}^n$) is the n -dimensional column vector with all entries equal to one (resp. zero). $I_n \in \mathbb{R}^{n \times n}$ is the n -dimensional identity matrix. A^T is the transpose of matrix A . $\text{diag}(A_1, \dots, A_p)$ is the "block diagonal" matrix with the k -th main diagonal block being matrix A_k . $\|x\|$ stands for the Euclidean norm of a vector x . $A \otimes B$ refers to the Kronecker product of two matrices A and B (Horn & Johnson, 1985).

2. Problem formulation

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a weighted directed graph of order N , where $\mathcal{V} = \{v_1, v_2, \dots, v_N\}$ is the set of nodes (or agents), $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of directed edges, and $\mathcal{A} = (a_{ij}) \in \mathbb{R}^{N \times N}$ is the weighted adjacency matrix. A directed edge of \mathcal{G} is denoted by $(v_j, v_i) \in \mathcal{E}$, indicating that agent v_j can obtain information from agent v_i . The entry $a_{ij} > 0$ if $(v_j, v_i) \in \mathcal{E}$; $a_{ij} = 0$ otherwise. Moreover, we assume $a_{ii} = 0$ for all i . The set of neighbors of agent v_i is denoted by $\mathcal{N}_i = \{v_j \in \mathcal{V} : (v_j, v_i) \in \mathcal{E}\}$. $\mathcal{L} = (l_{ij}) \in \mathbb{R}^{N \times N}$ is the Laplacian matrix of \mathcal{G} , where $l_{ij} = -a_{ji}$, $i \neq j$, and $l_{ii} = \sum_{j=1, j \neq i}^N a_{ij}$. $\sum_{j=1}^N a_{ij}$ is called the in-degree of agent v_i . A directed path is a sequence of distinct nodes $v_{i_1}, v_{i_2}, \dots, v_{i_k}$ in \mathcal{G} such that $(v_{i_{j-1}}, v_{i_j}) \in \mathcal{E}$ for $j=2, \dots, k$. \mathcal{G} is said to contain a spanning tree if there exists an agent (referred to as root) such that every other agent can be connected via a directed path starting from the root. The Laplacian matrix has the following property (see e.g. [Ren and Beard (2005), Lemma 3.3] and [Horn and Johnson (1985), Theorem 8.3.1]).

LEMMA 1 *If \mathcal{G} has a spanning tree, then the Laplacian matrix \mathcal{L} has exactly one zero eigenvalue with corresponding eigenvector $\mathbf{1}_N$, and all of the non-zero eigenvalues are with positive real parts. Moreover, \mathcal{L} has a non-negative left eigenvalue $\eta \in \mathbb{R}^N$ associated with the zero eigenvalue, satisfying $\eta^T \mathcal{L} = \mathbf{0}_N^T$ and $\eta^T \mathbf{1}_N = 1$.*

Consider a multi-agent system consisting of N agents with interaction graph delineated by a directed graph \mathcal{G} , where the group of agents is partitioned into p subgroups for some integer p . Without loss of generality, we assume $\mathcal{V}_1 = \{v_1, \dots, v_{N_1}\}$, $\mathcal{V}_2 = \{v_{N_1+1}, \dots, v_{N_1+N_2}\}$, \dots , $\mathcal{V}_p = \{v_{\sum_{k=1}^{p-1} N_k+1}, \dots, v_N\}$ and $\sum_{k=1}^p N_k = N$. Set $N_0 = 0$. The dynamics of agent v_i takes the following form

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \quad t \geq 0, \quad i = 1, 2, \dots, N \quad (1)$$

where $x_i(t) \in \mathbb{R}^n$ and $u_i(t) \in \mathbb{R}^m$ represent the state and control input of agent v_i ; $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ are constant system matrices. Motivated by Han et al. (2013) and Shang (2013b), we introduce the following consensus protocol with inter-group non-identical inputs:

$$u_i(t) = K \left(\sum_{v_j \in \mathcal{N}_i} a_{ij} (x_j(t) - x_i(t)) + w_i(t) \right) \quad (2)$$

where $K \in \mathbb{R}^{m \times n}$ is the constant consensus gain matrix to be designed, and $w_i(t) \in \mathbb{R}^n$ are external inputs satisfying $w_i(t) = w_k(t)$ if and only if $v_i, v_j \in \mathcal{V}_k$, $k=1, \dots, p$. In addition, we assume that $w_i(t)$, $i=1, \dots, N$ are bounded, i.e. $\|w_i(t)\| \leq c$ for all $t \geq 0$, where c is a constant. It is worth noting that the

linear time-invariant dynamics (1) with protocol (2) addresses not only the self-dynamics of the agent but also the interactions between the neighboring agents. Hence, it is more general than the integrator cases; see, e.g. Yu and Wang (2010), Xia and Cao (2011), and Sun et al. (2011).

