On the positivity, excitability and transparency properties of a class of time-varying bilinear dynamic systems under multiple point internal and external delays

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Abstract: This paper investigates formally the internal and external positivity, together with its excitability and transparency properties, of a class of bilinear time-varying continuous-time dynamic systems subject to (in general, non-commensurate) multiple internal and external point delays. The evolution operator is calculated in a closed form and the mentioned properties can be checked through direct testable expressions. The bilinear system class under consideration is driven by two inputs, so-called, the control input and the bilinear action input, which are not necessarily coincident, the second one taking account of the coupling state-input defining the bilinear terms.

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PUBLIC INTEREST STATEMENT

Positive systems are characterized by the property that the state and output components are non-negative for all time under non-negative inputs and non-negative initial conditions. In the context of differential, difference or hybrid equations, the solution is non-negative for non-negative forcing terms. Positivity is a relevant property inherent to biological and epidemic models and also to predator-prey-related problems, population evolution models, etc., often combined with the presence of delays either in the dynamics and/ or in the forcing terms or controls. On the other hand, the bilinear dynamic systems are a special class of non-linear systems where the non-linearity consists of products between state and input components. An example is the regulation of electrical motors when neither the current nor the voltage are constant for all time so that the electric torque has a bilinear nature. It seems important to mention that positivity is a property related to the time behavior of the various signals rather than to the frequency responses.
1. Introduction


Two classical books or recommended reading for those interested in the subject are Kaczorek (2001) and Farina and Rinaldi (2000) while more recent extensions have been discussed in Mailleret (2004). Bilinear systems are a special class of non-linear systems where the non-linearity consists of products between state and input components. In particular, the positivity theory has been extended from the continuous-time and discrete-time linear cases to mathematical models including hybrid continuous-time and discrete-time differential equations and hybrid dynamic systems (De la Sen, 2007) as well as to those eventually possessing internal (i.e. in the state) and external (i.e. in the inputs and/or outputs) delays. See, for instance, De la Sen (2007), De la Sen et al. (2011), Kao (2014), Mailleret (2004), Nickel and Rhandi (2005), Sau et al. (2016), de la Sen (2008b), Shen and Zheng (2015), Tingting et al. (2015), Zames (1966), and De la Sen and Ibeas (2008) and references therein. In particular, stability results for positive systems are given in De la Sen (2007), Kao (2014), Nickel and Rhandi (2005), Sau et al. (2016), Shen and Zheng (2015), Tingting et al. (2015) and Zames (1966), including the problem of hybrid systems with mixed coupled continuous-time and either discrete-time (arising from continuous-time discretization) or digital (arising from purely digital unrelated-to-continuous) dynamics, and some references therein while classical stability results are detailed in Kaczorek (2001).

It is convenient to point out so as to avoiding potential confusion that positivity is a property related to the time-behavior of the various signals (input, state, and output components) through time (De La Sen, 2002), rather than to the respective frequency. Two properties of interest in positive systems are the “excitability” and the “transparency”. The first one is related to the capability of all the state components to reach positive values in finite time from injection of some positive control with the system initially at rest. The second one is related to the capability of all the output components of the homogeneous system to reach positive values in finite time for any given positive initial condition. Another research subject of increasing interest is that of time-delay differential/difference systems and related dynamic systems. The main motivation is that the subjects related to time-delayed dynamics are of interest to mathematicians since such a dynamics is described by functionnal equations of difficult analysis while they are also of relevant interest to engineers since a number of physical models have inherent delays (sunflower daily motion equation, ship maneuver dynamics, war-peace models, transmission signal problems, biological problems, etc.). In particular, time-delayed systems with internal delays, i.e. those being present in the state-dynamics) possess infinitely many poles. The related background literature is exhaustive. See, for instance, Al-Omari (2015), De la Sen (2004), Ebihara (2015), Haldar et al. (2015), McCluskey et al. (2015), Yi, Ulsoy, and Nelson (2006), Domoshnitsky and Volinsky (2015) and De la Sen and Ibeas (2008) and references therein. It turns out that some systems which are positive, by nature, have also delayed dynamics. See, for instance, Al-Omari (2015), Haldar et al. (2015), and McCluskey et al. (2015). On the other hand, sums and differences of positive operators on separate Hilbert spaces have been investigated in Kittaneh (2004) through the derivation of norm inequalities.

This paper investigates formally the properties of positivity, excitability and transparency of a class of generalized bilinear dynamic systems with multiple internal delays (i.e. in the state dynamics) and two inputs, namely, the control input and the, so-called, bilinear input action which is coupled to the state implying the presence of bilinear terms in the dynamics. In some particular cases of interest in applications, the bilinear action can be identical to the control action itself (Feng, Chen, Sun, & Zhu,
2005; Goka, Tarn, & Zaborszky, 1973; Liao, Cannon, & Kouvaritakis, 2005; Tarn, 1972), as it is the case of regulation of electrical machines without maintaining either the supply voltage or the armature current being constant so that the electrical torque becomes bilinear by nature. This paper considers the case of a bilinear, in general, time-varying, dynamic system with both mentioned inputs being distinct, in general, and the dynamic system being positive while being subject to multiple internal delays. Section 2 investigates the positivity (or internal positivity, that is, that of the state dynamics) conditions as well as the external positivity (that is, the positivity of the output). Section 3 is devoted to the investigation and the formal characterization of the excitability (in the standard, strong and external senses) and the transparency properties. Finally, some auxiliary lemmas are given in Appendix A.

