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PURE MATHEMATICS | RESEARCH ARTICLE

Hermite–Hadamard’s inequality and Cauchy-type mean operators for positive C_0 -semigroups

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Abstract: The Hermite–Hadamard’s-type inequality is derived using superquadratic maps for positive operator semigroups. A methodical procedure was adopted to obtain the corresponding mean operators.

Subjects: Analysis - Mathematics; Operator Theory; Pure Mathematics; Foundations & Theorems

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1. Preambles

The idea of the main content of this note is not unusual, but is this decade’s active research area (Bakry, 2004; Wang, 2005; Wang, 2013). In recent years, there has been considerable interest in the generalization of functional and type-inequalities to the operator semigroups. To get a common abstract setting within which ordering among the elements can be considered, a Banach lattice was defined. So that the idea related to positivity can be generalized.

Any (real) vector space \mathfrak{G} is said to be an *ordered vector space*, if

$$A_1: f \leq g \Rightarrow f + h \leq g + h \text{ for all } f, g, h \in \mathfrak{G},$$

$$A_2: f \geq 0 \Rightarrow \alpha f \geq 0 \forall f \in \mathfrak{G} \text{ and } \alpha \geq 0,$$

Note that, A_1 , shows the translation invariance. Therefore, it indicates that the order among the elements of \mathfrak{G} is completely established by the positive part $\mathfrak{G}_+ = \{f \in \mathfrak{G}; f \geq 0\}$ of \mathfrak{G} . Equivalently, $f \leq g$ if and only if $g - f \in \mathfrak{G}_+$. Moreover, A_2 indicates that \mathfrak{G}_+ is a convex set and a cone with vertex 0. If a “supremum” $\sup(f, g)$ and thus an “infimum” $\inf(f, g)$ for any two elements $f, g \in \mathfrak{G}$ can be specified, an ordered vector space \mathfrak{G} is called a *vector lattice*. The compatibility axiom between norm and order is given briefly bellow:

$$|f| \leq |g| \Rightarrow \|f\| \leq \|g\|, \quad f, g \in \mathfrak{G}, \quad (1.1)$$

where $\sup(f, -f) = |f|$. This particular norm is called a lattice norm. Therefore, a vector lattice (\mathfrak{G}, \leq) establishing a lattice norm is called *Banach lattice*. If a Banach lattice satisfies that,

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PUBLIC INTEREST STATEMENT

This research is devoted to Cauchy-type mean Operators defined on superquadratic mappings and operator semigroups. Gul I Hina Aslam and Matloob Anwar have been generalizing the theory of inequalities for operator semigroups. In this paper, authors continue the study for superquadratic mappings.

$f, g \in \mathfrak{G}_+ \Rightarrow fg \in \mathfrak{G}_+$, it is said to be a *Banach lattice algebra*. A linear mapping $\mathfrak{A}: \mathfrak{G} \rightarrow \mathfrak{G}$ is *positive* ($\mathfrak{A} \geq 0$) if $\mathfrak{A}(f) \in \mathfrak{G}_+$, for all $f \in \mathfrak{G}_+$. The set of all positive linear mappings forms a convex cone in the space $L(\mathfrak{G})$ and defines the natural ordering in it. The absolute value of \mathfrak{A} is given below (if exists),

$$|\mathfrak{A}|(f) = \sup\{\mathfrak{A}(g): |g| \leq f\}, \quad (f \in \mathfrak{G}_+).$$

Therefore, $\mathfrak{A}: \mathfrak{G} \rightarrow \mathfrak{G}$ is positive iff $|\mathfrak{A}(f)| \leq \mathfrak{A}(|f|)$ for any $f \in \mathfrak{G}$.

Definition 1.1 A family of bounded linear operators $\{\mathfrak{A}(s)\}_{s \geq 0}$ on a Banach space \mathfrak{G} is called a (one parameter) C_0 -semigroup (or strongly continuous semigroup), if it satisfies

- (i) $\mathfrak{A}(s)\mathfrak{A}(t) = \mathfrak{A}(s+t)$ for all $s, t \in \mathbb{R}^+$.
- (ii) $\mathfrak{A}(0) = I$

and is strongly continuous in the sense that for every $f \in \mathfrak{G}$ the orbit maps

$$\zeta_f: s \rightarrow \zeta_f(s) = T(s)f$$

are continuous from \mathbb{R}_+ into \mathfrak{G} for every $f \in \mathfrak{G}$.

