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Hermite–Hadamard's inequality and Cauchy-type mean operators for positive C_0-semigroups

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Abstract: The Hermite–Hadamard's-type inequality is derived using superquadratic maps for positive operator semigroups. A methodical procedure was adopted to obtain the corresponding mean operators.

Subjects: Analysis - Mathematics; Operator Theory; Pure Mathematics; Foundations & Theorems

Keywords: Hermite–Hadamard's-type inequalities; positive semigroups; mean operators

1. Preambles

The idea of the main content of this note is not unusual, but is this decade's active research area (Bakry, 2004; Wang, 2005; Wang, 2013). In recent years, there has been considerable interest in the generalization of functional and type-inequalities to the operator semigroups. To get a common abstract setting within which ordering among the elements can be considered, a Banach lattice was defined. So that the idea related to positivity can be generalized.

Any (real) vector space $\mathfrak{E}$ is said to be an *ordered vector space*, if

A_1: $\mathfrak{f} \leq \mathfrak{g} \Rightarrow \mathfrak{f} + \mathfrak{h} \leq \mathfrak{g} + \mathfrak{h}$ for all $\mathfrak{f}, \mathfrak{g}, \mathfrak{h} \in \mathfrak{E}$,

A_2: $\mathfrak{f} \geq 0 \Rightarrow \alpha \mathfrak{f} \geq 0 \forall \mathfrak{f} \in \mathfrak{E}$ and $\alpha \geq 0$,

Note that, A_1, shows the translation invariance. Therefore, it indicates that the order among the elements of $\mathfrak{E}$ is completely established by the positive part $\mathfrak{E}_+ = \{\mathfrak{f} \in \mathfrak{E} : \mathfrak{f} \geq 0\}$ of $\mathfrak{E}$. Equivalently, $\mathfrak{f} \leq \mathfrak{g}$ if and only if $\mathfrak{g} - \mathfrak{f} \in \mathfrak{E}_+$. Moreover, A_2 indicates that $\mathfrak{E}_+$ is a convex set and a cone with vertex 0. If a "supremum" $\sup(\mathfrak{f}, \mathfrak{g})$ and thus an "infumum" $\inf(\mathfrak{f}, \mathfrak{g})$ for any two elements $\mathfrak{f}, \mathfrak{g} \in \mathfrak{E}$ can be specified, an ordered vector space $\mathfrak{E}$ is called a *vector lattice*. The compatibility axiom between norm and order is given briefly bellow:

$$|\mathfrak{f}| \leq |\mathfrak{g}| \quad \Rightarrow \quad \|\mathfrak{f}\| \leq \|\mathfrak{g}\|, \quad \mathfrak{f}, \mathfrak{g} \in \mathfrak{E}, \tag{1.1}$$

where $\sup(\mathfrak{f}, -\mathfrak{f}) = |\mathfrak{f}|$. This particular norm is called a lattice norm. Therefore, a vector lattice $(\mathfrak{E}, \leq)$ establishing a lattice norm is called *Banach lattice*. If a Banach lattice satisfies that,

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PUBLIC INTEREST STATEMENT

This research is devoted to Cauchy-type mean Operators defined on superquadratic mappings and operator semigroups. Gul I Hina Aslam and Matloob Anwar have been generalizing the theory of inequalities for operator semigroups. In this paper, authors continue the study for superquadratic mappings.

$\mathfrak{f}, \mathfrak{g} \in \mathfrak{E}_+ \quad \Rightarrow \quad \mathfrak{f}\mathfrak{g} \in \mathfrak{E}_+$, it is said to be a *Banach lattice algebra*. A linear mapping $\mathfrak{L}:\mathfrak{E} \to \mathfrak{E}$ is *positive* ($\mathfrak{L} \geq 0$) if $\mathfrak{L}(\mathfrak{f}) \in \mathfrak{E}_+$, for all $\mathfrak{f} \in \mathfrak{E}_+$. The set of all positive linear mappings forms a convex cone in the space $L(\mathfrak{E})$ and defines the natural ordering in it. The absolute value of $\mathfrak{L}$ is given bellow (if exists),

$$|\mathfrak{L}|(\mathfrak{f}) = \sup\{\mathfrak{L}(\mathfrak{g}):|\mathfrak{g}| \leq \mathfrak{f}\}, \quad (\mathfrak{f} \in \mathfrak{E}_+).$$

Therefore, $\mathfrak{L}:\mathfrak{E} \to \mathfrak{E}$ is positive iff $|\mathfrak{L}(\mathfrak{f})| \leq \mathfrak{L}(|\mathfrak{f}|)$ for any $\mathfrak{f} \in \mathfrak{E}$.