1.1. Notation

\[ R_{0+} = R_+ \cup \{0\}; \quad R_- = \{z \in R : z > 0\}, \]

\[ Z_{0+} = Z_+ \cup \{0\}; \quad Z_- = \{z \in Z : z > 0\}, \]

\[ R_{0-} = R_- \cup \{0\}; \quad R_+ = \{z \in R : z < 0\}, \]

\[ Z_{0-} = Z_- \cup \{0\}; \quad Z_+ = \{z \in Z : z < 0\}, \]

\[ q = \{1, 2, \ldots, q\}, \]

The superscript “\(T\)” denotes transposition,

\[ e_i \] is the unity vector of the Euclidean natural basis whose \(i\)th component is unity,

\[ M_i^T = e_i^T M \] and \( M_j = e_j^T M \) are the \(i\)th row and \(j\)th column of a real matrix \( M = \left( M_{ij} \right) \) of compatible,

If \( x, y \in R_{0+}^m \), respectively, \( x = \left( x_{ij} \right), y = \left( y_{ij} \right) \in R_{0+}^m \) then:

\( x \geq 0 \) (\( x \) is non-negative) means that \( x_i \geq 0 \), respectively, \( x_i \geq 0 ; \forall i, j \in \mathbb{N} \),

\( x > 0 \) (\( x \) is positive) means that \( x \neq 0 \) and \( x_i \geq 0 \), respectively, \( r x_i \geq 0 ; \forall i, j \in \mathbb{N} \),

\( x \succ 0 \) (\( x \) is strictly positive) means that \( x_i > 0 \), respectively, \( x_i > 0 ; \forall i, j \in \mathbb{N} \), that is, \( x \in R_+^m \),

\( x \succ 0 \) means \( x > 0 \) with non-strict positivity; i.e. there is some zero entry or component,

\( x \preceq 0 \iff (\neg x) \succeq 0 ; x < 0 \iff (\neg x) > 0 ; x \ll 0 \iff (\neg x) \gg 0 \),

\( x \preceq y \) \textcolor{red}{(or} \( y \preceq x \) \textcolor{red}{)} \iff \( x - y \succeq 0 ; x > y \) \textcolor{red}{(or} \( y \prec x \) \textcolor{red}{)} \iff \( x - y > 0 ; x \succ y \) \textcolor{red}{(or} \( y \ll x \) \textcolor{red}{)} \iff \( x - y \gg 0 \),

\textbf{Notation remark:} It has to be pointed out that \( x \) non-negative \((x \succeq 0)\), \( x \) positive \((x > 0)\) and \( x \) strictly positive \((x \gg 0)\) are referred to in some background literature, respectively, as positive, strictly positive and strongly positive. See, for instance, Mailleret (2004).

\[ M_0^n = \left\{ X = \left( X_{ij} \right) \in R_+^{n \times n} : \forall i, j(\neq i) \in \mathbb{N} \right\} \] is the set of Metzler matrices of \( n \)th order,

\( I_n \) is the \( n \)th identity matrix,

\[ C_q^n (\text{Dom} ; \text{Ran}) \] is the set of \( n \)th vector functions of class \( q \) of domain \( \text{Dom} \) and range \( \text{Ran} \),

\[ PC_q^n (\text{Dom} ; \text{Ran}) \] is the set of \( n \)th vector functions of class \( q \) of domain \( \text{Dom} \) and range \( \text{Ran} \), whose \( q \)th derivative (or the function itself if \( q = 0 \)) is everywhere piecewise continuous,
2. Problem formulation: Dynamic bilinear system with delays and positivity conditions

Consider the bilinear time-varying system:

\[ \dot{x}(t) = \sum_{j=0}^{q} \left( A_j(t) + v_j(t)E_j(t) \right) x(t-h_j) + \sum_{j=0}^{q} B_j(t)u(t-h_j) \tag{1} \]

\[ y(t) = C(t)x(t) + D(t)u(t); \quad \forall t \in R_{0+}, \tag{2} \]

where \( x(t) \in R^n, u(t) \in R^m \) and \( y(t) \in R^p \) are the state, piecewise-continuous control input and output vectors, respectively, and \( v_j(t) \in R^q \) are the bilinear piecewise-continuous action inputs; \( \forall j \in q \cup \{ 0 \} \).