The (infinitesimal) generator $A: \mathfrak{G} \supseteq D(A) \rightarrow R(A) \subseteq \mathfrak{G}$ of a strongly continuous semigroup is the densely defined closed linear operator

$$Af = \zeta_f'(0) = \lim_{h \rightarrow 0^+} \frac{1}{h} (\mathfrak{A}(h)f - f) \quad (f \in D(A))$$

$$D(A) = \left\{ f: \lim_{s \rightarrow 0^+} \frac{1}{s} (\mathfrak{A}(s)f - f) \text{ exists in } \mathfrak{G}. \right\}$$

For a positive C_0 -semigroup $\{\mathfrak{A}(s)\}_{s \geq 0}$, the positivity of the operators can be equivalent to:

$$|\mathfrak{A}(s)f| \leq \mathfrak{A}(s)|f|, \quad t \geq 0, \quad f \in \mathfrak{G}.$$

More details can be found in Nagel (1986). In accordance with the customary definition of power integral means, the power means for C_0 -group of operators are defined.

Definition 1.2 (Aslam & Anwar, 2015a) Let $\{\mathfrak{A}(s)\}_{s \in \mathbb{R}}$ be the C_0 -group of operators on a Banach space \mathfrak{X} . The power mean is given as

$$\mathcal{M}_r(\mathfrak{A}, f, s) = \begin{cases} \left\{ \frac{1}{s} \int_0^s [\mathfrak{A}(\varepsilon)]^r f d\varepsilon \right\}^{1/r}, & r \neq 0 \\ \exp\left[\frac{1}{s} \int_0^s \ln[\mathfrak{A}(\varepsilon)] f d\varepsilon\right], & r = 0. \end{cases} \quad (1.2)$$

where $f \in \mathfrak{X}$ and $s \in \mathbb{R}$.

2. Main Results

In Banić and Varošanec (2008), the Hermite–Hadamard’s-type inequality for positive linear functionals is derived. Few mean-value theorems that ultimately lead to new means of Cauchy type are given in Abramovich et al. (2010).

In this note, we generalize the Hermite–Hadamard’s-type inequality for positive C_0 -semigroup. We also give some generalized mean value theorems and define related mean operators.

Throughout the section, \mathfrak{G} denotes the real Banach lattice algebra, until and unless stated otherwise.

Definition 2.1 (Aslam & Anwar, 2015b) An operator $\psi: \mathfrak{G}_+ \rightarrow \mathfrak{G}$ is superquadratic, given that for all $g_1 \geq 0$ there exists a constant vector $C(g_1)$ such that

$$\psi(f_1) - \psi(g_1) - \psi(|f_1 - g_1|) \geq C(g_1)(f_1 - g_1) \tag{2.1}$$

for all $f_1 \geq 0$.

Theorem 2.2 Let $\{\mathfrak{T}(s)\}_{s \geq 0}$ be a positive C_0 -semigroup on \mathfrak{G} ; then for an integrable superquadratic operator $\psi: \mathfrak{G}_+ \rightarrow \mathfrak{G}$, we have

$$\psi\left[\frac{1}{s} \int_0^s [\mathfrak{T}(\epsilon)]f d\epsilon\right] + \frac{1}{s} \int_0^s \psi\left[\left|[\mathfrak{T}(\epsilon)]f - \frac{1}{s} \int_0^s [\mathfrak{T}(\epsilon)]f d\epsilon\right|\right] d\epsilon \leq \frac{1}{s} \int_0^s \psi[\mathfrak{T}(\epsilon)]f d\epsilon, \quad f \in \mathfrak{G}_+. \tag{2.2}$$