Definition 1 The multi-agent system (1) under the control law (2) is said to achieve group consensus if there exists a consensus gain K , such that for any $x_i(0) \in \mathbb{R}^n$,

$$\lim_{t \rightarrow \infty} \|x_i(t) - x_j(t)\| = 0, \quad \text{for all } v_i, v_j \in \mathcal{V}_k, \quad k = 1, \dots, p$$

and

$$\lim_{t \rightarrow \infty} \|x_i(t) - x_j(t)\| > 0, \quad \text{for each pair of } v_i \in \mathcal{V}_k \text{ and } v_j \in \mathcal{V}_l \text{ with } k \neq l$$

In addition, if there exist positive numbers κ , C , and t_0 such that $\|x_i(t) - x_j(t)\| \leq Ce^{-\kappa t}$ for all $v_i, v_j \in \mathcal{V}_k$ ($k = 1, \dots, p$) and $t > t_0$, we say the consensus is achieved exponentially fast (with rate at least κ).

The Laplacian matrix of \mathcal{G} takes the following block matrix form:

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \cdots & \mathcal{L}_{1p} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \cdots & \mathcal{L}_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{L}_{p1} & \mathcal{L}_{p2} & \cdots & \mathcal{L}_{pp} \end{bmatrix} \quad (3)$$

where $\mathcal{L}_{kk} \in \mathbb{R}^{N_k \times N_k}$ specifies the information exchange within subgroup \mathcal{V}_k , and $\mathcal{L}_{kl} \in \mathbb{R}^{N_k \times N_l}$ specifies the information exchange from subgroup \mathcal{V}_l to \mathcal{V}_k .

ASSUMPTION 1 For all $k, l = 1, \dots, p$, \mathcal{L}_{kl} has a constant row sum.

Note that Assumption 1 means $\mathcal{L}_{kl} \mathbf{1}_{N_l} = c_{kl} \mathbf{1}_{N_k}$ for some constant $c_{kl} \in \mathbb{R}$. Such an assumption is widely made in most of the literature pertaining to group consensus problems; among them, many further assume that $c_{kl} = 0$ for all k, l —called the in-degree balanced condition [see e.g. (Feng et al., 2014; Qin & Yu, 2013; Sun et al., 2011; Xia & Cao, 2011; Yu & Wang, 2010)].

3. Main results

In this section, we tackle multi-agent system (1) under protocol (2). The objective is to derive simple sufficient conditions for achieving group consensus for any coupling strength among agents.

Below, we first present a lemma, which will be needed in the convergence analysis in Theorem 1. We believe that the result is also interesting in its own right.

LEMMA 2 Consider an $nN \times nN$ matrix $\tilde{\Omega}$ given by

$$\tilde{\Omega} = \begin{bmatrix} \tilde{\Omega}_{11} & \tilde{\Omega}_{12} & \cdots & \tilde{\Omega}_{1p} \\ \tilde{\Omega}_{21} & \tilde{\Omega}_{22} & \cdots & \tilde{\Omega}_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\Omega}_{p1} & \tilde{\Omega}_{p2} & \cdots & \tilde{\Omega}_{pp} \end{bmatrix}$$

where $\tilde{\Omega}_{kl} = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \cdots & \Omega_{1N_l} \\ \Omega_{21} & \Omega_{22} & \cdots & \Omega_{2N_l} \\ \vdots & \vdots & \ddots & \vdots \\ \Omega_{N_k 1} & \Omega_{N_k 2} & \cdots & \Omega_{N_k N_l} \end{bmatrix}$ and $\Omega_{ij} \in \mathbb{R}^{n \times n}$ for $1 \leq i \leq N_k, 1 \leq j \leq N_l$, and $1 \leq k, l \leq p$. Define

an nN -dimensional vector $\tilde{\Gamma} = ((\mathbf{1}_{N_1} \otimes \Gamma_1)^T, \dots, (\mathbf{1}_{N_p} \otimes \Gamma_p)^T)^T$ with $\Gamma_k \in \mathbb{R}^n$ for $1 \leq k \leq p$. Assume that for

