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PURE MATHEMATICS | RESEARCH ARTICLE

A new generalized contraction and its application in dynamic programming

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Abstract: By using the idea of Pata [*V. Pata*, A fixed point theorem in metric spaces. *J. Fixed Point Theory Appl. 10 (2011) 299–305.*] we establish a new common fixed point theorem and as an application, we prove the existence and uniqueness of common solutions for a class of system of functional equations arising in dynamic programming.

Subjects: Applied Mathematics; Non-Linear Systems; Dynamical Systems

Keywords: dynamic programming; functional equations; contractive mapping; common fixed point
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1. Introduction

It is clear that fixed point theory is one of the powerful tools in solving nonlinear analysis problems such as integral and differential equations. The Banach contraction mapping principle is one of the pivotal results in fixed point theory and it has a board set of applications. Generalization of the above principle has been proved by various authors (see, for example, (Agarwal, Meehan, & O'Regan, 2001; Gordji & Ramezani, 2011; Harjani & Sadarangani, 2010)). In particular, recently, (Pata, 2011) improves the Banach principle. In this article, we present a common fixed point theorem by using the idea of Pata. As an application, the existence and uniqueness of common solution for a system of functional equations arising in dynamic programming are given.

2. Main results

Let (X, d) be a complete metric space. Selecting an arbitrary $x_0 \in X$. We denote

$$\| x \| = d(x, x_0)$$

M. Ramezani

ABOUT THE AUTHORS

M. Ramezani has some papers in this field such as the following: 1. A generalization of Geraghty's fixed point theorem in partially ordered metric spaces andapplications to ordinary differential equations, Fixed Point Theory and Applications, 2. Pata type fixed point theorems of multivalued operators in ordered metric spaces with applications to hyperbolic differential inclusions., U.P.B. Sci. Bull., Series A., 3. Presic- Kannan- Rus fixed point theorem on partially ordered metric spaces, Fixed Point Theory.

H. Ramezani is a member of statistical research group in Payam noor University of Mashhad and he has a ISC published paper in Mathematics and Statistic by title: The estimation of a compound Poisson process by used wavelets.

PUBLIC INTEREST STATEMENT

One of the powerful tools in solving nonlinear equations and dynamic programming problems is the Banach contraction principle and its generalizations. In this manuscript, we solve a dynamic programming by a generalization of Banach's fixed point theorem. Also, we give some examples to support our main results.

for all $x \in X$. We consider the function $\psi : [0,1] \to [0,\infty)$ such that ψ is an increasing function vanishing with continuity at zero. We will also consider the (vanishing) sequence depending on $\alpha \geq 1$ by the following.

$$w_n(\alpha) = \left(\frac{\alpha}{n}\right)^n \sum_{k=1}^{n} \psi\left(\frac{\alpha}{k}\right).$$

Our first result is the following.

Theorem 2.1. *Let $f, g : X \to X$ and $\Lambda \geq 0$, $\alpha \geq 1$, $\beta \in [0,\alpha]$ be fixed constants.*

Suppose

$$d(fx, gy) \leq (1-\varepsilon)M(x,y) + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1+ \| x \| + \| y \|]^{\beta} \tag{2.1}$$

for all $x, y \in X$ and $\varepsilon \in (0,1]$, where

$$M(x,y) = \max\left\{d(x,y), d(x,fx), d(y,gy), \frac{[d(x,gy) + d(y,fx)]}{2}\right\}.$$

Then f and g have a unique common fixed point.

Proof. Let x_0 be an arbitrary element in X. We introduce the sequences

$$c_n = \| x_n \|, \quad fx_{2n} = x_{2n+1}, \quad gx_{2n+1} = x_{2n+2}$$

for all $n \in \mathbb{N} \cup \{0\}$. Note that

$$\begin{aligned} M(x_{2n}, x_{2n+1}) &= \max\left\{d(x_{2n}, x_{2n+1}), d(x_{2n}, fx_{2n}), d(x_{2n+1}, gx_{2n+1}), \frac{d(x_{2n}, gx_{2n+1}) + d(x_{2n+1}, fx_{2n})}{2}\right\} \\ &= \max\left\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \frac{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})}{2}\right\} \\ &\leq \max\left\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \frac{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})}{2}\right\} \\ &= \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}. \end{aligned}$$