Proof Let ψ be a superquadratic mapping, then (2.1) holds for all $f, g \in \mathfrak{G}_+$. Choosing $f_1 = [\mathfrak{T}(\epsilon)]f$ and $g_1 = \frac{1}{s} \int_0^s [\mathfrak{T}(\epsilon)]f d\epsilon$ in (2.1) we get

$$\begin{aligned} \psi([\mathfrak{T}(\epsilon)]f) \geq & \psi\left[\frac{1}{s} \int_0^s [\mathfrak{T}(\epsilon)]f d\epsilon\right] + C\left[\frac{1}{s} \int_0^s [\mathfrak{T}(\epsilon)]f d\epsilon\right] \left[[\mathfrak{T}(\epsilon)]f - \frac{1}{s} \int_0^s [\mathfrak{T}(\epsilon)]f d\tau \right] \\ & + \psi\left[\left| [\mathfrak{T}(\epsilon)]f - \frac{1}{s} \int_0^s [\mathfrak{T}(\epsilon)]f d\epsilon \right|\right] \end{aligned}$$

By integrating from $0 \rightarrow s$, we obtain,

$$\begin{aligned} \int_0^s \psi([\mathfrak{T}(\epsilon)]f) d\epsilon \geq & t \cdot \psi\left[\frac{1}{s} \int_0^s [\mathfrak{T}(\epsilon)]f d\epsilon\right] + C\left[\frac{1}{s} \int_0^s [\mathfrak{T}(\epsilon)]f d\epsilon\right] \left[\int_0^s [\mathfrak{T}(\epsilon)]f d\epsilon - t \left\{ \frac{1}{s} \int_0^s [\mathfrak{T}(\epsilon)]f d\epsilon \right\} \right] \\ & + \int_0^s \psi\left[\left| [\mathfrak{T}(\epsilon)]f - \frac{1}{s} \int_0^s [\mathfrak{T}(\epsilon)]f d\epsilon \right|\right] d\epsilon, \end{aligned}$$

or

$$\int_0^s \psi([\mathfrak{T}(\epsilon)]f) d\epsilon \geq s \cdot \psi\left[\frac{1}{s} \int_0^s [\mathfrak{T}(\epsilon)]f d\epsilon\right] + \int_0^s \psi\left[\left| [\mathfrak{T}(\epsilon)]f - \frac{1}{s} \int_0^s [\mathfrak{T}(\epsilon)]f d\epsilon \right|\right] d\epsilon.$$

By multiplying $1/s$, we finally get the assertion (2.2).

Definition 2.3 Let $\{\mathfrak{T}(s)\}_{s \geq 0}$ be a positive C_0 -semigroup of operators defined on \mathfrak{G} ; then for an integrable operator $\psi: \mathfrak{G}_+ \rightarrow \mathfrak{G}$, we define another operator $\Lambda_\psi: \mathfrak{G}_+ \rightarrow \mathfrak{G}$

$$\Lambda_\psi := \frac{1}{s} \int_0^s \psi[\mathfrak{T}(\epsilon)]f d\epsilon - \psi\left[\frac{1}{s} \int_0^s [\mathfrak{T}(\epsilon)]f d\epsilon\right] - \frac{1}{s} \int_0^s \psi\left[\left| [\mathfrak{T}(\epsilon)]f - \frac{1}{s} \int_0^s [\mathfrak{T}(\epsilon)]f d\epsilon \right|\right] d\epsilon, \quad f \in \mathfrak{G}_+. \tag{2.3}$$

If ψ is continuous superquadratic mapping, then by (2.2), $\Lambda_\psi \geq 0$. □

To simplify expressions, we replace $\frac{1}{s} \int_0^s [\mathfrak{T}(\epsilon)]f d\epsilon$ by $\mathcal{M}_1(s)$. Therefore, Λ_ψ can be written as

$$\Lambda_\psi := \frac{1}{s} \int_0^s \psi[\mathfrak{T}(\epsilon)]f d\epsilon - \psi[\mathcal{M}_1(s)] - \frac{1}{s} \int_0^s \psi\left[\left| [\mathfrak{T}(\epsilon)]f - \mathcal{M}_1(s) \right|\right] d\tau$$

The operator analogue of Abramovich et al. (2004, Lemma 3.1) is given in Aslam and Anwar (2015b).