\[ A_j: R_{0+} \to R^{n \times m}; \quad \forall j \in q \cup \{ 0 \}, \quad B_j: R_{0+} \to R^{n \times m}; \quad \forall j \in q \cup \{ 0 \} \]

\[ E_j: R_{0+} \to R^q; \quad \forall j \in q \cup \{ 0 \}, \quad C: R_{0+} \to R^n, \quad D: R_{0+} \to R^p \]

are piecewise-continuous matrices of bounded entries, where \( q = \{ 1, 2, \ldots, q \} \) and \( q = \{ 1, 2, \ldots, q \} \), \( h_j, h' \in q \cup \{ 0 \} \) are internal and external point delays, subject to \( h_j \leq h' \leq h; \quad \forall j \in q \cup \{ 0 \} \) and \( h_j \leq h' \leq h; \quad \forall j \in q \cup \{ 0 \} \) with \( h_0 = h'_0 = 0, h = h = h' \) and \( h_j = h' \) for the initial conditions of (1) are defined by \( x(t) \equiv \varphi(t) \), where \( \varphi: [-h, 0) \to R^n \) is piecewise continuous with \( \varphi(0) = x(0) = x_0 \) and \( u(-t) = 0, \quad v_j(-t) = 0; \quad \forall j \in q \cup \{ 0 \}, \quad \forall t \in R_{0+} \).

A system of similar structure of (1)-(2) parameterized by \( A_{j}^{(G)}(t); \quad \forall i \in q \cup \{ 0 \}, \quad E_{j}^{(G)}(t), \quad B_{j}^{(G)}(t); \quad \forall j \in q \cup \{ 0 \}, \quad C^{(G)}(t) \) and \( D^{(G)}(t); \quad \forall t \in R_{0+} \), whose entries are zero if the corresponding ones of (1)-(2) are zero and unity otherwise is said to be the associated system to (1)-(2). The inputs \( u^{(G)}(t) \) and \( v_j^{(G)}(t) \) are defined with unity and zero components, if the corresponding ones of \( u(t) \) and \( v_j(t) \) are non-zero or zero, respectively, for each given \( t \in R_{+} \). The above matrices are said to be the associated matrices to the corresponding ones of (1)-(2).

THEOREM 1. Assume that \( A_j(t) = A_{0j} + \hat{A}_j(t_h) \), where \( A_{0j} \) is constant; \( \forall j \in q \cup \{ 0 \} \). Then, the following properties hold:

(i) The solution of (1) is given explicitly by:

\[ x(t) = \psi(t)x_0 + \sum_{j=0}^{q} \int_{h_j}^{t} \psi(t-\sigma)A_{0j}\varphi(\sigma)d\sigma + \int_{0}^{t} \psi(t-\sigma)\left[ \sum_{j=0}^{q} \left( \hat{A}_j(\sigma) + v_j(\sigma)E_j(\sigma) \right) x \left( \sigma-h_j \right) \right] d\sigma \]

\[ + \sum_{j=0}^{q} \int_{0}^{t} \psi(t-\sigma)B_{j}(\sigma)u \left( \sigma-h_j \right) d\sigma; \quad \forall t \in R_{+} \tag{3} \]

and \( x(t) = \varphi(t) \) for \( t \in [-h, 0] \), which is unique in \( R_{+} \cup [-h, 0] \) and continuously differentiable in \( R_{+} \), for each given piecewise continuous vector function \( \varphi: [-h, 0] \to R^n \) continuous at \( t = 0 \) of initial conditions with \( x_0 = x(0) = \varphi(0) \) and for each given piecewise-continuous inputs \( u: R_{+} \cup [-h', 0] \to R^m \), with \( u(-t) = 0; \quad \forall t \in R_{0+} \), and \( v_j: R_{0+} \to R^q; \quad \forall j \in q \cup \{ 0 \} \) such that the fundamental matrix \( \psi: [-h, 0] \cup R_{+} \to R^{n \times n} \) satisfies the homogeneous auxiliary time-invariant delayed system:

\[ \psi(t) = \sum_{j=0}^{q} A_{0j} \psi(t-h_j)U(t-h_j); \quad \forall t \in R_{+} \tag{4} \]

where \( u(t) \) is the Heaviside function.
(ii) The fundamental matrix satisfying (4) is continuously differentiable in $\mathbb{R}_+$ and defined by:

$$
\Psi(t) = \begin{cases} 
    e^{A_0t} \left( I_n + \sum_{j=1}^q \int_{h_j}^0 e^{-A_0\sigma} A_j \Psi(\sigma-h) d\sigma \right); & \forall t \in \mathbb{R}_+, \\
    0; & t \in [-h, 0) 
\end{cases}
$$

which becomes $\Psi(t) = e^{A_0t}$ for $t \in [0, h_1]$.