If

$$x_{2n_0+1} = x_{2n_0+2}$$

for some $n_0 \in \mathbb{N}$, by using definition of sequence $\{x_n\}$ we see that $x_{2n_0+2} = gx_{2n_0+1}$ and x_{2n_0+1} is a fixed point of g. Also,

$$\begin{aligned} d(fx_{2n_0+1}, x_{2n_0+1}) &= d(fx_{2n_0+1}, gx_{2n_0+1}) \\ &\leq (1-\varepsilon)d(x_{2n_0+1}, fx_{2n_0+1}) + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1+ \| x_{2n_0+1} \| + \| x_{2n_0+1} \|]^{\beta} \end{aligned}$$

for all $\varepsilon \in [0,1]$. So,

$$d(fx_{2n_0+1}, x_{2n_0+1}) \leq \Lambda\varepsilon^{\alpha-1}\psi(\varepsilon)[1 + 2 \| x_{2n_0+1} \|]^{\beta}$$

for all $\varepsilon \in (0,1]$. If $\varepsilon \to 0$ then $x_{2n_0+1} = fx_{2n_0+1}$. Hence x_{n_0+1} is a common fixed point of f and g. Now, we suppose that

$$x_{2n+1} \neq x_{2n+2}, \quad \forall n \in \mathbb{N}. \tag{2.2}$$

We divide the proof into several steps.

Step1: $\{d(x_n, x_{n-1})\}$ is a nondecreasing sequence.

Let n be fixed, if

$$\max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} = d(x_{2n+1}, x_{2n+2})$$

therefore,

$$d(x_{2n+1}, x_{2n+2}) \le (1-\varepsilon)d(x_{2n+1}, x_{2n+2}) + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1+\parallel x_{2n} \parallel + \parallel x_{2n+1} \parallel]^{\beta}$$

for all $\varepsilon \in (0, 1]$. Hence,

$$d(x_{2n+1}, x_{2n+2}) \le \Lambda\varepsilon^{\alpha-1}\psi(\varepsilon)[1+\parallel x_{2n} \parallel + \parallel x_{2n+1} \parallel]^{\beta}$$

for all $\varepsilon \in (0, 1]$. If $\varepsilon \to 0$ then

$$d(x_{2n+1}, x_{2n+2}) = 0$$

that is a contradiction to (2.2). Therefore, we have

$$\max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} = d(x_{2n}, x_{2n+1}).$$

Hence,

$$d(x_{2n+1}, x_{2n+2}) \le (1-\varepsilon)d(x_{2n}, x_{2n+1}) + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1+\parallel x_{2n} \parallel + \parallel x_{2n+1} \parallel]^{\beta}$$

for all $\varepsilon \in (0, 1]$. This show that

$$d(x_{2n+1}, x_{2n+2}) \le d(x_{2n}, x_{2n+1}).$$

Thus, the sequence $\{d(x_n, x_{n+1})\}$ is nonincreasing.

Step 2: The sequence c_n is bounded.

Fix $n \in \mathbb{N}$. By using Step 1 we have

$$d(x_{2n+1}, x_{2n+2}) \le d(x_{2n}, x_{2n+1}) \le \ldots \le d(x_0, x_1).$$

By the above and the triangle inequality we have

$$\begin{aligned} c_{2n+1} &= d(x_{2n+1}, x_0) \le d(x_{2n+1}, x_{2n+2}) + d(x_{2n+2}, x_1) + d(x_1, x_0) = d(x_{2n+2}, x_1) + 2c_1 \\ &= d(gx_{2n+1}, fx_0) + 2c_1. \end{aligned}$$

Therefore, as $\beta \le \alpha$ we infer that

$$\begin{aligned} c_{2n+1} &\le (1-\varepsilon)M(x_{2n+1}, x_0) + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1+\parallel x_{2n+1} \parallel + \parallel x_0 \parallel]^{\beta} + 2c_1 \\ &\le (1-\varepsilon)(c_{2n+1} + c_1) + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1 + c_{2n+1}]^{\beta} + 2c_1 \\ &\le (1-\varepsilon)c_{2n+1} + a\varepsilon^{\alpha}\psi(\varepsilon)c_{2n+1}^{\alpha} + b \end{aligned}$$

for some $a, b > 0$. Accordingly,

$$\varepsilon c_{2n+1} \le a\varepsilon^{\alpha}\psi(\varepsilon)c_{2n+1}^{\alpha} + b$$

If there is a subsequence $c_{2n_k+1} \to \infty$, the choice $\varepsilon = \varepsilon_1 = (1+b)/c_{2n_k+1}$ leads to the contradiction

$$1 \le a(1+b)^{\alpha}\psi(\varepsilon_1) \to 0.$$

Similarly, we have

$$c_{2n+2} \le d(x_{2n+3}, x_2) + d(x_2, x_1) + 2c_1 \le d(x_{2n+3}, x_2) + 3c_1$$