LEMMA 2.4 Let $\psi: \mathbb{G}_+ \rightarrow \mathbb{G}$ be continuously differentiable and $\psi(0) \leq 0$. If ψ' is super-additive or $f \rightarrow \frac{\psi'(g)}{g}$, $g \in \mathbb{G}_+$, is increasing, then ψ is superquadratic. \square

LEMMA 2.5 Let $\psi \in C^2[\mathbb{G}_+]$ and $u, U \in \mathbb{G}$ be such that

$$u \leq \left(\frac{\psi'(g)}{g} \right)' = \frac{f\psi''(g) - \psi'(g)}{g^2} \leq U, \quad \text{for all } g > 0. \quad (2.4)$$

Consider the operators $\psi_1, \psi_2: \mathbb{G}_+ \rightarrow \mathbb{G}$ defined as:

$$\psi_1(g) = \frac{Ug^3}{3} - \psi(g), \quad \psi_2 = \psi(g) - \frac{ug^3}{3}$$

Then, the mappings $g \rightarrow \frac{\psi_1'(g)}{g}$ and $g \rightarrow \frac{\psi_2'(g)}{g}$ are increasing. If also $\psi_i(0) = 0, i = 1, 2$, then these are superquadratic mappings.

Proof Using (2.4), it can be noted that the mappings $g \rightarrow \frac{\psi_1'(g)}{g}$ and $g \rightarrow \frac{\psi_2'(g)}{g}$ are increasing. Moreover, if $\psi_i(0) = 0, i = 1, 2$, Lemma (2.4) implies these to be superquadratic. \square

Theorem 2.6 Let $\{\mathfrak{A}(s)\}_{s \geq 0}$ be a positive C_0 -semigroup on \mathbb{G} and $\frac{\psi'}{f} \in C^1(\mathbb{G}_+)$ and $\psi(0) = 0$, then the following inequality holds

$$\Lambda_\psi = \frac{\rho\psi''(\rho) - \psi'(\rho)}{3\rho^2} \{(\mathcal{M}_3(s))^3 - (\mathcal{M}_1(s))^3\} - \frac{1}{s} \int_0^s |\mathfrak{A}(\epsilon)f - m_s|^3 d\epsilon \quad (2.5)$$

Here, m_s denotes the arithmetic mean of $\{\mathfrak{A}(s)\}_{s \geq 0}$.

Proof Suppose the conditions in Lemma 2.5 hold for all $f \in \mathbb{G}_+$. Using ψ_1 instead of ψ in (2.2), we get

$$\frac{1}{s} \int_0^s \psi[\mathfrak{A}(\epsilon)]f d\epsilon - \psi[\mathcal{M}_1(s)] - \frac{1}{s} \int_0^s \psi \left[\left| [\mathfrak{A}(\epsilon)]f - \mathcal{M}_1(s) \right| \right] d\epsilon \leq \frac{U}{3} \{(\mathcal{M}_3(s))^3 - (\mathcal{M}_1(s))^3\} - \frac{1}{s} \int_0^s |\mathfrak{A}(\epsilon)f - m_s|^3 d\epsilon$$

Similarly, using ψ_2 instead of ψ in (2.2), we get

$$\frac{1}{s} \int_0^s \psi[\mathfrak{A}(\epsilon)]f d\epsilon - \psi[\mathcal{M}_1(s)] - \frac{1}{s} \int_0^s \psi \left[\left| [\mathfrak{A}(\epsilon)]f - \mathcal{M}_1(s) \right| \right] d\epsilon \geq \frac{u}{3} \{(\mathcal{M}_3(s))^3 - (\mathcal{M}_1(s))^3\} - \frac{1}{s} \int_0^s |\mathfrak{A}(\epsilon)f - m_s|^3 d\epsilon$$