Definition 1.1 A family of bounded linear operators $\{\mathfrak{T}(s)\}_{s\geq 0}$ on a Banach space $\mathfrak{E}$ is called a (one parameter) C_0-semigroup (or strongly continuous semigroup), if it satisfies

(i) $\mathfrak{T}(s)\mathfrak{T}(t) = \mathfrak{T}(s+t)$ for all $s, t \in \mathbb{R}^+$.
(ii) $\mathfrak{T}(0) = I$

and is strongly continuous in the sense that for every $\mathfrak{f} \in \mathfrak{E}$ the orbit maps

$$:\zeta_{\mathfrak{f}}:s \to \zeta_{\mathfrak{f}}(s): = T(s)\mathfrak{f}$$

are continuous from $\mathbb{R}_+$ into $\mathfrak{E}$ for every $\mathfrak{f} \in \mathfrak{E}$.

The (infinitesimal) generator $A:\mathfrak{E} \supseteq D(A) \to R(A) \subseteq \mathfrak{E}$ of a strongly continuous semigroup is the densely defined closed linear operator

$$A\mathfrak{f} = \dot{\zeta}_{\mathfrak{f}}(0) = \lim_{h\to 0^+} \frac{1}{s}\left(\mathfrak{T}(s)\mathfrak{f} - \mathfrak{f}\right) \quad (\mathfrak{f} \in D(A))$$
$$D(A) = \left\{\mathfrak{f}: \lim_{s\to 0^+} \frac{1}{s}\left(\mathfrak{T}(s)\mathfrak{f} - \mathfrak{f}\right) \text{ exists in} \mathfrak{E}.\right\}$$

For a positive C_0-semigroup $\{\mathfrak{T}(s)\}_{s\geq 0}$, the positivity of the operators can be equivalent to:

$$|\mathfrak{T}(s)\mathfrak{f}| \leq \mathfrak{T}(s)|\mathfrak{f}|, \quad t \geq 0, \quad \mathfrak{f} \in \mathfrak{E}.$$

More details can be found in Nagel (1986). In accordance with the customary definition of power integral means, the power means for C_0-group of operators are defined.

Definition 1.2 (Aslam & Anwar, 2015a) Let $\{\mathfrak{T}(s)\}_{s\in\mathbb{R}}$ be the C_0-group of operators on a Banach space $\mathcal{X}$. The power mean is given as

$$\mathcal{M}_r(\mathfrak{T}, \mathfrak{f}, s) = \begin{cases} \left\{\frac{1}{s}\int_0^s [\mathfrak{T}(\varepsilon)]^r \mathfrak{f} d\varepsilon\right\}^{1/r}, & r \neq 0 \\ exp[\frac{1}{s}\int_0^s ln[\mathfrak{T}(\varepsilon)]\mathfrak{f} d\varepsilon], & r = 0. \end{cases} \tag{1.2}$$

where $\mathfrak{f} \in \mathcal{X}$ and $s \in \mathbb{R}$.

2. Main Results

In Banić and Varošanec (2008), the Hermite–Hadamard's-type inequality for positive linear functionals is derived. Few mean-value theorems that ultimately lead to new means of Cauchy type are given in Abramovich et al. (2010).

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In this note, we generalize the Hermite–Hadamard's-type inequality for positive C_0-semigroup. We also give some generalized mean value theorems and define related mean operators.

Throughout the section, $\mathfrak{E}$ denotes the real Banach lattice algebra, until and unless stated otherwise.

Definition 2.1 (Aslam & Anwar, 2015b) An operator $\psi:\mathfrak{E}_+ \to \mathfrak{E}$ is superquadratic, given that for all $\mathfrak{g}_1 \geq 0$ there exists a constant vector $C(\mathfrak{g}_1)$ such that

$$\psi(\mathfrak{f}_1) - \psi(\mathfrak{g}_1) - \psi(|\mathfrak{f}_1 - \mathfrak{g}_1|) \geq C(\mathfrak{g}_1)(\mathfrak{f}_1 - \mathfrak{g}_1) \tag{2.1}$$

for all $\mathfrak{f}_1 \geq 0$.