Therefore,

$$\begin{aligned}c_{2n+2} &\le d(x_{2n+3}, x_2) + 3c_1 = d(fx_{2n+2}, x_1) + 3c_1\\ &\le (1-\varepsilon)M(x_{2n+2}, x_1) + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1+ \parallel x_{2n+2} \parallel + \parallel x_1 \parallel]^{\beta} + 3c_1\\ &\le (1-\varepsilon)(c_{2n+2} + 2c_1) + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1 + c_{2n+2}]^{\beta} + 3c_1\\ &\le (1-\varepsilon)c_{2n+2} + a'\varepsilon^{\alpha}\psi(\varepsilon)c_{2n+2}^{\alpha} + b'\end{aligned}$$

for some $a', b' > 0$. Accordingly,

$$\varepsilon c_{2n+2} \le a'\varepsilon^{\alpha}\psi(\varepsilon)c_{2n+2}^{\alpha} + b'$$

If there is a subsequence $c_{2n_k+2} \to \infty$, the choice $\varepsilon = \varepsilon_2 = (1+b)/c_{2n_k+2}$ leads to the contradiction $1 \le a'(1+b')^{\alpha}\psi(\varepsilon_2) \to 0$.

Set $C = \sup_{n\in\mathbf{N}}\Lambda(1+2c_n)^{\beta} < \infty$.

Step 3: The sequence $\{x_n\}$ is Cauchy.

Since $\{d(x_{2n}, x_{2n+1})\}$ is decreasing thus

$$d(x_{2n}, x_{2n+1}) \to r \ge 0$$

If $r > 0$, then

$$d(x_{2n}, x_{2n+1}) = d(gx_{2n-1}, fx_{2n}) \le (1-\varepsilon)M(x_{2n+1}, x_{2n}) + C\varepsilon^{\alpha}\psi(\varepsilon) \le (1-\varepsilon)d(x_{2n+1}, x_{2n}) + C\varepsilon^{\alpha}\psi(\varepsilon)$$

for all $n \in \mathbb{N}$, and $\varepsilon \in (0, 1]$. As $n \to \infty$, we have

$$r \le (1-\varepsilon)r + C\varepsilon^{\alpha}\psi(\varepsilon)$$

for all $\varepsilon \in (0, 1]$. So

$$r \le C\varepsilon^{\alpha-1}\psi(\varepsilon)$$

for all $\varepsilon \in (0, 1]$. As $\varepsilon \to 0$ we get $r = 0$ and this is a contradiction, therefore $r = 0$.

Hence

$$\lim_{n\to\infty} d(x_{2n}, x_{2n+1}) = 0. \tag{2.3}$$

To show that $\{x_n\}$ is Cauchy sequence, it is sufficient to show that the subsequence $\{x_{2n}\}$ of $\{x_n\}$ is a Cauchy sequence in view of (2.3). If $\{x_{2n}\}$ is not Cauchy, there exist an $\delta > 0$ and monotone increasing sequence of natural numbers $\{2m_k\}$ and $\{2n_k\}$ such that $n_k > m_k$,

$$d(x_{2m_k}, x_{2n_k}) \ge \delta \quad \textit{and} \quad d(x_{2m_k}, x_{2n_k-2}) < \delta. \tag{2.4}$$

From (2.4), we get

$$\begin{aligned}\delta &\le d(x_{2m_k}, x_{2n_k}) \le d(x_{2m_k}, x_{2n_k-2}) + d(x_{2n_k-2}, x_{2n_k-1}) + d(x_{2n_k-1}, x_{2n_k})\\ &\le \delta + d(x_{2n_k-2}, x_{2n_k-1}) + d(x_{2n_k-1}, x_{2n_k}).\end{aligned}$$

Letting $k \to \infty$ and using (2.3) we have

$$\lim_{n\to\infty} d(x_{2m_k}, x_{2n_k}) = \delta. \tag{2.5}$$

Letting $k \to \infty$ and using (2.3), (2.4) and (2.5), we get

$$|d(x_{2m_k}, x_{2n_k+1}) - d(x_{2m_k}, x_{2n_k})| \le d(x_{2n_k}, x_{2n_k+1}).$$