By combining the above two inequalities and using intermediate value theorem Ali (1997), we have existence of $\rho \in \mathbb{G}_+$ such that (2.5) holds. \square **Theorem 2.7** Let $\{\mathfrak{A}(s)\}_{s \geq 0}$ be a positive C_0 -semigroup on \mathbb{G} and $\frac{\psi'}{f}, \frac{\phi'}{f} \in C^1(\mathbb{G}_+)$ such that, $\psi(0) = \phi(0) = 0$, we have

$$\frac{\Lambda_\psi}{\Lambda_\phi} = \frac{\rho\psi''(\rho) - \psi'(\rho)}{\xi\phi''(\rho) - \phi'(\rho)} = \mathbf{F}(\rho), \quad \rho \in \mathbb{G}_+, \quad (2.6)$$

given the denominator is not zero. If \mathbf{F}^{-1} exists, then

$$\rho = \mathbf{F}^{-1} \left(\frac{\Lambda_\psi}{\Lambda_\phi} \right), \quad \Lambda_\phi \neq 0, \quad (2.7)$$

Proof Consider a function $\Omega = c_1\psi - c_2\phi$, where

$$c_1 = \Lambda_\phi, \quad c_2 = \Lambda_\psi.$$

Then, for $f \in \mathbb{G}_+$

$$\frac{\Omega'}{f} = c_1 \frac{\psi'}{f} - c_2 \frac{\phi'}{f} \in C^1(\mathbb{G}_+).$$

we can calculate that $\Lambda_\Omega = 0$ and by Lemma (2.5) with $\psi = \Omega$ we have

$$[c_1(\rho\psi''(\rho) - \psi'(\rho)) - c_2(\rho\phi''(\rho) - \phi'(\rho))] \left[(\mathcal{M}_3(t))^3 - (\mathcal{M}_1(t))^3 - \frac{1}{s} \int_0^s |\mathfrak{A}(\tau)f - m_t|^3 d\tau \right] = 0, \quad f \in \mathbb{G}_+.$$

Since $\psi = f^3$ is superquadratic mapping and $\{\mathfrak{A}(s)\}_{s \geq 0}$ is positive, we have

$$\frac{c_2}{c_1} = \frac{\rho\psi''(\rho) - \psi'(\rho)}{\rho\phi''(\rho) - \phi'(\rho)} = \frac{\Lambda_\psi}{\Lambda_\phi}, \quad \rho \in \mathbb{G}_+,$$

given the denominator is not zero, proof follows. □

Let \mathcal{G} be the set of all invertible bounded linear operators $H: \mathbb{G} \rightarrow \mathbb{G}$. For a C_0 -semigroup of positive operators $\{\mathfrak{A}(s)\}_{s \geq 0} \subset B(\mathbb{G})$ defined on \mathbb{G} and $H \in \mathcal{G}$, the quasi-arithmetic mean is given as Aslam and Anwar (2015a)

$$\mathcal{M}_H^\circ(\mathfrak{A}, f, s) = H^{-1} \left\{ \frac{1}{s} \int_0^s H[\mathfrak{A}(\varepsilon)f] d\varepsilon \right\}, \quad f \in \mathbb{G}_+, s \geq 0. \tag{2.8}$$

By Bellini-Morante and McBride (1998, Lemma 1.85), $B(\mathbb{G})$ is closed under composition of operators, therefore the above expressions exist and $\mathcal{M}_H^\circ(\mathfrak{A}, f, s) \in \mathbb{G}$. To avoid complexity among the expressions, let

$$C^2\mathcal{G}(\mathbb{G}) = \{H: H \in \mathcal{G}, \quad H'' \text{ exists in Gateaux sense}\}.$$

Theorem 2.8 Let $\{\mathfrak{A}(s)\}_{s \geq 0}$ be a positive C_0 -semigroup of operators defined on \mathbb{G} and $H, F, K \in C^2\mathcal{G}(\mathbb{G})$. Let for $f \in \mathbb{G}_+$, $\frac{H \circ F^{-1}(f)}{f}, \frac{K \circ F^{-1}(f)}{f} \in C^1(\mathbb{G})$ with $H \circ F^{-1}(0) = 0 = K \circ F^{-1}(0)$, then for $f \in \mathbb{G}_+$ holds for some $\xi \in \mathbb{G}_+$, provided the denominator does not vanish.