Theorem 2.2 Let $\{\mathfrak{T}(s)\}_{s\geq 0}$ be a positive C_0-semigroup on $\mathfrak{E}$; then for an integrable superquadratic operator $\psi:\mathfrak{E}_+ \to \mathfrak{E}$, we have

$$\psi\left[\frac{1}{s}\int_0^s [\mathfrak{T}(\varepsilon)]\mathfrak{f}d\varepsilon\right] + \frac{1}{s}\int_0^s \psi\left[\left|[\mathfrak{T}(\varepsilon)]f - \frac{1}{s}\int_0^s [\mathfrak{T}(\varepsilon)]\mathfrak{f}d\varepsilon\right|\right]d\varepsilon \leq \frac{1}{s}\int_0^s \psi[\mathfrak{T}(\varepsilon)]\mathfrak{f}d\varepsilon, \quad f \in \mathfrak{E}_+. \tag{2.2}$$

Proof Let ψ be a superquadratic mapping, then (2.1) holds for all $\mathfrak{f}, \mathfrak{g} \in \mathfrak{E}_+$. Choosing $\mathfrak{f}_1 = [\mathfrak{T}(\varepsilon)]\mathfrak{f}$ and $\mathfrak{g}_1 = \frac{1}{s}\int_0^s [\mathfrak{T}(\varepsilon)]\mathfrak{f}d\varepsilon$ in (2.1) we get

$$\begin{aligned}\psi[[\mathfrak{T}(\varepsilon)]\mathfrak{f}] \geq{}& \psi\left[\frac{1}{s}\int_0^s [\mathfrak{T}(\varepsilon)]\mathfrak{f}d\varepsilon\right] + C\left[\frac{1}{s}\int_0^s [\mathfrak{T}(\varepsilon)]\mathfrak{f}d\varepsilon\right]\left[[\mathfrak{T}(\varepsilon)]\mathfrak{f} - \frac{1}{s}\int_0^s [\mathfrak{T}(\varepsilon)]\mathfrak{f}d\tau\right] \\ &+ \psi\left[\left|[\mathfrak{T}(\varepsilon)]\mathfrak{f} - \frac{1}{t}\int_0^s [\mathfrak{T}(\varepsilon)]\mathfrak{f}d\varepsilon\right|\right]\end{aligned}$$

By integrating from $0 \to s$, we obtain,

$$\begin{aligned}\int_0^s \psi[[\mathfrak{T}(\varepsilon)]\mathfrak{f}]d\varepsilon \geq{}& t.\psi\left[\frac{1}{s}\int_0^s [\mathfrak{T}(\varepsilon)]\mathfrak{f}d\varepsilon\right] + C\left[\frac{1}{s}\int_0^s [\mathfrak{T}(\varepsilon)]\mathfrak{f}d\varepsilon\right]\left[\int_0^s [\mathfrak{T}(\varepsilon)]\mathfrak{f}d\varepsilon - t\left\{\frac{1}{s}\int_0^s [\mathfrak{T}(\varepsilon)]\mathfrak{f}d\varepsilon\right\}\right] \\ &+ \int_0^s \psi\left[\left|[\mathfrak{T}(\varepsilon)]\mathfrak{f} - \frac{1}{s}\int_0^s [\mathfrak{T}(\varepsilon)]\mathfrak{f}d\varepsilon\right|\right]d\varepsilon,\end{aligned}$$

or

$$\int_0^s \psi[[\mathfrak{T}(\varepsilon)]\mathfrak{f}]d\varepsilon \geq s.\psi\left[\frac{1}{s}\int_0^s [\mathfrak{T}(\varepsilon)]\mathfrak{f}d\varepsilon\right] + \int_0^s \psi\left[\left|[\mathfrak{T}(\varepsilon)]\mathfrak{f} - \frac{1}{s}\int_0^s [\mathfrak{T}(\varepsilon)]\mathfrak{f}d\varepsilon\right|\right]d\varepsilon.$$

By multiplying 1/s, we finally get the assertion (2.2).

Definition 2.3 Let $\{\mathfrak{T}(s)\}_{s\geq 0}$ be a positive C_0-semigroup of operators defined on $\mathfrak{E}$; then for an integrable operator $\psi:\mathfrak{E}_+ \to \mathfrak{E}$, we define another operator $\Lambda_\psi:\mathfrak{E}_+ \to \mathfrak{E}$

$$\Lambda_\psi := \frac{1}{s}\int_0^s \psi[\mathfrak{T}(\varepsilon)]\mathfrak{f}d\varepsilon - \psi\left[\frac{1}{s}\int_0^s [\mathfrak{T}(\varepsilon)]\mathfrak{f}d\varepsilon\right] - \frac{1}{s}\int_0^s \psi\left[\left|[\mathfrak{T}(\varepsilon)]\mathfrak{f} - \frac{1}{s}\int_0^s [\mathfrak{T}(\varepsilon)]\mathfrak{f}d\varepsilon\right|\right]d\varepsilon, \quad \mathfrak{f} \in \mathfrak{E}_+. \tag{2.3}$$

If ψ is continuous superquadratic mapping, then by (2.2), $\Lambda_\psi \geq 0$. □

To simplify expressions, we replace $\frac{1}{s}\int_0^s [\mathfrak{T}(\varepsilon)]\mathfrak{f}d\varepsilon$ by $\mathcal{M}_1(s)$. Therefore, Λ_ψ can be written as

$$\Lambda_\psi := \frac{1}{s}\int_0^s \psi[\mathfrak{T}(\varepsilon)]\mathfrak{f}d\varepsilon - \psi[\mathcal{M}_1(s)] - \frac{1}{s}\int_0^s \psi\left[\left|[\mathfrak{T}(\varepsilon)]\mathfrak{f} - \mathcal{M}_1(s)\right|\right]d\tau$$

The operator analogue of Abramovich et al. (2004, Lemma 3.1) is given in Aslam and Anwar (2015b).