Hence,

$$\lim_{n\to\infty} d(x_{2n_k+1}, x_{2m_k}) = \delta. \tag{2.6}$$

Letting $k \to \infty$ and using (2.3) and (2.6), we have

$$\left|d(x_{2m_k-1}, x_{2n_k}) - d(x_{2m_k}, x_{2n_k})\right| \le d(x_{2m_k}, x_{2m_k-1}),$$

which implies that

$$\lim d(x_{2n_k}, x_{2m_k-1}) = \delta. \tag{2.7}$$

Putting $x = x_{2n_k}$ and $y = x_{2m_k-1}$ in (2.1) we have

$$\begin{aligned} d(fx_{2n_k}, gx_{2m_k-1}) &\le (1-\varepsilon)M(x_{2n_k}, x_{2m_k-1}) + C\varepsilon^{\alpha}\psi(\varepsilon) \\ &= (1-\varepsilon)\max\Big\{d(x_{2n_k}, x_{2m_k-1}), d(x_{2n_k}, x_{2n_k+1}), d(x_{2m_k-1}, x_{2m_k}), \\ &\quad \frac{d(x_{2n_k}, x_{2m_k}) + d(x_{2m_k-1}, x_{2n_k+1})}{2}\Big\} \\ &\le (1-\varepsilon)\max\Big\{d(x_{2n_k}, x_{2m_k-1}), d(x_{2n_k}, x_{2n_k+1}), d(x_{2m_k-1}, x_{2m_k}), \\ &\quad \frac{d(x_{2n_k}, x_{2m_k}) + d(x_{2m_k-1}, x_{2m_k}) + d(x_{2m_k}, x_{2n_k+1})}{2}\Big\} \end{aligned}$$

for all $\varepsilon \in (0, 1]$.

Letting $k \to \infty$ and using (2.3), (2.4), (2.5), (2.6) and (2.7) we get

$$\delta \le (1-\varepsilon)\delta + C\varepsilon^{\alpha}\psi(\varepsilon)$$

for all $\varepsilon \in (0, 1]$. Thus

$$\delta \le C\varepsilon^{\alpha-1}\psi(\varepsilon).$$

If $\varepsilon \to 0$ then we have $\delta = 0$ and it is a contradiction, therefore $\{x_{2n}\}$ is a Cauchy sequence.

Step 4: $fx_* = x_*$.

Since X is complete, there exists $x_* \in X$ such that $x_n \to x_*$ as $n \to \infty$, so $x_{2n} \to x_*$ and $x_{2n+1} \to x_*$ so,

$$d(fx_*, gx_{2n+1}) \le (1-\varepsilon)M(x_*, x_{2n+1}) + C\varepsilon^{\alpha}\psi(\varepsilon)$$

for all $\varepsilon \in [0, 1]$.

$$M(x_*, x_{2n+1}) = \max\left\{d(x_*, x_{2n+1}), d(x_*, fx_*), d(x_{2n+1}, x_{2n+2}), \frac{d((x_*, x_{2n+2}) + d(x_{2n+1}, fx_*)}{2}\right\}.$$

As $n \to \infty$ we have

$$d(fx_*, x_*) \le (1-\varepsilon)d(x_*, fx_*) + C\varepsilon^{\alpha}\psi(\varepsilon)$$

for all $\varepsilon \in (0, 1]$. So

$$d(x_*, fx_*) \le C\varepsilon^{\alpha-1}\psi(\varepsilon)$$

for all $\varepsilon \in (0, 1]$. If $\varepsilon \to 0$ then we get

$$d(x_*, fx_*) \to 0.$$

Hence $fx_* = x_*$.

Step 5: $gx_* = x_*$.

Now we show that x_* is a fixed point of g too,

$$0 < d(x_*, gx_*) = d(fx_*, gx_*) \\ \leq (1-\varepsilon)\max\{d(x_*, x_*), d(x_*, fx_*), d(x_*, gx_*), d(x_*, gx_*) + d(x_*, fx_*)/2\} + k\varepsilon\psi(\varepsilon)$$

where $k > 0$. So,

$$d(x_*, gx_*) \leq (1-\varepsilon)d(x_*, gx_*) + k\varepsilon\psi(\varepsilon).$$

This implies that

$$d(x_*, gx_*) \leq k\psi(\varepsilon)$$

where $\varepsilon \in (0, 1]$. Since ψ is increasing and continuous at zero, then $\psi(0) = 0$ and

$$d(x_*, gx_*) = 0.$$

Therefore $x_* = gx_*$.

Step 6: Uniqueness of x_*.