(iii) The fundamental matrix satisfies the following constraint for any $t_j, t_j^+ \in \mathbb{R}_+$:

$$
\Psi(t_j) = \Psi(t_j - t_j) \Psi(t_j) + \sum_{j=1}^q \int_{h_j}^0 \Psi(t_j - t_j - \sigma) \Psi(t_j + \sigma) d\sigma
$$

**Proof** By convenience, one first prove Property (ii). One gets directly from (6) that $\Psi(t) = 0$ for $t \in [-h, 0)$ and $\Psi(0) = I_n$. By taking derivatives with respect to time, one gets:

$$
\Psi(t) = A_0 e^{A_0t} \left( I_n + \sum_{j=1}^q \int_{h_j}^0 e^{-A_0\sigma} A_j \Psi(\sigma-h) d\sigma \right) + \sum_{j=1}^q e^{A_0(t-h_j)} A_j \Psi(t-h_j)
$$

so that (5) holds. Thus, the first expression of (5) is a solution of (4) for all $t \in \mathbb{R}_+$ subject to initial conditions $\Psi(t) = 0$ for $t \in [-h, 0)$ and $\Psi(0) = I_n$. The solution is unique from Picard-Lindelöff theorem since (4) is time-invariant what implies that $\Psi(t) = \sum_{j=0}^q A_j \Psi(t-h_j)$ is locally Lipschitz-continuous in any open interval of its definition domain $[-h, 0) \cup \mathbb{R}_+$. The proof of Property (ii) is as follows. Take time derivatives in (3) using (4) and then (3) again to recover the values $x(t-h_j)$ for $t \geq h_j$ with $x(t) = \varphi(t)$ for $t \in [-h, 0]$ to yield:

$$
x(t) = \Psi(t) x_0 + \sum_{j=1}^q \int_{h_j}^0 \Psi(t-h \sigma) A_j \varphi(\sigma) d\sigma
$$

which coincides with (1) so that (3) is a solution of (1) for the function of initial conditions $\varphi: [-h, 0] \rightarrow \mathbb{R}^n$. Such a solution is unique as it follows under close arguments to those of Property (i) to prove the uniqueness of the fundamental matrix being based on Picard-Lindelöff theorem since
the matrix functions parameterizing (1) and the control and bilinear action inputs are piecewise continuous, and the function of initial conditions is piecewise continuous.

To prove Property (iii), take the homogeneous time-invariant auxiliary delay system:

\[ z(t) = \sum_{j=0}^{\infty} \left( A_j(t) + v_j(t)E_j(t) \right) x(t - h_j) \]

with initial conditions \( \varphi(t) = 0 \) for \( t \in [-h, 0) \) and \( z_0 = z(0) \neq 0 \). For any \( t_j \geq t_1 \geq 0 \), one has from (3):

\[
z(t_2) = \Psi(t_2)z_0 = \Psi(t_2 - t_1)z(t_1) + \sum_{j=0}^{\infty} \int_{t-h_j}^{t_2} \Psi(t_2 - t_3 - \sigma)z(t_3 + \sigma) \, d\sigma
\]

with any fundamental matrix \( \Psi : [-h_j, 0] \cup R_+ \to R^n \) defined for any given subset \( L \) such that \( \{0\} \subseteq L \subseteq \tilde{q} \cup \{0\} \), so that \( L = (\tilde{q} \cup \{0\}) \setminus L \) and \( h_L = \max_{j \in L} h_j \) as follows:

\[ \Psi_L(t) = \left\{ \begin{array}{cl}
\begin{aligned}
e^{A_0 t} (I_n + \sum_{j \in L} \int_{0}^{t} e^{-A_0 \sigma} A_0 \Psi \sigma - h_j) d\sigma) & ; & t \in R_+ \\
0 & ; & t \in [-h_j, 0)
\end{aligned}
\end{array} \right. \]

satisfying the homogeneous auxiliary time-invariant dynamics:

\[ \Psi_L(t) = \sum_{j \in L} A_0 \Psi \left( t - h_j \right) U \left( t - h_j \right) ; & \forall t \in R_+ \tag{10} \]

Note that all the dynamics which is not present in the selected homogeneous auxiliary system is incorporated as a forced term in the solution using the superposition principle. Note that if \( L = \{0\} \) then the fundamental matrix \( \Psi(t) = e^{A_0 t} \) is a \( C_1 \)-semigroup of infinitesimal generator \( A_0 \) and if \( L = \tilde{q} \cup \{0\} \) then \( \Psi(t) = \Psi(t) \) as addressed in Theorem 1. In general, the associated evolution operators \( T : R_0 \to L(R^n) \) which defines the unique state- trajectory solution of the homogeneous system (1) are not strongly continuous evolution operators since they do not satisfy the semigroup property.
The following result establishes that any fundamental matrix of the form (9) is positive for all time if \(A_{00}\) is a Metzler matrix and \(A_{0j} > 0; \forall j \in \bar{q}\).

**Lemma 1.** Assume that \(A_{00} \in M^+_\mathbb{R}\) and \(A_{0j} > 0; \forall j \in \bar{q}\). Then, \(\Psi_L(t) > 0; \forall t \in \mathcal{R}_{0\bar{r}},\) irrespective of the delays, for any \(L\) with \(\{0\} \subseteq L \subseteq \bar{q} \cup \{0\} \).