LEMMA 2.4 *Let $\psi:\mathfrak{E}_+ \to \mathfrak{E}$ be continuously differentiable and $\psi(0) \le 0$. If ψ' is super-additive or $\mathfrak{f} \to \frac{\psi'(\mathfrak{g})}{\mathfrak{g}}$, $\mathfrak{g} \in \mathfrak{E}_+$, is increasing, then ψ is superquadratic.* □

LEMMA 2.5 *Let $\psi \in C^2[\mathfrak{E}_+]$ and $u, U \in \mathfrak{E}$ be such that*

$$u \le \left(\frac{\psi'(\mathfrak{g})}{\mathfrak{g}}\right)' = \frac{\mathfrak{f}\psi''(\mathfrak{g}) - \psi'(\mathfrak{g})}{\mathfrak{g}^2} \le U, \quad \text{for all } \mathfrak{g} > 0. \tag{2.4}$$

Consider the operators $\psi_1, \psi_2:\mathfrak{E}_+ \to \mathfrak{E}$ defined as:

$$\psi_1(\mathfrak{g}) = \frac{U\mathfrak{g}^3}{3} - \psi(\mathfrak{g}), \quad \psi_2 = \psi(\mathfrak{g}) - \frac{u\mathfrak{g}^3}{3}$$

Then, the mappings $\mathfrak{g} \to \frac{\psi_1'(\mathfrak{g})}{\mathfrak{g}}$ and $\mathfrak{g} \to \frac{\psi_2'(\mathfrak{g})}{\mathfrak{g}}$ are increasing. If also $\psi_i(0) = 0, i = 1, 2$, then these are superquadratic mappings.

Proof Using (2.4), it can be noted that the mappings $\mathfrak{g} \to \frac{\psi_1'(\mathfrak{g})}{\mathfrak{g}}$ and $\mathfrak{g} \to \frac{\psi_2'(\mathfrak{g})}{\mathfrak{g}}$ are increasing. Moreover, if $\psi_i(0) = 0, i = 1, 2$, Lemma (2.4) implies these to be superquadratic. □

Theorem 2.6Let $\{\mathfrak{T}(s)\}_{s\ge 0}$ be a positive C_0-semigroup on $\mathfrak{E}$ and $\frac{\psi'}{\mathfrak{f}} \in C^1(\mathfrak{E}_+)$ and $\psi(0) = 0$, then the following inequality holds

$$\Lambda_\psi = \frac{\rho\psi''(\rho) - \psi'(\rho)}{3\rho^2}\{(\mathcal{M}_3(s))^3 - (\mathcal{M}_1(s))^3 - \frac{1}{s}\int_0^s |\mathfrak{T}(\varepsilon)\mathfrak{f} - m_s|^3 d\varepsilon\} \tag{2.5}$$

Here, m_s denotes the arithmetic mean of $\{\mathfrak{T}(s)\}_{s\ge 0}$.

Proof Suppose the conditions in Lemma 2.5 hold for all $\mathfrak{f} \in \mathfrak{E}_+$. Using ψ_1 instead of ψ in (2.2), we get

$$\frac{1}{s}\int_0^s \psi[\mathfrak{T}(\varepsilon)]\mathfrak{f}d\varepsilon - \psi[\mathcal{M}_1(s)] - \frac{1}{s}\int_0^s \psi\left[\left|[\mathfrak{T}(\varepsilon)]\mathfrak{f} - \mathcal{M}_1(s)\right|\right]d\varepsilon \le \frac{U}{3}\{(\mathcal{M}_3(s))^3 - (\mathcal{M}_1(s))^3 - \frac{1}{s}\int_0^s |\mathfrak{T}(\varepsilon)\mathfrak{f} - m_s|^3 d\varepsilon\}$$

Similarly, using ψ_2 instead of ψ in (2.2), we get

$$\frac{1}{s}\int_0^s \psi[\mathfrak{T}(\varepsilon)]\mathfrak{f}d\varepsilon - \psi[\mathcal{M}_1(s)] - \frac{1}{s}\int_0^s \psi\left[\left|[\mathfrak{T}(\varepsilon)]\mathfrak{f} - \mathcal{M}_1(s)\right|\right]d\varepsilon \ge \frac{u}{3}\{(\mathcal{M}_3(s))^3 - (\mathcal{M}_1(s))^3 - \frac{1}{s}\int_0^s |\mathfrak{T}(\varepsilon)\mathfrak{f} - m_s|^3 d\varepsilon\}$$