If there exists $y_* \in X$ that $y_* = fy_* = gy_*$, then

$$d(x_*, y_*) = d(fx_*, gy_*) \leq (1-\varepsilon)d(x_*, gy_*) + k\varepsilon\psi(\varepsilon) = (1-\varepsilon)d(x_*, y_*) + k\varepsilon\psi(\varepsilon).$$

Setting $\varepsilon = 0$,

$$d(x_*, y_*) = 0$$

and so $y_* = x_*$

Theorem 2.2. *Let $\rho : [0, \infty) \to [0, \infty)$ be a continuous function satisfying the inequality $\rho(r) < r$ for every $r > 0$, suppose that for every x, y with, $x \neq y$*

$$d(fx, gy) \leq \rho(M(x, y)) \tag{2.8}$$

where

$$M(x, y) = \max\left\{d(x, y), d(x, fx), d(y, gy), \frac{d(x, gy) + d(y, fx)}{2}\right\}.$$

Then f and g have a unique common fixed point x_* and $d(x_*, x_n) \to 0$, where $\{x_n\}$ is the sequence is defined in Theorem 2.1.

Proof. We introduce the intreate sequences

$$fx_{2n} = x_{2n+1}, gx_{2n+1} = x_{2n+2}.$$

Note that

$$M(x_{2n}, x_{2n+1}) = \max\left\{d(x_{2n}, x_{2n+1}), d(x_{2n}, fx_{2n}), d(x_{2n+1}, gx_{2n+1}), \frac{d(x_{2n}, gx_{2n+1}) + d(x_{2n+1}, fx_{2n})}{2}\right\} \\ = \max\left\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \frac{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})}{2}\right\} \\ \leq \max\left\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \frac{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})}{2}\right\} \\ = \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}).\}$$

Let $n \in \mathbb{N}$ be fixed, if

$$\max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} = d(x_{2n+1}, x_{2n+2})$$

Therefore,

$$d(x_{2n+1}, x_{2n+2}) = d(fx_{2n}, gx_{2n+1}) \leq (d(x_{2n+1}, x_{2n+2})) < d(x_{2n+1}, x_{2n+2}).$$

That is a contradiction. Therefore we have

$$\max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} = d(x_{2n}, x_{2n+1}).$$

So,

$$d(x_{2n+1}, x_{2n+2}) \leq d(x_{2n}, x_{2n+1}).$$

Hence, $\{d(x_{2n}, x_{2n+1})\}$ is a decreasing sequence that converges monotically to some $r \geq 0$.

If $r > 0$, then

$$d(x_{2n}, x_{2n+1}) = d(gx_{2n-1}, fx_{2n}) \leq (d(x_{2n-1}, x_{2n})).$$

As $n \to \infty$, we have $r \leq \rho(r) < r$ and this is a contradiction, therefore $r = 0$. Hence

$$\lim d(x_{2n}, x_{2n+1}) = 0 \tag{2.9}$$

To show that $\{x_n\}$ is Cauchy sequence, it is sufficient to show that the subsequence $\{x_{2n}\}$ of $\{x_n\}$ is Cauchy sequence in view of (2.9). If $\{x_{2n}\}$ is not Cauchy there exist an $\delta > 0$ and natural numbers n_k and m_k such that $n_k > m_k$, and

$$d(x_{2m_k}, x_{2n_k}) \geq \delta \quad \textit{and} \quad d(x_{2m_k}, x_{2n_k-2}) < \delta. \tag{2.10}$$

From (2.10),

$$\begin{aligned}\delta \leq d(x_{2m_k}, x_{2n_k}) &\leq d(x_{2m_k}, x_{2n_k-2}) + d(x_{2n_k-2}, x_{2n_k-1}) + d(x_{2n_k-1}, x_{2n_k}) \\ &\leq \delta + d(x_{2n_k-2}, x_{2n_k-1}) + d(x_{2n_k-1}, x_{2n_k}).\end{aligned}$$

Letting $k \to \infty$ and using (2.9) we have

$$\lim_{k\to\infty} d(x_{2m_k}, x_{2n_k}) = \delta. \tag{2.11}$$

Letting $k \to \infty$ and using (2.9) and (2.10)

$$\left|d(x_{2m_k}, x_{2n_k+1}) - d(x_{2m_k}, x_{2n_k})\right| \leq d(x_{2n_k}, x_{2n_k-1}).$$

Hence,

$$\lim_{k\to\infty} d(x_{2n_k+1}, x_{2m_k}) = \delta. \tag{2.12}$$

Letting $k \to \infty$ and using (2.9) and (2.10)

$$\left|d(x_{2m_k-1}, x_{2n_k}) - d(x_{2m_k}, x_{2n_k})\right| \leq d(x_{2m_k}, x_{2m_k-1}).$$