**Proof.** Since \(A_{00} \in M^+_\mathbb{R}\), \(e^{A_{00}t} > 0; \forall t \in \mathcal{R}_{0\bar{r}}\), (Lemma A.2, Appendix A). Now, from (9), \(\Psi_1(t) = e^{A_{00}t} > 0; \forall t \in \mathcal{R}_{0\bar{r}},\) the result is proved. Otherwise, and since all the entries of the fundamental matrix are everywhere continuous in \(\mathcal{R}_{0\bar{r}},\) assume that there is some \(t_1 > h_1\) such that \(\Psi_1(t) > 0; \forall t \in [0, t_1)\) and at least one entry of the fundamental matrix \(\Psi_{1j}(t_j) < 0\) for some \(i, j \in \bar{n}\). Then, there exists a connected real interval \([t_2, t_1] \in \mathcal{R},\) such that \(\Psi_{1j}(t_j) \supset [t_2, t_1] \rightarrow \mathcal{R}\) in view of (9) since \(e^{A_{00}t} > 0; \forall t \in \mathcal{R}_{0\bar{r}}\) and \(A_{0j} > 0; \forall j \in \bar{q}\). But this contradicts \(\Psi_1(t) > 0; \forall t \in [0, t_1).\) Thus, \(\Psi_1(t) > 0; \forall t \in \mathcal{R}_{0\bar{r}},\) and the result is proved. \(\square\)

It can be pointed out that in the internal delay-free case, i.e. \(q = 0, \Psi'(t) = e^{A_{00}t} > 0; \forall t \in \mathcal{R}_{0\bar{r}},\) since \(A_{00} \in M^+_\mathbb{R},\) so that Lemma 1 still holds. This property also holds in the time-delay case if \(t \in [0, h_1]\). Note that Lemma 1 is independent of the bilinear dynamics of (1). Some specific definitions are first needed and now established for the subsequent formal framework and note that the matrices of dynamics, functions of initial conditions and control input and bilinear input action are always assumed to satisfy the piecewise continuity and absolute piecewise continuity assumptions, without giving specific “ad hoc” indications, given when defining the dynamic system (1)–(2).

**Definitions.**

1. (1) The bilinear system (1)–(2) is said to be internally positive (abbreviated to positive) if for any \(\varphi: [-h, 0] \rightarrow \mathcal{R}_{0\bar{r}},\) every \(u: \mathcal{R}_{0\bar{r}} \rightarrow \mathcal{R}_{0\bar{r}}\), and every \(v: \mathcal{R}_{0\bar{r}} \rightarrow \mathcal{R}_{0\bar{r}}; \forall j \in \bar{q} \cup \{0\},\) one has \(x: \mathcal{R}_{0\bar{r}} \rightarrow \mathcal{R}_{0\bar{r}}\) and \(y: \mathcal{R}_{0\bar{r}} \rightarrow \mathcal{R}_{0\bar{r}}\).

2. The bilinear system (1)–(2) is said to be positive with respect to some \(C_\varphi\) such that \(\mathcal{R}_{\varphi 1\text{in}}^{\varphi 1\text{in}} \supset C_\varphi \supset \mathcal{R}_{\varphi 1\text{in}}^{\varphi 1\text{in}}\) (abbreviated to \(C_\varphi\)-positive) if for any \(\varphi: [-h, 0] \rightarrow \mathcal{R}_{0\bar{r}},\) every \(u: \mathcal{R}_{0\bar{r}} \rightarrow \mathcal{R}_{0\bar{r}}\), and every \(v: \mathcal{R}_{0\bar{r}} \rightarrow C_\varphi\) \((\subseteq \mathcal{R}_{\varphi 1\text{in}}^{\varphi 1\text{in}})\) (abbreviated to \(v \in C_\varphi\)), where \(v(t) = (v^1(t), v^2(t), \ldots, v^p(t))\) with \(\forall j \in \bar{q} \cup \{0\},\) one has \(x: \mathcal{R}_{0\bar{r}} \rightarrow \mathcal{R}_{0\bar{r}}\) and \(y: \mathcal{R}_{0\bar{r}} \rightarrow \mathcal{R}_{0\bar{r}}\).

3. The bilinear system (1)–(2) is said to be externally positive (respectively, \(C_\varphi\)-externally positive) if for every \(u: \mathcal{R}_{0\bar{r}} \rightarrow \mathcal{R}_{0\bar{r}}\), and every \(v \in C_\varphi\), one has \(y: \mathcal{R}_{0\bar{r}} \rightarrow \mathcal{R}_{0\bar{r}}\), provided that \(\varphi(t) = 0\) for \(t \in [-h, 0]\).