By combining the above two inequalities and using intermediate value theorem Ali (1997), we have existence of $\rho \in \mathfrak{E}_+$ such that (2.5) holds. □Theorem 2.7Let $\{\mathfrak{T}(s)\}_{s\ge 0}$ be a positive C_0-semigroup on $\mathfrak{E}$ and $\frac{\psi'}{\mathfrak{f}}, \frac{\phi'}{\mathfrak{f}} \in C^1(\mathfrak{E}_+)$ such that, $\psi(0) = \phi(0) = 0$, we have

$$\frac{\Lambda_\psi}{\Lambda_\phi} = \frac{\rho\psi''(\rho) - \psi'(\rho)}{\xi\boldsymbol{\phi}''(\rho) - \boldsymbol{\rho}'(\rho)} = \mathbf{F}(\rho), \quad \rho \in \mathfrak{E}_+, \tag{2.6}$$

given the denominator is not zero. If $\mathbf{F}^{-1}$ exists, then

$$\rho = \mathbf{F}^{-1}\left(\frac{\Lambda_\psi}{\Lambda_\phi}\right), \quad \Lambda_\phi \ne 0, \tag{2.7}$$

Proof Consider a function $\Omega = c_1\psi - c_2\phi$, where

$$c_1 = \Lambda_\phi, \quad c_2 = \Lambda_\psi.$$

Then, for $\mathfrak{f} \in \mathfrak{E}_+$

$$\frac{\Omega'}{\mathfrak{f}} = c_1\frac{\psi'}{\mathfrak{f}} - c_2\frac{\phi'}{\mathfrak{f}} \in C^1(\mathfrak{E}_+).$$

we can calculate that $\Lambda_\Omega = 0$ and by Lemma (2.5) with $\psi = \Omega$ we have

$$[c_1(\rho\psi''(\rho) - \psi'(\rho)) - c_2(\rho\phi''(\rho) - \phi'(\rho))]\left[(\mathcal{M}_3(t))^3 - (\mathcal{M}_1(t))^3 - \frac{1}{s}\int_0^s |\mathfrak{T}(\tau)\mathfrak{f} - m_t|^3 d\tau\right] = 0, \quad \mathfrak{f} \in \mathfrak{E}_+.$$

Since $\psi = \mathfrak{f}^3$ is superquadratic mapping and $\{\mathfrak{T}(s)\}_{s\geq 0}$ is positive, we have

$$\frac{c_2}{c_1} = \frac{\rho\psi''(\rho) - \psi'(\rho)}{\rho\phi''(\rho) - \phi'(\rho)} = \frac{\Lambda_\psi}{\Lambda_\phi}, \quad \rho \in \mathfrak{E}_+,$$

given the denominator is not zero, proof follows. □

Let $\mathcal{G}$ be the set of all invertible bounded linear operators $H{:}\mathfrak{E} \to \mathfrak{E}$. For a C_0-semigroup of positive operators $\{\mathfrak{T}(s)\}_{s\geq 0} \subset B(\mathfrak{E})$ defined on $\mathfrak{E}$ and $H \in \mathcal{G}$, the quasi-arithmetic mean is given as Aslam and Anwar (2015a)

$$\mathcal{M}_H^\circ(\mathfrak{T}, \mathfrak{f}, s) = H^{-1}\left\{\frac{1}{s}\int_0^s H[\mathfrak{T}(\varepsilon)\mathfrak{f}]d\varepsilon\right\}, \quad \mathfrak{f} \in \mathfrak{E}_+, s \geq 0. \tag{2.8}$$

By Belleni-Morante and McBride (1998, Lemma 1.85), $B(\mathfrak{E})$ is closed under composition of operators, therefore the above expressions exist and $\mathcal{M}_H^\circ(\mathfrak{T}, \mathfrak{f}, s) \in \mathfrak{E}$. To avoid complexity among the expressions, let

$$C^2\mathcal{G}(\mathfrak{E}) = \{H{:}H \in \mathcal{G}, \quad H'' \text{ exists in Gateaux sense}\}.$$