$$\begin{aligned} & \frac{H(\mathcal{M}_H^\circ(\mathfrak{A}, f, s)) - H(\mathcal{M}_F^\circ(\mathfrak{A}, f, s)) - H(\mathcal{M}_H^\circ(F^{-1}[F[\mathfrak{A}(\varepsilon)f] - F\mathcal{M}_F^\circ(\mathfrak{A}, f, s)], f, s))}{K(\mathcal{M}_H^\circ(\mathfrak{A}, f, s)) - K(\mathcal{M}_F^\circ(\mathfrak{A}, f, s)) - K(\mathcal{M}_K^\circ(F^{-1}[F[\mathfrak{A}(\varepsilon)f] - F\mathcal{M}_F^\circ(\mathfrak{A}, f, s)], f, s))} \\ &= \frac{F(\xi)\{H''(\xi)F'(\xi) - H'(\xi)F''(\xi) - H'(\xi)[F'(\xi)]^2\}}{F(\xi)\{K''(\xi)F'(\xi) - K'(\xi)F''(\xi) - K'(\xi)[F'(\xi)]^2\}} \end{aligned} \tag{2.9}$$

Proof By setting the operators ψ and ϕ in Theorem 2.7, such that

$$\psi = H \circ F^{-1}, \quad \phi = K \circ F^{-1} \quad \text{and} \quad \mathfrak{A}(\varepsilon)f = F[\mathfrak{A}(\varepsilon)f], \quad f \in \mathbb{G}_+,$$

where $H, F, K \in C^2\mathcal{G}(\mathbb{G})$. We find that there exists $\rho \in \mathbb{G}_+$, such that

$$\begin{aligned} & \frac{\mathbf{H}(\mathcal{M}_H^\circ(\mathfrak{A}, \mathfrak{f}, s)) - \mathbf{H}(\mathcal{M}_F^\circ(\mathfrak{A}, \mathfrak{f}, s)) - \mathbf{H}(\mathcal{M}_H^\circ(\mathbf{F}^{-1}|\mathbf{F}[\mathfrak{A}(\varepsilon)\mathfrak{f}] - \mathbf{F}\mathcal{M}_F^\circ(\mathfrak{A}, \mathfrak{f}, s)|, \mathfrak{f}, s))}{\mathbf{K}(\mathcal{M}_H^\circ(\mathfrak{A}, \mathfrak{f}, s)) - \mathbf{K}(\mathcal{M}_F^\circ(\mathfrak{A}, \mathfrak{f}, s)) - \mathbf{K}(\mathcal{M}_K^\circ(\mathbf{F}^{-1}|\mathbf{F}[\mathfrak{A}(\varepsilon)\mathfrak{f}] - \mathbf{F}\mathcal{M}_F^\circ(\mathfrak{A}, \mathfrak{f}, s)|, \mathfrak{f}, s))} \\ &= \frac{\rho\{\mathbf{H}''(\mathbf{F}^{-1}\rho)\mathbf{F}'(\mathbf{F}^{-1}\rho) - \mathbf{H}'(\mathbf{F}^{-1}\rho)\mathbf{F}''(\mathbf{F}^{-1}\rho) - \mathbf{H}'(\mathbf{F}^{-1}\rho)[\mathbf{F}'(\mathbf{F}^{-1}\rho)]^2\}}{\rho\{\mathbf{K}''(\mathbf{F}^{-1}\rho)\mathbf{F}'(\mathbf{F}^{-1}\rho) - \mathbf{K}'(\mathbf{F}^{-1}\rho)\mathbf{F}''(\mathbf{F}^{-1}\rho) - \mathbf{K}'(\mathbf{F}^{-1}\rho)[\mathbf{F}'(\mathbf{F}^{-1}\rho)]^2\}}, \end{aligned}$$

Hence, by putting $\mathbf{F}^{-1}(\rho) = \omega$ for some $\mu \in \mathfrak{G}$, the assertion (2.9) follows directly. □