all k, l , there exist some $\alpha_{kl} \in \mathbb{R}^{n \times n}$ satisfying $\tilde{\Omega}_{kl}(\mathbf{1}_{N_l} \otimes I_n) = \mathbf{1}_{N_k} \otimes \alpha_{kl}$ i.e. $\tilde{\Omega}_{kl}$ "as a block matrix" has a constant row sum. Let $\alpha = (\alpha_{kl}) \in \mathbb{R}^{np \times np}$. Then

$$e^{\tilde{\Omega}} \cdot \tilde{\Gamma} = \left((\mathbf{1}_{N_1} \otimes b_1)^T, \dots, (\mathbf{1}_{N_p} \otimes b_p)^T \right)^T$$

where $b_k = \sum_{l=1}^p (e^\alpha)_{kl} \Gamma_l$ for $1 \leq k \leq p$. Here, $(e^\alpha)_{kl}$ means the (k, l) -block if we partition e^α in conformity with that of α .

Proof From the straightforward calculation, we know that

$$\tilde{\Omega} \cdot \left(\text{diag}(\mathbf{1}_{N_1}, \dots, \mathbf{1}_{N_p}) \otimes I_n \right) = \left(\text{diag}(\mathbf{1}_{N_1}, \dots, \mathbf{1}_{N_p}) \otimes I_n \right) \cdot \alpha$$

Since the expansion $e^M = \sum_{i=0}^{\infty} \frac{1}{i!} M^i$ holds for any square matrix M , we further obtain

$$e^{\tilde{\Omega}} \cdot \left(\text{diag}(\mathbf{1}_{N_1}, \dots, \mathbf{1}_{N_p}) \otimes I_n \right) = \left(\text{diag}(\mathbf{1}_{N_1}, \dots, \mathbf{1}_{N_p}) \otimes I_n \right) \cdot e^\alpha$$

If we partition $e^{\tilde{\Omega}}$ as $e^{\tilde{\Omega}} = \begin{bmatrix} \Theta_{11} & \Theta_{12} & \dots & \Theta_{1p} \\ \Theta_{21} & \Theta_{22} & \dots & \Theta_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \Theta_{p1} & \Theta_{p2} & \dots & \Theta_{pp} \end{bmatrix}$ in conformity with that of $\tilde{\Omega}$, then it is easy to check

$$\Theta_{kl}(\mathbf{1}_{N_l} \otimes I_n) = \mathbf{1}_{N_k} \otimes (e^\alpha)_{kl} \tag{4}$$

where $(e^\alpha)_{kl}$ is defined as above.

Using (4) and the properties of Kronecker product, we derive that

$$\begin{aligned} e^{\tilde{\Omega}} \cdot \tilde{\Gamma} &= \begin{bmatrix} \sum_{l=1}^p \Theta_{1l}(\mathbf{1}_{N_l} \otimes \Gamma_l) \\ \vdots \\ \sum_{l=1}^p \Theta_{pl}(\mathbf{1}_{N_l} \otimes \Gamma_l) \end{bmatrix} = \begin{bmatrix} \sum_{l=1}^p \Theta_{1l}(\mathbf{1}_{N_l} \otimes I_n)(\mathbf{1} \otimes \Gamma_l) \\ \vdots \\ \sum_{l=1}^p \Theta_{pl}(\mathbf{1}_{N_l} \otimes I_n)(\mathbf{1} \otimes \Gamma_l) \end{bmatrix} \\ &= \begin{bmatrix} \sum_{l=1}^p (\mathbf{1}_{N_1} \otimes (e^\alpha)_{1l})(\mathbf{1} \otimes \Gamma_l) \\ \vdots \\ \sum_{l=1}^p (\mathbf{1}_{N_p} \otimes (e^\alpha)_{pl})(\mathbf{1} \otimes \Gamma_l) \end{bmatrix} = \begin{bmatrix} \mathbf{1}_{N_1} \otimes b_1 \\ \vdots \\ \mathbf{1}_{N_p} \otimes b_p \end{bmatrix} \end{aligned}$$

as desired.