Hence,

$$\lim_{k\to\infty} d(x_{2n_k}, x_{2m_k-1}) = \delta \tag{2.13}$$

Putting $x = x_{2n_k}$ and $y = x_{2m_k-1}$ in (2.8) we have

$$
\begin{aligned}
d(fx_{2n_k}, gx_{2m_k-1}) &\le \rho(m(x_{2n_k}, x_{2m_k-1})) \\
&\le \rho(\max\{d(x_{2n_k}, x_{2m_k-1}), d(x_{2n_k}, x_{2n_k+1}), d(x_{2m_k-1}, x_{2m_k}) \\
&\quad \frac{d(x_{2n_k}, x_{2m_k}) + d(x_{2m_k-1}, x_{2n_k+1})}{2}\}) \\
&\le \rho(\max\{d(x_{2n_k}, x_{2m_k-1}), d(x_{2n_k}, x_{2n_k+1}), d(x_{2m_k-1}, x_{2m_k}) \\
&\quad \frac{d(x_{2n_k}, x_{2m_k}) + d(x_{2m_k-1}, x_{2n_k}) + d(x_{2n_k}, x_{2n_k+1})}{2}\}).
\end{aligned}
$$

Letting $k \to \infty$ and using (2.9), (2.10), (2.11), (2.12) and (2.13) we get

$$\delta \le \rho(\delta) < \delta$$

and this is a contradiction. Hence, $\{x_{2n}\}$ is a Cauchy sequence. Since X is complete, there exists $x_* \in X$ such that $x_n \to x_*$ as $n \to \infty$. Hence, $x_{2n} \to x_*$ and $x_{2n+1} \to x_*$. Assume that $fx_* \ne x_*$,

$$d(fx_*, gx_{2n+1}) \le \rho(m(x_*, x_{2n+1}))$$

as $n \to \infty$ we have

$$d(fx_*, x_*) \le \rho(d(x_*, fx_*)) < d(x_*, fx_*)$$

and this is a contradiction. Suppose that $gx_* \ne x_*$, therefore

$$
\begin{aligned}
d(x_*, gx_*) &= d(fx_*, gx_*) \le \rho(m(x_*, x_*)) \\
&= \rho\left(\max\left(d(x_*, x_*), d(x_*, fx_*), d(x_*, gx_*), \frac{d(x_*, gx_*), d(x_*, fx_*)}{2}\right)\right) = \rho(d(x_*, gx_*)) < d(x_*, gx_*)
\end{aligned}
$$

that is a contradiction, therefore x_* is a fixed point of g too.

If there exists $y_* \in X$ that $y_* \ne x_*$ and

$$y_* = fy_* = gy_*,$$

then,

$$
\begin{aligned}
d(x_*, y_*) &= d(fx_*, gy_*) \le \rho(M(x_*, y_*)) \le \rho(\max(d(x_*, y_*), d(x_*, x_*), d(y_*, y_*), \frac{d(x_*, y_*), d(y_*, x_*)}{2})) \\
&= \rho(d(x_*, y_*)) < d(x_*, y_*),
\end{aligned}
$$

and it is a contradiction which completes the proof.

Example 2.3. *Let $X = [1, \infty)$ with Euclidean metric and $f, g : X \to X$ be two mappins by defined $f(x) = g(x) = x - 2x^{\frac{1}{2}} + 4x^{\frac{1}{4}} - 2$. It is clear that $x^* = 1$ is a unique common fixed point of f and g. For all $x, y \in [1, \infty)$ with $y > x$ we have*

$$fy - gx = (y - x) - 2[y^{\frac{1}{2}} - x^{\frac{1}{2}}] + 4[y^{\frac{1}{4}} - x^{\frac{1}{4}}] = (y - x) - F(x, y).$$

where $F(x, y) = 2[y^{\frac{1}{2}} - x^{\frac{1}{2}}] - 4[y^{\frac{1}{4}} - x^{\frac{1}{4}}]$. For every $\varepsilon \in (0, 1]$ standard calculations show that

$$-\varepsilon(y - x) + \varepsilon^2(2x + (y - x))^{\frac{3}{2}} + F(x, y) \ge 0.$$

Hence,

$$|fy - gx| = (y - x) - F(x, y) \le (1 - \varepsilon)(y - x) + \varepsilon^2(2x + (y - x))^{\frac{3}{2}} \le (1 - \varepsilon)M(x, y) + \varepsilon^2(2x + (y - x))^{\frac{3}{2}}$$

by putting $\Lambda = \alpha = \beta = 1, x_0 = 1$ and $\psi(r) = r$ we can see all assumptions of Theorem 2.1 hold. Therefore, the existence and uniqueness of common fixed point of f and g implies from Theorem 2.1.