A set \(C_\varphi\), as defined in Definitions 1.2 and 1.3 is (in general) non-properly inclusive of \(\mathcal{R}_{\varphi 1\text{in}}^{\varphi 1\text{in}}\) in the sense that \(C_\varphi\)-positivity (respectively, \(C_\varphi\)-external positivity) implies positivity (respectively, external positivity). Note that if the system is \(C_\varphi\)-positive then it is also \(C_\varphi\)-externally positive since \(C_\varphi\)-positivity implies output positivity for zero initial conditions if \(v \in C_\varphi\) from the above definitions. The converse is not true, in general. Note that the positivity properties of a system (1)–(2) are kept by its associated system.

**Theorem 2.** Assume that

A.1. \(A_{00} \in M^+_\mathbb{R}\) and \(A_{0j} > 0; \forall j \in \bar{q}\).

A.2. \(\bar{A}_i(t) > 0, B_i(t) > 0; \forall i \in \bar{q} \cup \{0\}, \forall t \in \mathcal{R}_{0\bar{r}}.\)

A.3. \(C(t) > 0, D(t) > 0; \forall t \in \mathcal{R}_{0\bar{r}}.\)

Then, the bilinear system (1)–(2) is \(C_\varphi\)-positive and then also positive, where

\[C_\varphi = \left\{ v: \mathcal{R}_{0\bar{r}} \rightarrow \mathcal{R}_{\varphi 1\text{in}}^{\varphi 1\text{in}} : \left(\frac{\bar{A}_i(t)}{E_j(t)}\right) \leq -\min_{k \in \bar{n}} \left(\frac{\bar{A}_k(t)}{E_j(t)}\right), \forall k \in \bar{n}, \forall j \in \bar{q} \cup \{0\}, \forall t \in \mathcal{R}_{0\bar{r}} \} \]
Proof. Since \( A_{q_i} \in M_{q_i}^2 \) and \( A_{q_i} \) are positive (Assumption A.1), \( \forall j \in \bar{q} \) (Assumption A.1) then \( \Psi(t) > 0; \forall t \in R_{0^+} \), irrespective of the delays, from Lemma 1. In addition, one has from (3) Assumption A.2, that \( x(t) \geq 0; \forall t \in R_{0^+} \) provided that \( \tilde{A} \tilde{t} > 0 \), \( B(t) \geq 0; \forall t \in \bar{q} \cup \{0\} \), \( \forall j \in \bar{q} \cup \{0\} \), \( \forall t \in R_{0^+} \) (Assumption A.2) for any given \( \Psi(t) > 0, t \in \{t \leq 0\} \) and \( u(t) \geq 0; \forall t \in R_{0^+} \) provided that \( v \in C \) since then \( \tilde{A} \tilde{t} + \tilde{v} \tilde{t} E(t) \geq 0 \), \( \forall j \in \bar{q} \cup \{0\} \), \( \forall t \in R_{0^+} \). This follows directly via a contradiction argument as follows. Assume that there is some \( t_r > 0 \) such that \( x(t_r) < 0 \) and \( x(t) > 0 \) for \( t \in \{t \leq 0\} \) with \( x(t) = \tilde{v}(t) \) for \( t \in \{t \leq 0\} \). If some component of \( x(t) \) is negative, such a first time \( t = t_r \) has to exist that \( x(t) \) is everywhere continuously differentiable subject to non-negative initial conditions and non-negative controls. But, the assumption, this is impossible from (3) unless \( x(t) < 0 \) for some \( i \in \bar{q} \) and \( t < t_r \), a contradiction. Furthermore, \( y(t) \geq 0 \) from (2) and (3) since \( x(t) \geq 0, u(t) \geq 0 \); \( \forall t \in R_{0^+} \) since Assumption A.3 holds. As a result, the bilinear system (1)–(2) is \( C \) positive. Since \( C \supset R_{0^+} \) then it is also positive. Property (i) has been proved.

\[ x(t) = \sum_{j=0}^{q} (A_{q_j}x(t - h_j) + \tilde{v}(t)E(t)y(t - h_j)) + \sum_{j=q}^{q} B(t)u(t - h_j) \]  
\[ y(t) = C(t)x(t) + D(t)u(t); \forall t \in R_{0^+} \]

could be considered instead of (1)–(2) where the bilinear dynamics contribution results from a coupling in-between the output and the action input and the linear dynamics matrices are constant. In this case, the following result close to Theorem 2 holds:

**THEOREM 3.** The following properties hold:

(i) If Assumptions A.1 and A.3 of Theorem 2 hold together with

\[ A_4. E_i(t) \geq 0, B_j(t) \geq 0; \forall i \in \bar{q} \cup \{0\}, \forall j \in \bar{q} \cup \{0\}, \forall t \in R_{0^+}. \]

Then, the bilinear system (11)–(12) is positive (then also externally positive) for every \( \forall j(t) \geq 0 \); \( \forall q \in \bar{q} \cup \{0\}, \forall t \in R_{0^+}. \)