Set $x(t) = (x_1^T(t), \dots, x_N^T(t))^T$. The system (1) under protocol (2) can be recast in the following matrix form

$$\dot{x}(t) = (I_N \otimes A - \mathcal{L} \otimes BK)x(t) + (I_N \otimes BK)w(t) \tag{5}$$

where $w(t) = \left((\mathbf{1}_{N_1} \otimes w_1(t))^T, \dots, (\mathbf{1}_{N_p} \otimes w_p(t))^T \right)^T$ and \mathcal{L} is given by (3). If (A, B) is stabilizable, there exists a non-negative definite matrix P , such that the algebraic Riccati equation

$$A^T P + PA - PBB^T P + I_n = 0 \tag{6}$$

holds, and all eigenvalues of $A - BB^T P$ are in the open left half-plane (Ogata, 2010). Let $\lambda_1 = 0, \lambda_2, \dots, \lambda_N$ be the eigenvalues of Laplacian \mathcal{L} .

Based on the above preparations, we are now in a position to get the main result concerning group consensus of system (5).

THEOREM 1 Under Assumption 1, if (A, B) is stabilizable and \mathcal{G} contains a spanning tree, then the gain matrix can be designed as $K = \sigma \cdot B^T P$, where $\sigma \geq (\min_{2 \leq i \leq N} \{\text{Re}(\lambda_i)\})^{-1}$ and P is given in (6), such that the multi-agent system (1) under protocol (2) can achieve group consensus exponentially fast.

Proof The solution of (5) can be written as

$$x(t) = e^{(I_N \otimes A - \mathcal{L} \otimes BK)t} x(0) + \int_0^t e^{(I_N \otimes A - \mathcal{L} \otimes BK)(t-s)} (I_N \otimes BK) w(s) ds \quad (7)$$

Since \mathcal{G} contains a spanning tree, by Lemma 1, we know that $\lambda_2, \dots, \lambda_N$ are in the open right half-plane. Furthermore, there exists an invertible matrix Q taking the form $Q = [1_N \ Y]$ and $Q^{-1} = [\eta \ Z]^T$ with $\eta = (\eta_1, \dots, \eta_N)^T$ defined in Lemma 1, $Y, Z \in \mathbb{R}^{N \times (N-1)}$ such that \mathcal{L} is similar to a Jordan canonical form, i.e.

$$Q^{-1} \mathcal{L} Q = \text{diag}(0, J)$$

where J is a $(N-1) \times (N-1)$ -dimensional upper triangular matrix, whose principal diagonal elements consist of $\lambda_i, i = 2, \dots, N$.

Since (A, B) is stabilizable, from the comments above Theorem 1, $\sigma \geq (\min_{2 \leq i \leq N} \{\text{Re}(\lambda_i)\})^{-1}$, and the property that all the eigenvalues of $A - (a + ib)BB^T P$, ($i^2 = -1$) are in the open left half-plane for any $a \geq 1$ and $b \in \mathbb{R}$ (Ma & Zhang, 2010), we derive that all the eigenvalues of $A - \lambda_i BK, i = 2, \dots, N$ are in the open left half-plane. Thus, we have

$$\begin{aligned} e^{(I_N \otimes A - \mathcal{L} \otimes BK)t} &= (Q \otimes I_n) \begin{bmatrix} e^{At} & 0 \\ 0 & e^{(I_{N-1} \otimes A - J \otimes BK)t} \end{bmatrix} (Q^{-1} \otimes I_n) \\ &\rightarrow (1_N \eta^T) \otimes e^{At} \end{aligned}$$

as $t \rightarrow \infty$. Hence,

$$e^{(I_N \otimes A - \mathcal{L} \otimes BK)t} x(0) \rightarrow ((1_N \eta^T) \otimes e^{At}) x(0) \quad (8)$$

exponentially fast as $t \rightarrow \infty$.

Note that

$$\begin{aligned} &\int_0^t e^{(I_N \otimes A - \mathcal{L} \otimes BK)(t-s)} (I_N \otimes BK) w(s) ds \\ &= \int_0^t (Q \otimes I_n) \begin{bmatrix} e^{A(t-s)} & 0 \\ 0 & e^{(I_{N-1} \otimes A - J \otimes BK)(t-s)} \end{bmatrix} (Q^{-1} \otimes I_n) (I_N \otimes BK) w(s) ds \\ &= \int_0^t (Q \otimes I_n) \begin{bmatrix} e^{A(t-s)} & 0 \\ 0 & 0 \end{bmatrix} (Q^{-1} \otimes I_n) (I_N \otimes BK) w(s) ds \\ &+ \int_0^t (Q \otimes I_n) \begin{bmatrix} 0 & 0 \\ 0 & e^{(I_{N-1} \otimes A - J \otimes BK)(t-s)} \end{bmatrix} (Q^{-1} \otimes I_n) (I_N \otimes BK) w(s) ds \\ &:= T_1(t) + T_2(t) \end{aligned}$$