Theorem 2.8 Let $\{\mathfrak{T}(s)\}_{s\geq 0}$ be a positive C_0-semigroup of operators defined on $\mathfrak{E}$ and $\mathbf{H}, \mathbf{F}, \mathbf{K} \in C^2\mathcal{G}(\mathfrak{E})$. Let for $\mathfrak{f} \in \mathfrak{E}_+, \frac{\mathbf{H}\circ\mathbf{F}^{-1}(\mathfrak{f})}{\mathfrak{f}}, \frac{\mathbf{K}\circ\mathbf{F}^{-1}(\mathfrak{f})}{\mathfrak{f}} \in C^1(\mathfrak{E})$ with $\mathbf{H}\circ\mathbf{F}^{-1}(0) = 0 = \mathbf{K}\circ\mathbf{F}^{-1}(0)$, then for $\mathfrak{f} \in \mathfrak{E}_+$
holds for some $\xi \in \mathfrak{E}_+$, provided the denominator does not vanish.

$$\begin{aligned}&\frac{\mathbf{H}(\mathcal{M}_\mathbf{H}^\circ(\mathfrak{T}, \mathfrak{f}, s)) - \mathbf{H}(\mathcal{M}_\mathbf{F}^\circ(\mathfrak{T}, \mathfrak{f}, s)) - \mathbf{H}(\mathcal{M}_\mathbf{H}^\circ(\mathbf{F}^{-1}|\mathbf{F}[\mathfrak{T}(\varepsilon)\mathfrak{f}] - \mathbf{F}\mathcal{M}_\mathbf{F}^\circ(\mathfrak{T}, \mathfrak{f}, s)|, \mathfrak{f}, s))}{\mathbf{K}(\mathcal{M}_\mathbf{H}^\circ(\mathfrak{T}, \mathfrak{f}, s)) - \mathbf{K}(\mathcal{M}_\mathbf{F}^\circ(\mathfrak{T}, \mathfrak{f}, s)) - \mathbf{K}(\mathcal{M}_\mathbf{K}^\circ(\mathbf{F}^{-1}|\mathbf{F}[\mathfrak{T}(\varepsilon)\mathfrak{f}] - \mathbf{F}\mathcal{M}_\mathbf{F}^\circ(\mathfrak{T}, \mathfrak{f}, s)|, \mathfrak{f}, s))}\\ &\quad = \frac{\mathbf{F}(\xi)\{\mathbf{H}''(\xi)\mathbf{F}'(\xi) - \mathbf{H}'(\xi)\mathbf{F}''(\xi) - \mathbf{H}'(\xi)[\mathbf{F}'(\xi)]^2\}}{\mathbf{F}(\xi)\{\mathbf{K}''(\xi)\mathbf{F}'(\xi) - \mathbf{K}'(\xi)\mathbf{F}''(\xi) - \mathbf{K}'(\xi)[\mathbf{F}'(\xi)]^2\}},\end{aligned} \tag{2.9}$$

Proof By setting the operators ψ and ϕ in Theorem 2.7, such that

$$\psi = \mathbf{H}\circ\mathbf{F}^{-1}, \quad \phi = \mathbf{K}\circ\mathbf{F}^{-1} \quad \textit{and} \quad \mathfrak{T}(\varepsilon)\mathfrak{f} = \mathbf{F}[\mathfrak{T}(\varepsilon)\mathfrak{f}], \quad \mathfrak{f} \in \mathfrak{E}_+,$$

where $\mathbf{H}, \mathbf{F}, \mathbf{K} \in C^2\mathcal{G}(\mathfrak{E})$. We find that there exists $\rho \in \mathfrak{E}_+$, such that

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$$\frac{\mathbf{H}(\mathcal{M}^{\circ}_{\mathbf{H}}(\mathfrak{T},\mathfrak{f},s))-\mathbf{H}(\mathcal{M}^{\circ}_{\mathbf{F}}(\mathfrak{T},\mathfrak{f},s))-\mathbf{H}(\mathcal{M}^{\circ}_{\mathbf{H}}(\mathbf{F}^{-1}|\mathbf{F}[\mathfrak{T}(\varepsilon)\mathfrak{f}]-\mathbf{F}\mathcal{M}^{\circ}_{\mathbf{F}}(\mathfrak{T},\mathfrak{f},s)|,\mathfrak{f},s))}{\mathbf{K}(\mathcal{M}^{\circ}_{\mathbf{H}}(\mathfrak{T},\mathfrak{f},s))-\mathbf{K}(\mathcal{M}^{\circ}_{\mathbf{F}}(\mathfrak{T},\mathfrak{f},s))-\mathbf{K}(\mathcal{M}^{\circ}_{\mathbf{K}}(\mathbf{F}^{-1}|\mathbf{F}[\mathfrak{T}(\varepsilon)\mathfrak{f}]-\mathbf{F}\mathcal{M}^{\circ}_{\mathbf{F}}(\mathfrak{T},\mathfrak{f},s)|,\mathfrak{f},s))}$$
$$=\frac{\rho\{\mathbf{H}''(\mathbf{F}^{-1}\rho)\mathbf{F}'(\mathbf{F}^{-1}\rho)-\mathbf{H}'(\mathbf{F}^{-1}\rho)\mathbf{F}''(\mathbf{F}^{-1}\rho)-\mathbf{H}'(\mathbf{F}^{-1}\rho)[\mathbf{F}'(\mathbf{F}^{-1}\rho)]^2\}}{\rho\left\{\mathbf{K}''(\mathbf{F}^{-1}\rho)\mathbf{F}'(\mathbf{F}^{-1}\rho)-\mathbf{K}'(\mathbf{F}^{-1}\rho)\mathbf{F}''(\mathbf{F}^{-1}\rho)-\mathbf{K}'(\mathbf{F}^{-1}\rho)[\mathbf{F}'(\mathbf{F}^{-1}\rho)]^2\right\}},$$