(ii) Assume that

\[ A_5. C(t)\Psi(t - \sigma)\Psi(\sigma)E_i(\sigma) \geq 0, C(t)\Psi(t - \sigma)B_j(\sigma) + D(t) \geq 0, C(t)\Psi(t - \sigma)B_j(\sigma); \forall i \in \bar{q} \cup \{0\}, \forall j \in \bar{q} \cup \{0\} \text{ for any } \sigma \in \{0, t_r\}; \forall t \in R_{0^+}. \]

Then, the bilinear system (11)–(22) is externally positive for every \( \forall j(t) \geq 0; \forall q \in \bar{q} \cup \{0\}, \forall t \in R_{0^+}. \)

Proof. Note that the solution (3) to (1) is replaced with

\[ x(t) = \Psi(t)x_0 + \sum_{j=0}^{q} \int_{-h_j}^{0} \Psi(t - \sigma)A_{q_j} \Psi(\sigma) d\sigma \]
\[ + \sum_{j=0}^{q} \int_{0}^{t} \Psi(t - \sigma)A_{q_j} \Psi(\sigma)E_j(\sigma) \left[ C \left( \sigma - h_j \right) x \left( \sigma - h_j \right) + D \left( \sigma - h_j \right) u \left( \sigma - h_j \right) \right] d\sigma \]
\[ + \sum_{j=0}^{q} \int_{0}^{t} \Psi(t - \sigma)B_j(\sigma)u \left( \sigma - h_j \right) d\sigma; \forall t \in R_{0^+}. \]

From (13) and (2), one gets:

\[ y(t) = C(t)\Psi(t) \left[ x_0 + \sum_{j=0}^{q} \int_{-h_j}^{0} \Psi(t - \sigma)A_{q_j} \Psi(\sigma) d\sigma \right] + \sum_{j=0}^{q} \int_{0}^{t} C(t)\Psi(t - \sigma)A_{q_j} \Psi(\sigma)E_j(\sigma) y \left( t - h_j \right) d\sigma \]
\[ + \sum_{j=0}^{q} \int_{0}^{t} C(t)\Psi(t - \sigma)B_j(\sigma)u \left( \sigma - h_j \right) d\sigma + D(t)u(t); \forall t \in R_{0^+}. \]
The proof of Property (i) is similar to the proof of Theorem 2 and thus omitted. To prove Property (ii), take zero initial conditions in (13) to yield the “zero-state” output

\[y_{x0}(t) = \sum_{j=0}^{q} \int_0^t C(t)\phi(t - \sigma) V_j(\sigma) d\sigma + \sum_{j=0}^{q} \int_0^t C(t)\phi(t - \sigma) B_j(\sigma) u(\sigma - h_j) d\sigma + \int_0^t C(t)\phi(t - \sigma) D(t) u(t) d\sigma; \forall t \in \mathbb{R}_+\]

(15)

Thus, it follows from (14) under continuity arguments of the zero state output that the system is externally positive if Assumption A.5 holds.

\[\square\]

The subsequent result establishes that

**Corollary 2.** Assume that (1)-(2) is time-invariant in the sense that \(A_{00}, \hat{A}_j \in \mathbb{R}^{m\times n}, v_j \in \mathbb{Q} \cup \{0\}, B_{0j}, \hat{B}_j \in \mathbb{R}^{n\times q}, v_j \in \mathbb{Q} \cup \{0\}, E_j \in \mathbb{R}^{n\times q}, \forall j \in \mathbb{Q} \cup \{0\}, C \in \mathbb{R}^{n\times q}, D \in \mathbb{R}^{n\times q}, \forall j \in \mathbb{Q} \cup \{0\}, C \in \mathbb{R}^{n\times q}, D \in \mathbb{R}^{n\times q}\). Assume also that \(h_i \geq h_{i0}\) with \(h_{i0}\) being sufficiently large. Then, a necessary and sufficient condition for the time-invariant resulting bilinear system to be positive if \(v: \mathbb{R}^{n\times q} \to \mathbb{R}^{n\times q}\) is that \((A_{00} + \hat{A}_j) \in M_0^p, (A_{00} + \hat{A}_j) \geq 0, E_j > 0 B_j \geq 0; \forall j \in \mathbb{Q} \cup \{0\}, C > 0, D \geq 0.\)

**Proof.** The sufficiency part of the proof follows directly from Theorem 2 since the time-invariant bilinear system is a particular case of (1)-(2). To prove the necessity part, first note that the solution of (3) becomes in this particular case by incorporating the constant \(\hat{A}_j\) matrix to the evolution operator

\[x(t) = \sum_{j=1}^{q} \int_{-h_j}^t \int_0^t C(t)\phi(t - \sigma) (A_{00} + \hat{A}_j) \psi(\sigma) d\sigma d\sigma
\]

\[+ \sum_{j=0}^{q} \int_0^t \int_0^t C(t)\phi(t - \sigma) B_j(\sigma) u(\sigma - h_j) d\sigma d\sigma; \forall t \in \mathbb{R}_+\]