Recall that $N_0 = 0$, and by straightforward calculation, we obtain

$$T_1(t) = 1_N \otimes \left(\sum_{k=1}^p \left(\sum_{i=N_0+\dots+N_{k-1}+1}^{N_0+\dots+N_k} \eta_i \right) \int_0^t e^{A(t-s)} BK w_k(s) ds \right) \quad (9)$$

Based on Assumption 1 and the fact that $w_i(t)$, $i=1, \dots, N$ are intra-group identical and inter-group non-identical inputs, we can think of $I_N \otimes A - \mathcal{L} \otimes BK$ as $\tilde{\Omega}$ and think of $(I_N \otimes BK)w(s)$ as $\tilde{\Gamma}(s) = \left((1_{N_1} \otimes \Gamma_1(s))^T, \dots, (1_{N_p} \otimes \Gamma_p(s))^T \right)^T$ with $\Gamma_k(s) = BKw_k(s)$ for $k=1, \dots, p$ in Lemma 2. In the light of Lemma 2, the second term on the right-hand side of (7) can be written as $\Psi(t) := \left((1_{N_1} \otimes \Psi_1(t))^T, \dots, (1_{N_p} \otimes \Psi_p(t))^T \right)^T$, where $\Psi_k(t) \in \mathbb{R}^n$, $1 \leq k \leq p$. In other words,

$$T_1(t) + T_2(t) = \Psi(t) \tag{10}$$

Since all the eigenvalues of $A - \lambda_i BK$, $i=2, \dots, N$ are in the open left half-plane and $w(t)$ is bounded, we know that the limit $\lim_{t \rightarrow \infty} T_2(t)$ exists and is a constant matrix. It is clear that the convergence is exponentially fast, and we write $\Phi := \lim_{t \rightarrow \infty} T_2(t)$.

Therefore, by virtue of (7-10), we obtain that

$$\begin{aligned} x(t) &\rightarrow ((1_N \eta^T) \otimes e^{At}) x(0) \\ &+ 1_N \otimes \left(\sum_{k=1}^p \left(\sum_{i=N_0+\dots+N_{k-1}+1}^{N_0+\dots+N_k} \eta_i \right) \int_0^t e^{A(t-s)} BKw_k(s) ds \right) + \Phi \\ &= ((1_N \eta^T) \otimes e^{At}) x(0) + \left((1_{N_1} \otimes \Psi_1(t))^T, \dots, (1_{N_p} \otimes \Psi_p(t))^T \right)^T \end{aligned}$$

exponentially fast as $t \rightarrow \infty$. Thus, for $v_i, v_j \in \mathcal{V}_k$ ($k=1, \dots, p$), $\|x_i(t) - x_j(t)\| \rightarrow 0$ as t tends to infinity. By the basic inequality $\|x_i - x_j\| \leq \|x_i - c\| + \|x_j - c\|$, it is easy to see that the convergence is exponentially fast. For $v_i \in \mathcal{V}_k$ and $v_j \in \mathcal{V}_l$, $k \neq l$, with the inter-group non-identical inputs $w_i(t)$, one can have $\lim_{t \rightarrow \infty} \|x_i(t) - x_j(t)\| > 0$. This completes the proof.

Remark 1 The design of feedback matrix K in Theorem 1 has a highly desirable feature—it uncouples the effects of the agent dynamics and the network topology. Specifically, each agent constructs a gain $B^T P$ by only using (6), and scales it by a multiplicative factor σ taking into account of the interaction topology.

Remark 2 The control signals $\{w_i(t)\}_{i=1}^N$ play a key role in achieving group consensus. The intra-group identical and inter-group non-identical property facilitates the transformation to an explicitly solvable system via Lemma 2. Moreover, if each subgroup \mathcal{V}_k ($k=1, \dots, p$) admits a control node (or leader), which generates the desired target trajectory $s_k(t)$ autonomously, then our consensus protocol (2) becomes similar to the general track control protocol studied in (Zhang, Lewis, & Das, 2011) by taking $w_i(t) = c_k(s_k(t) - x_i(t))$ for any $v_i \in \mathcal{V}_k$ ($i=1, \dots, N$; $k=1, \dots, p$), where $c_k > 0$. It is reasonable to conjecture that $s_k(t)$ would be the ultimate consensus trajectory for subgroup \mathcal{V}_k under some connectivity conditions.