The above theorem enables us to introduce new means. Set

$$\Gamma(\omega) = \frac{\mathbf{F}(\omega)\{\mathbf{H}''(\omega)\mathbf{F}'(\omega) - \mathbf{H}'(\omega)\mathbf{F}''(\omega) - \mathbf{H}'(\omega)[\mathbf{F}'(\omega)]^2\}}{\mathbf{F}(\omega)\{\mathbf{K}''(\omega)\mathbf{F}'(\omega) - \mathbf{K}'(\omega)\mathbf{F}''(\omega) - \mathbf{K}'(\omega)[\mathbf{F}'(\omega)]^2\}},$$

and when $\mathbf{F} \in \mathcal{G}(\mathfrak{G})$

$$\omega = \Gamma^{-1}\left(\frac{\mathbf{H}(\mathcal{M}_H^\circ(\mathfrak{A}, \mathfrak{f}, s)) - \mathbf{H}(\mathcal{M}_F^\circ(\mathfrak{A}, \mathfrak{f}, s)) - \mathbf{H}(\mathcal{M}_H^\circ(\psi^{-1}|\mathbf{F}[\mathfrak{A}(\varepsilon)\mathfrak{f}] - \mathbf{F}\mathcal{M}_F^\circ(\mathfrak{A}, \mathfrak{f}, s)|, \mathfrak{f}, s))}{\mathbf{K}(\mathcal{M}_H^\circ(\mathfrak{A}, \mathfrak{f}, s)) - \mathbf{K}(\mathcal{M}_F^\circ(\mathfrak{A}, \mathfrak{f}, s)) - \mathbf{K}(\mathcal{M}_K^\circ(\mathbf{F}^{-1}|\mathbf{F}[\mathfrak{A}(\varepsilon)\mathfrak{f}] - \mathbf{F}\mathcal{M}_F^\circ(\mathfrak{A}, \mathfrak{f}, s)|, \mathfrak{f}, s))}\right)$$

Remark 2.9 For a Banach lattice algebra $(\mathfrak{G}, \|\cdot\|)$, from Theorem 2.8 we have that

$$m \leq \left\| \frac{\mathbf{H}(\mathcal{M}_H^\circ(\mathfrak{A}, \mathfrak{f}, s)) - \mathbf{H}(\mathcal{M}_F^\circ(\mathfrak{A}, \mathfrak{f}, s)) - \mathbf{H}(\mathcal{M}_H^\circ(\mathbf{F}^{-1}|\mathbf{F}[\mathfrak{A}(\varepsilon)\mathfrak{f}] - \mathbf{F}\mathcal{M}_F^\circ(\mathfrak{A}, \mathfrak{f}, s)|, \mathfrak{f}, s))}{\mathbf{K}(\mathcal{M}_H^\circ(\mathfrak{A}, \mathfrak{f}, s)) - \mathbf{K}(\mathcal{M}_F^\circ(\mathfrak{A}, \mathfrak{f}, s)) - \mathbf{K}(\mathcal{M}_K^\circ(\mathbf{F}^{-1}|\mathbf{F}[\mathfrak{A}(\varepsilon)\mathfrak{f}] - \mathbf{F}\mathcal{M}_F^\circ(\mathfrak{A}, \mathfrak{f}, s)|, \mathfrak{f}, s))} \right\| \leq M,$$

where m and M are, respectively, the minimum and maximum values of

$$\left\| \frac{\mathbf{F}(\omega)\{\mathbf{H}''(\omega)\mathbf{F}'(\omega) - \mathbf{H}'(\omega)\mathbf{F}''(\omega) - \mathbf{H}'(\omega)[\mathbf{F}'(\omega)]^2\}}{\mathbf{F}(\omega)\{\mathbf{K}''(\omega)\mathbf{F}'(\omega) - \mathbf{K}'(\omega)\mathbf{F}''(\omega) - \mathbf{K}'(\omega)[\mathbf{F}'(\omega)]^2\}} \right\|, \quad \omega \in \mathfrak{G}.$$

□

Next we give a significant result which leads us to define the Cauchy-type mean operators on C_0 -group of operators.