Proposition 2.4. *If X is a bounded space, Theorem 2.1 and Theorem 2.2 imply each other.*

Proof. Suppose for simplicity that $diam(X) \le 1$. Then for all $x, y \in X$, $\| x \| \le, \| y \| \le 1$ and the inquality (2.1) reads

$$d(fx,gy) \le (1-\varepsilon)M(x,y) + \Lambda\varepsilon^{\alpha}\psi(\varepsilon). \tag{2.14}$$

We first prove the implication (2.14) $\Rightarrow$ (2.8). Define $\varphi(r) := \psi^{-1}(\nu r)$ with $\nu > 0$ suitably small and the function ρ by

$$\rho(r) = r - r\varphi(r)[1 - \nu\Lambda[\varphi(r)]^{\alpha-1}].$$

It is obvious that $\rho(r) < r$ for all $r > 0$, also, the continuity of ψ implies the continuity of ρ. Since ψ is defined on $[0,1]$, then we can choose $\varepsilon = \varphi(M(x,y))$ in (2.14). The inequality of (2.14) and definition of ρ follows that

$$\begin{aligned} d(fx,gy) &\le (1-\varepsilon)M(x,y) + \Lambda\varepsilon^{\alpha}\psi(\varepsilon) = [1-\varphi(M(x,y))]M(x,y) + \Lambda\,\nu\, M(x,y)\varphi(M(x,y))^{\alpha} \\ &= [1-\varphi(M(x,y))](\rho(M(x,y)) + M(x,y)\varphi(M(x,y))[1-\nu\Lambda[\varphi(M(x,y))]^{\alpha-1}]) \\ &\quad + \Lambda\nu M(x,y)\varphi(M(x,y))^{\alpha}. \end{aligned}$$

By algebraic calculations we conclude that $d(fx,gy) \le \rho(M(x,y))$ and so the inequality (2.8) is hold.

Conversely, let (2.8) holds. Since ρ is continuous and $\rho(r) < r$ on $(0,1]$, we can easily construct a strictly increasing continuous function μ such that $\mu(0) = 0$ and

$$\mu(r) \le 1 - \frac{\rho(r)}{r} \qquad \forall r \in (0,1]$$

Let $x,y \in X$ with $m(x,y) = r \in (0,1]$ be given. By virtue of (2.8),

$$d(fx,gy) \le \rho(r) \le r - r\mu(r) < r$$

If $\mu(r) \ge \varepsilon$, we readily obtain

$$d(fx,gy) \le (1-\varepsilon)r$$

Whereas if $\mu(r) < \varepsilon$, we get

$$d(fx,gy) < (1-\varepsilon)r + \varepsilon r < (1-\varepsilon)r + \varepsilon\mu^{-1}(\varepsilon)$$

Collecting the two inequalities (2.14) follows for $\Lambda = 1$ and $\psi(\varepsilon) = \mu^{-1}(\varepsilon)$ if $\varepsilon \le \mu(1)$, $\psi(\varepsilon) = 1$ otherwise.

Corollary 2.5. *Let (X,d) be a complete metric space. Selecting an arbitrary $x_0 \in X$. We denote*

$$\| x \| = d(x, x_0) \quad \forall x \in X$$

Let $f,g : X \to X$ and $\Lambda \ge 0$, $\alpha \ge 1$, $\beta \in [0,\alpha]$ be fixed constants. Suppose

$$d(fx,gy) \le (1-\varepsilon)M(x,y) + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1+ \| x \| + \| y \|]^{\beta}$$

for all $x,y \in X$ and $\varepsilon \in [0,1]$, where

$$M(x,y) = max\{d(x,y), [d(x,fx) + d(y,gy)]/2, [d(x,gy) + d(y,fx)]/2\}.$$

Then f and g have a unique common fixed point.

Proof. Since $[d(x,fx) + d(y,gy)]/2$ is the average of $d(x,fx)$ and $d(y,gy)$, then

$$[d(x,fx) + d(y,gy)]/2 \le d(x,fx) \quad or \quad d(y,gy).$$

The result is seen by Theorem 2.1.

3. Application

Let X and Y be Banach spaces, $S \subseteq X$ be the state space, $D \subseteq Y$ be the decision space(for more details, the reader can see (Harjani & Sadarangani, 2010; Li, Fu, Liu, & Kang, 2008)), and i_x be the identity mapping on X. $B(S)$ denotes the set of all bounded real-valued functions on S and $d(f,g) = \sup\{|f(x) - g(x)| : x \in S\}$. It is clear that $(B(S), d)$ is a complete metric space.