If we take $n=m$, $A=0$, and $B=I_n$, the system (5) reduces to the case of integrator agents. We then have the following corollary.

COROLLARY 1 Under Assumption 1, if \mathcal{G} contains a spanning tree, then the gain matrix can be designed as $K = \sigma I_n$, where $\sigma \geq (\min_{2 \leq i \leq N} \{\text{Re}(\lambda_i)\})^{-1}$, such that the multi-agent system

$$\dot{x}_i(t) = u_i(t), \quad t \geq 0, \quad i=1, 2, \dots, N$$

under protocol (2) can achieve group consensus exponentially fast.

4. Simulation

In this section, we present numerical simulations to illustrate the validity of the proposed theoretical results.

Figure 1. Network topology \mathcal{G} .

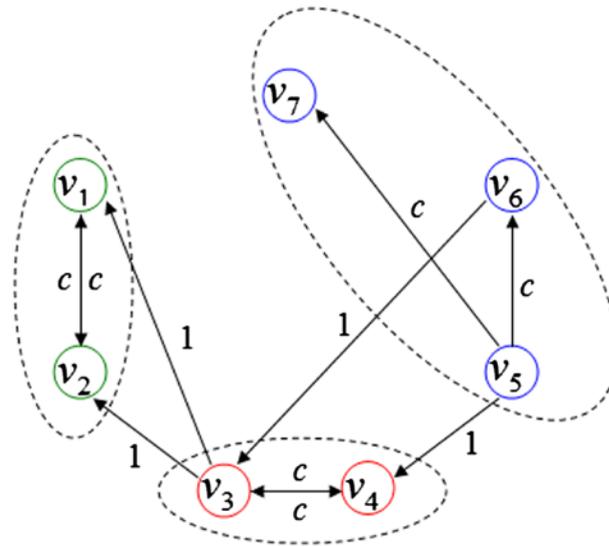


Figure 2. The state trajectories for the case of $c=1$.

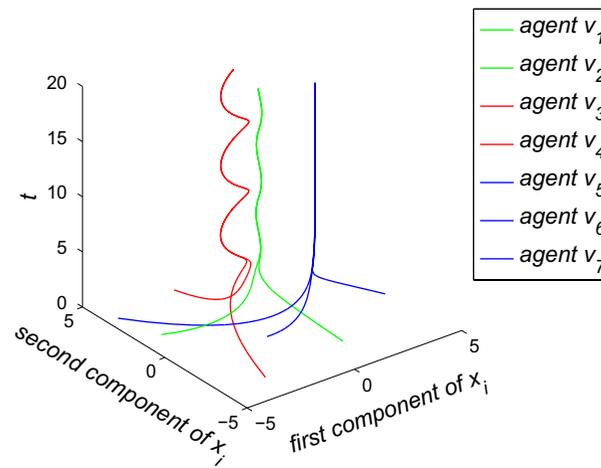


Figure 3. The state trajectories for the case of $c=.01$.

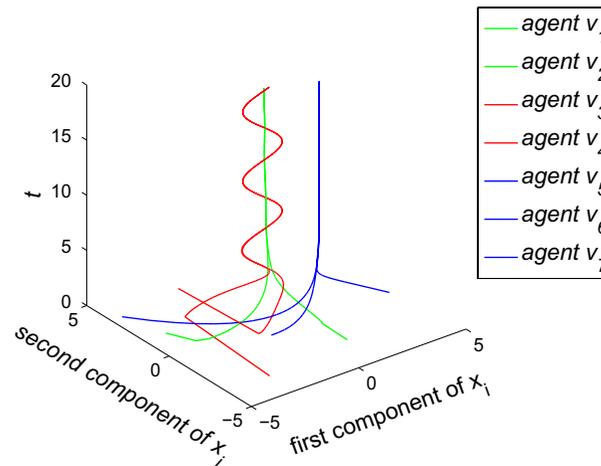
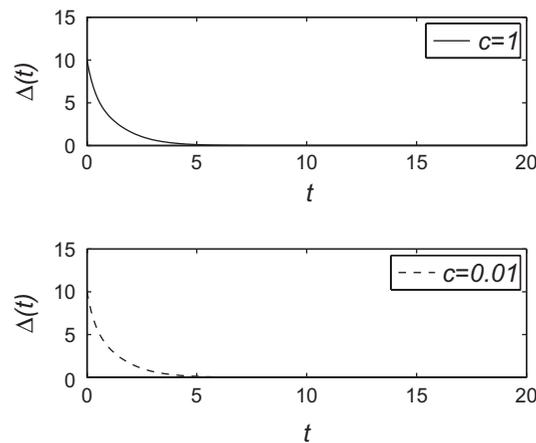


Figure 4. The dynamical behavior of $\Delta(t)$.