COROLLARY 2.10 *Let all the conditions of Theorem 2.8 be satisfied. For $r, n, l \in \mathbb{R}_+$ such that $r \neq l; l \neq 2s$, we have*

$$\frac{\mathcal{M}_r^l(\mathfrak{A}, \mathfrak{f}, s) - \mathcal{M}_n^l(\mathfrak{A}, \mathfrak{f}, s) - \mathcal{M}_l^r(|[\mathfrak{A}(\tau)\mathfrak{f}]^n - \mathcal{M}_n^n(\mathfrak{A}, \mathfrak{f}, s)|^{\frac{1}{2}}, \mathfrak{f}, s)}{\mathcal{M}_l^l(\mathfrak{A}, \mathfrak{f}, s) - \mathcal{M}_n^l(\mathfrak{A}, \mathfrak{f}, s) - \mathcal{M}_l^l(|[\mathfrak{A}(\varepsilon)\mathfrak{f}]^n - \mathcal{M}_n^n(\mathfrak{A}, \mathfrak{f}, s)|^{\frac{1}{2}}, \mathfrak{f}, s)} = \frac{r(r-2n)}{l(l-2n)} \omega^{r-l} \tag{2.10}$$

where $\mathcal{M}_l(\mathfrak{A}, \mathfrak{f}, s)$ is defined by (1.2). The assertion (2.10) holds for some μ , provided that the denominators do not vanish.

Proof For $r, n, l \in \mathbb{R}_+$ and $\mathfrak{f} \in \mathfrak{G}_+$, if we set

$$\mathbf{H}(\mathfrak{f}) = \mathfrak{f}^r, \quad \mathbf{F}(\mathfrak{f}) = \mathfrak{f}^n, \quad \mathbf{K}(\mathfrak{f}) = \mathfrak{f}^l$$

in Theorem (2.8), the assertion in (2.10) follows directly.

Finally, we are able to define mean operators of the Cauchy type on positive C_0 -semigroup on Banach lattice algebra \mathfrak{G} .

Definition 2.11 Let $r, n, l \in \mathbb{R}_+$ and $\{\mathfrak{A}(s)\}_{s \geq 0} \subset B(\mathfrak{E})$ be a positive C_0 -semigroup on a Banach lattice algebra \mathfrak{E} . Then,

$$\mathfrak{M}_r^{l,n}(\mathfrak{A}, \mathfrak{f}, s) = \left(\frac{l(l-2n) \mathcal{M}_r^l(\mathfrak{A}, \mathfrak{f}, s) - \mathcal{M}_r^n(\mathfrak{A}, \mathfrak{f}, s) - \mathcal{M}_r^l(|[\mathfrak{A}(\varepsilon)\mathfrak{f}]^n - \mathcal{M}_r^n(\mathfrak{A}, \mathfrak{f}, s)|^{\frac{1}{2}}, \mathfrak{f}, s)}{r(r-2n) \mathcal{M}_r^l(\mathfrak{A}, \mathfrak{f}, s) - \mathcal{M}_r^n(\mathfrak{A}, \mathfrak{f}, s) - \mathcal{M}_r^l(|[\mathfrak{A}(\varepsilon)\mathfrak{f}]^n - \mathcal{M}_r^n(\mathfrak{A}, \mathfrak{f}, s)|^{\frac{1}{2}}, \mathfrak{f}, s)} \right)^{\frac{1}{r-1}}. \quad (2.11)$$

is a family of mean operators of the Cauchy type on C_0 -semigroup of positive operators. This definition is true for all $r \neq l \neq n \neq 0$ and other cases can be taken as limiting cases.

3. Conclusion

In this note, a Hermite-Hadamard type inequality has been proved for a positive C_0 -semigroup and a superquadratic mapping defined on a Banach lattice algebra. A methodic way has been adopted to prove the corresponding mean value theorems, which enabled us to define a new set of mean operators. These mean operators are inspired from Cauchy-type mean.

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