Hence, by putting $\mathbf{F}^{-1}(\rho)=\omega$ for some $\mu\in\mathfrak{E}$, the assertion (2.9) follows directly. □

The above theorem enables us to introduce new means. Set

$$\Gamma(\omega)=\frac{\mathbf{F}(\omega)\{\mathbf{H}''(\omega)\mathbf{F}'(\omega)-\mathbf{H}'(\omega)\mathbf{F}''(\omega)-\mathbf{H}'(\omega)[\mathbf{F}'(\omega)]^2\}}{\mathbf{F}(\omega)\{\mathbf{K}''(\omega)\mathbf{F}'(\omega)-\mathbf{K}'(\omega)\mathbf{F}''(\omega)-\mathbf{K}'(\omega)[\mathbf{F}'(\omega)]^2\}},$$

and when $\mathbf{F}\in\mathcal{G}(\mathfrak{E})$

$$\omega=\Gamma^{-1}\left(\frac{\mathbf{H}(\mathcal{M}^{\circ}_{\mathbf{H}}(\mathfrak{T},\mathfrak{f},s))-\mathbf{H}(\mathcal{M}^{\circ}_{\mathbf{F}}(\mathfrak{T},\mathfrak{f},s))-\mathbf{H}(\mathcal{M}^{\circ}_{\mathbf{H}}(\psi^{-1}|\mathbf{F}[\mathfrak{T}(\varepsilon)\mathfrak{f}]-\mathbf{F}\mathcal{M}^{\circ}_{\mathbf{F}}(\mathfrak{T},\mathfrak{f},s)|,\mathfrak{f},s))}{\mathbf{K}(\mathcal{M}^{\circ}_{\mathbf{H}}(\mathfrak{T},\mathfrak{f},s))-\mathbf{K}(\mathcal{M}^{\circ}_{\mathbf{F}}(\mathfrak{T},\mathfrak{f},s))-\mathbf{K}(\mathcal{M}^{\circ}_{\mathbf{K}}(\mathbf{F}^{-1}|\mathbf{F}[\mathfrak{T}(\varepsilon)\mathfrak{f}]-\mathbf{F}\mathcal{M}^{\circ}_{\mathbf{F}}(\mathfrak{T},\mathfrak{f},s)|,\mathfrak{f},s))}\right)$$

Remark 2.9 For a Banach lattice algebra $(\mathfrak{E},\|.\|)$, from Theorem 2.8 we have that

$$m\le\left\|\frac{\mathbf{H}(\mathcal{M}^{\circ}_{\mathbf{H}}(\mathfrak{T},\mathfrak{f},s))-\mathbf{H}(\mathcal{M}^{\circ}_{\mathbf{F}}(\mathfrak{T},\mathfrak{f},s))-\mathbf{H}(\mathcal{M}^{\circ}_{\mathbf{H}}(\mathbf{F}^{-1}|\mathbf{F}[\mathfrak{T}(\varepsilon)\mathfrak{f}]-\mathbf{F}\mathcal{M}^{\circ}_{\mathbf{F}}(\mathfrak{T},\mathfrak{f},s)|,\mathfrak{f},s))}{\mathbf{K}(\mathcal{M}^{\circ}_{\mathbf{H}}(\mathfrak{T},\mathfrak{f},s))-\mathbf{K}(\mathcal{M}^{\circ}_{\mathbf{F}}(\mathfrak{T},\mathfrak{f},s))-\mathbf{K}(\mathcal{M}^{\circ}_{\mathbf{K}}(\mathbf{F}^{-1}|\mathbf{F}[\mathfrak{T}(\varepsilon)\mathfrak{f}]-\mathbf{F}\mathcal{M}^{\circ}_{\mathbf{F}}(\mathfrak{T},\mathfrak{f},s)|,\mathfrak{f},s))}\right\|\le M,$$