Consider the multi-agent system with $N=7$ agents divided into three subgroups $\mathcal{V}_1 = \{v_1, v_2\}$, $\mathcal{V}_2 = \{v_3, v_4\}$, and $\mathcal{V}_3 = \{v_5, v_6, v_7\}$. The communication topology among agents is shown in Figure 1. The inter-group weights and intra-group weights are set to be 1 and c , respectively, for some positive c . The Laplacian matrix is given by

$$\mathcal{L} = \begin{bmatrix} 1+c & -c & -1 & 0 & 0 & 0 & 0 \\ -c & 1+c & -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1+c & -c & 0 & -1 & 0 \\ 0 & 0 & -c & 1+c & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -c & c & 0 \\ 0 & 0 & 0 & 0 & -c & 0 & c \end{bmatrix}$$

Take $n=2$, $m=1$, and let the agent dynamics (1) be specified as

$$A = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The pair (A, B) is stabilizable. The inputs are chosen as $w_1(t) = w_2(t) = ((t+1)^{-1}, 2(t+1)^{-1})^T$, $w_3(t) = w_4(t) = (\sin(t), 3 + \cos(t))^T$, and $w_5(t) = w_6(t) = w_7(t) = (2, -2)^T$. Therefore, all conditions in Theorems 1 hold. From (6), we solve $P = \begin{bmatrix} 0.6569 & 0.1716 \\ 0.1716 & 0.4142 \end{bmatrix}$. The initial states $x_i(0)$, $i=1, \dots, 7$ are taken randomly in $[-5, 5]^2$.

We consider two cases of coupling strength: $c=1$ and $c=.01$. In view of Theorem 1, the gain matrices are solved as $K=(.1716, .4142)$ and $K=(17.16, 41.42)$, respectively. Define $\Delta(t) = \max_{1 \leq k \leq 3} \max_{v_i, v_j \in \mathcal{V}_k} \|x_i(t) - x_j(t)\|$ as a measure of the discrepancy of states within the same groups. The dynamical behaviors of the states $x_i(t)$, $i=1, \dots, 7$ are shown in Figures 2 and 3, respectively. Figure 4 shows the corresponding error trajectories, indicating that the group consensus is achieved for both cases.

5. Conclusion

This paper studies the group consensus problem of a network of continuous-time agents with linear dynamics, whose interaction topology is directed and fixed. Based on algebraic graph theory and matrix theory, external adapted inputs are introduced to realize group consensus exponentially fast. Comparing the existing works on group consensus, we highlight the following features of our result:

- We study the case where each agent has dynamics of a continuous linear time-invariant system. Such systems include the single-integrator, double-integrator, and higher order integrator dynamics as special cases.
- The external control inputs contribute to the intra-group synchronization and inter-group separation, without imposing complicated algebraic criteria (Shang, 2013c; Sun et al., 2011; Tan et al., 2011; Xia & Cao, 2011; Yu & Wang, 2010) or restrictive topological constraints (Liu & Wong, 2013; Ma et al., 2014; Qin & Yu, 2013).
- Only non-negative weights are assigned to the communication links, which have practical applicability. In Yu and Wang (2010), Xia and Cao (2011), Sun et al. (2011), Shang (2013c), Tan et al. (2011), and Qin and Yu (2013), negative weights are essentially needed for group consensus.

Moreover, the group consensus can be achieved by designing appropriate control gains for any given magnitude of the coupling strengths among the agents. Numerical simulations are presented to illustrate the effectiveness of our theoretical results.

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Author details

Yilun Shang¹

E-mail: shylmath@hotmail.com

¹ Einstein Institute of Mathematics, Hebrew University of Jerusalem, Jerusalem 91904, Israel; SUTD-MIT International Design Center, Singapore University of Technology and Design, Singapore 138682, Singapore.

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