In this section we study the existence and uniqueness of a common solution of the following system of functional equations arising in dynamic programming:

$$f_i(x) = \sup_{y \in D}\{u(x,y) + H_i(x,y,f_i(T(x,y)))\}, \quad \forall x \in S, i \in \{1,2\}, \tag{3.1}$$

where $u : S \times D \to \mathbb{R}$, $T : S \times D \to S$, and $H_i : S \times D \times \mathbb{R} \to \mathbb{R}$ for $i \in \{1,2\}$

Theorem 3.1. *Suppose that the following conditions are satisfied:*

(*a*1) *u and H_i are bounded for $i \in \{1,2\}$;*

(*a*2) *There exist two mappings A_1 and A_2 defined by*

$$A_i g_i(x) = \sup_{y \in D}\{u(x,y) + H_i(x,y,g_i(T(x,y)))\}, \quad \forall x \in S, g_i \in B(S), i \in \{1,2\},$$

satisfying

$$|H_1(x,y,g(t)) - H_2(x,y,h(t))| \leq (1-\varepsilon)\max\{d(g,h), d(g,A_1g), d(h,A_2h), \frac{1}{2}[d(g,A_2h) + d(h,A_1g)]\} + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1+ \| g \| + \| h \||]^{\beta}$$

for all $(x,y) \in S \times D$, $g,h \in B(S)$, $t \in S$ where $\varepsilon \in (0,1], \Lambda, \psi$ are as in Theorem 2.1;

(*a*3) $A_1(B(S)) \subseteq B(S),\ A_2(B(S)) \subseteq B(S)$

(*a*4) *There exists some $A_i \in \{A_1, A_2\}$ such that for any sequence $\{h_n\}_{n\geq 1} \subseteq B(S)$ and $h \in B(S)$,*

$$\lim_{n\to\infty}\sup_{x\in S}|h_n(x) - h(x)| = 0 \Rightarrow \lim_{n\to\infty}\sup_{x\in S}|A_ih_n(x) - A_ih(x)| = 0,$$

Then the system of functional Equations (3.1) has a unique common solution in $B(S)$.

Proof. It follows from (*a*1) to (*a*4) that A_1 and A_2 is continuous self-mappings of $B(S)$. Let $\delta > 0$ be given. For any $g, h \in B(S)$, $x \in S$, there exist $y, z \in D$ such that

$$A_1g(x) < u(x,y) + H_1(x,y,g(T(x,y)) + \delta \tag{3.2}$$

$$A_2h(x) < u(x,z) + H_2(x,z,h(T(x,y)) + \delta \tag{3.3}$$

Not that

$$A_1g(x) \geq u(x,z) + H_1(x,z,g(T(x,z)) \tag{3.4}$$

$$A_2h(x) \geq u(x,y) + H_2(x,z,h(T(x,y)) \tag{3.5}$$

It follows from (3.2), (3.5) and (*a*2) that

$$\begin{aligned} A_1g(x) - A_2h(x) &< H_1(x,y,g(T(x,y)) - H_2(x,y,h(T(x,y)) + \delta \\ &\leq (1-\varepsilon)\max\{d(g,h), d(g,A_1g), d(h,A_2h), \frac{1}{2}[d(g,A_2h) + d(h,A_1g)]\} \\ &+ \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1+ \| g \| + \| h \||]^{\beta} + \delta. \end{aligned} \tag{3.6}$$

In view of (3.3), (3.4) and (*a*2) that

$$\begin{aligned} A_1g(x) - A_2h(x) &> H_1(x,z,g(T(x,z)) - H_2(x,z,h(T(x,z)) - \delta \\ &\geq -(1-\varepsilon)\max\{d(g,h), d(g,A_1g), d(h,A_2h), \frac{1}{2}[d(g,A_2h) - d(h,A_1g)]\} \\ &+ \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1+ \| g \| + \| h \||]^{\beta} - \delta \end{aligned} \tag{3.7}$$

(3.6) and (3.7) ensure that

$$\begin{aligned} d(A_1g(x) - A_2h(x)) &= \sup_{x\in S}|A_1g(x) - A_2h(x)| \\ &\le (1-\varepsilon)\max\{d(g,h), d(g,A_1g), d(h,A_2h), \frac{1}{2}[d(g,A_2h) + d(h,A_1g)]\} \\ &\quad + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1+\parallel g\parallel + \parallel h\parallel]^{\beta} + \delta \end{aligned} \tag{3.8}$$

It follows from (3.8) that Theorem 2.1 implies that A_1 and A_2 have a unique common fixed point $\nu \in B(S)$, that $\nu(x)$ is a unique common solution of the system of functional Equations (3.1).

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