where m and M are, respectively, the minimum and maximum values of

$$\left\|\frac{\mathbf{F}(\omega)\{\mathbf{H}''(\omega)\mathbf{F}'(\omega)-\mathbf{H}'(\omega)\mathbf{F}''(\omega)-\mathbf{H}'(\omega)[\mathbf{F}'(\omega)]^2\}}{\mathbf{F}(\omega)\{\mathbf{K}''(\omega)\mathbf{F}'(\omega)-\mathbf{K}'(\omega)\mathbf{F}''(\omega)-\mathbf{K}'(\omega)[\mathbf{F}'(\omega)]^2\}}\right\|,\quad \omega\in\mathfrak{E}.$$

□

Next we give a significant result which leads us to define the Cauchy-type mean operators on C_0-group of operators.

COROLLARY 2.10 *Let all the conditions of Theorem 2.8 be satisfied. For* $r,n,l\in\mathbb{R}_+$ *such that* $r\neq l;l\neq 2s$, *we have*

$$\frac{\mathcal{M}^r_r(\mathfrak{T},\mathfrak{f},s)-\mathcal{M}^r_n(\mathfrak{T},\mathfrak{f},s)-\mathcal{M}^r_r(|[\mathfrak{T}(\tau)\mathfrak{f}]^n-\mathcal{M}^n_n(\mathfrak{T},\mathfrak{f},s)|^{\frac{1}{s}},\mathfrak{f},s)}{\mathcal{M}^l_l(\mathfrak{T},\mathfrak{f},s)-\mathcal{M}^l_n(\mathfrak{T},\mathfrak{f},s)-\mathcal{M}^l_l(|[\mathfrak{T}(\varepsilon)\mathfrak{f}]^n-\mathcal{M}^n_n(\mathfrak{T},\mathfrak{f},s)|^{\frac{1}{s}},\mathfrak{f},s)}=\frac{r(r-2n)}{l(l-2n)}\omega^{r-l}\tag{2.10}$$

where $\mathcal{M}_r(\mathfrak{T},\mathfrak{f},s)$ is defined by (1.2). The assertion (2.10) holds for some μ, provided that the denominators do not vanish.

Proof For $r,n,l\in\mathbb{R}_+$ and $\mathfrak{f}\in\mathfrak{E}_+$, if we set

$$\mathbf{H}(\mathfrak{f})=\mathfrak{f}^r,\quad \mathbf{F}(\mathfrak{f})=\mathfrak{f}^n,\quad \mathbf{K}(\mathfrak{f})=\mathfrak{f}^l$$

in Theorem (2.8), the assertion in (2.10) follows directly.

Finally, we are able to define mean operators of the Cauchy type on positive C_0-semigroup on Banach lattice algebra $\mathfrak{E}$.

Definition 2.11 Let $r, n, l \in \mathbb{R}_+$ and $\{\mathfrak{T}(s)\}_{s \geq 0} \subset B(\mathfrak{E})$ be a positive C_0-semigroup on a Banach lattice algebra $\mathfrak{E}$. Then,

$$\mathfrak{M}_r^{l,n}(\mathfrak{T}, \mathfrak{f}, s) = \left(\frac{l(l-2n)}{r(r-2n)} \frac{\mathcal{M}_r^r(\mathfrak{T}, \mathfrak{f}, s) - \mathcal{M}_n^r(\mathfrak{T}, \mathfrak{f}, s) - \mathcal{M}_r^r(|[\mathfrak{T}(\varepsilon)\mathfrak{f}]^n - \mathcal{M}_n^n(\mathfrak{T}, \mathfrak{f}, s)|^{\frac{1}{s}}, \mathfrak{f}, s)}{\mathcal{M}_l^l(\mathfrak{T}, \mathfrak{f}, s) - \mathcal{M}_n^l(\mathfrak{T}, \mathfrak{f}, s) - \mathcal{M}_l^l(|[\mathfrak{T}(\varepsilon)\mathfrak{f}]^n - \mathcal{M}_n^n(\mathfrak{T}, \mathfrak{f}, s)|^{\frac{1}{s}}, \mathfrak{f}, s)} \right)^{\frac{1}{r-l}}. \tag{2.11}$$

is a family of mean operators of the Cauchy type on C_0-semigroup of positive operators. This definition is true for all $r \neq l \neq n \neq 0$ and other cases can be taken as limiting cases.

3. Conclusion

In this note, a Hermite-Hadamard type inequality has been proved for a positive C_0-semigroup and a superquadratic mapping defined on a Banach lattice algebra. A methodic way has been adopted to prove the corresponding mean value theorems, which enabled us to define a new set of mean operators. These mean operators are inspired from Cauchy-type mean.

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