(16.a)

\[= \sum_{j=0}^{q} \int_{-h_j}^t \int_0^t C(t)\phi(t - \sigma) (A_{00} + \hat{A}_j) \psi(\sigma) d\sigma d\sigma
\]

\[+ \sum_{j=0}^{q} \int_0^t \int_0^t C(t)\phi(t - \sigma) B_j(\sigma) u(\sigma - h_j) d\sigma d\sigma; \forall t \in \mathbb{R}_+\]

(16.b)

where \((A_{00})_{ij}, (\hat{A}_j)_{ij}\) and \((B_j)_{ij}\) are the kth columns of \(A_{00}, \hat{A}_j\) and \(B_j\), respectively, for \(j \in \mathbb{Q}, i \in \mathbb{Q} \cup \{0\}, k \in \mathbb{N}\) and \(i \in \mathbb{N}\) and \(\psi_i(\cdot), x_i(\cdot), E_j, v_j(\cdot)\) and \(u_i(\cdot)\) are the kth and lth components of \(\psi(\cdot), x(\cdot), E(\cdot)\) and \(v(\cdot), u(\cdot)\), respectively, for \(j \in \mathbb{Q} \cup \{0\}\).

with

\[\psi(t) = \begin{cases} e^{\delta_{\hat{A}_0 + \hat{A}_j} t} (I + \int_{-h_j}^t \int_0^t C(t)\phi(t - \sigma) (A_{00} + \hat{A}_j) \psi(\sigma - h_j) d\sigma) ; & \forall t \in \mathbb{R}_+ \\
0; & \forall t \in [-h, 0) \end{cases}
\]

(17)

Assume that \((A_{00} + \hat{A}_j) \in M_0^p\) and take \(u(t) = 0, v(t) = 0\), \(v_j \in \mathbb{Q} \cup \{0\}, (A_{00} + \hat{A}_j) \in M_0^p\). \(\psi(t) = 0\) for \(t \in [-h, 0)\) and \(x_0 = x(0^-) = \phi(0^+) > 0\) so that (16) results in \(x(t) = \psi(t)x_0; \forall t \in \mathbb{R}_+\).

Since \((A_{00} + \hat{A}_j) \notin M_0^p\) there is some \(t \in \mathbb{R}_+\) such that \(e^{\delta_{\hat{A}_0 + \hat{A}_j} t}\) is not positive and \(e^{\delta_{\hat{A}_0 + \hat{A}_j} t} > 0\) for \(t \in [0, t_j)\). Furthermore, \(\psi(t) = e^{\delta_{\hat{A}_0 + \hat{A}_j} t} \) for \(t \in [0, h_i)\). Also, if the lower-bound \(h_{i0}\) of the smaller delay \(h_i\) is large enough such that \(h_{i0} \geq t\), then \(x(t_j) = \psi(t_j)x_0 = e^{\delta_{\hat{A}_0 + \hat{A}_j} t_j} x_0\) and there is some pair \((i, j) \in \mathbb{N} \times \mathbb{N}\) such that \((\psi(t_j))_{ij} = (e^{\delta_{\hat{A}_0 + \hat{A}_j} t_j})_{ij} < 0\). Thus,
\[ x_i(t) = \sum_{k(i)}^{n} \psi_k(t_1) x_{0k} - \left[ \psi_j(t_1) \right] x_0 < 0 \]

for any \( x_0 > 0 \) fulfilling the constraint \( x_0 > \frac{\sum_{k(i)}^{n} \psi_k(t_1) x_{0k}}{\psi_j(t_1)} \) resulting in the non-positivity of \( x(t_1) \) so that the system is not positive if \( (A_{00} + \bar{A}_0) \notin M_0^p \) and \( h_1 \) is sufficiently large.

Now, assume that \( (A_{00} + \bar{A}_0) \notin M_0^p \) while there is some \( j \in q \) such that \( (A_{0j} + \bar{A}_j) \geq 0 \) fails, so that the entry \( (A_{0j} + \bar{A}_j)_{rs} = (A_0)_{rs} + (\bar{A}_j)_{rs} < 0 \) for some \( r, s \in n \). Now, take \( x_0 = 0 \) and \( u(t) = 0 \), \( v_j(t) = 0 \); \( \forall j \in q \cup \{0\} \), \( \forall t \in R_0 \), and \( \bar{A} \) piecewise-continuous function of initial conditions:

\[ \varphi(\sigma) = \varphi[j] > 0 \] for \( \sigma \in [-h_j, -h_j + \gamma_j] \) and some \( \gamma_j \in (0, 0] \), \( \varphi(\sigma) = 0 \), \( \sigma \in [-h_j + \gamma_j, -h_j] \) and \( \varphi(\sigma) = 0 \) for \( \sigma \in [-h_j, -h_j + \gamma_j] \), \( \forall (i \neq j) \in q \). Thus, one gets from (16.b) that the state-trajectory solution becomes:

\[
x(t) = \sum_{i=1}^{n} \sum_{k(i)}^{n} \int_{-h_j}^{t \to \infty} \psi(t - \sigma)((A_{0})_{j} + (\bar{A})_{j}) \varphi(\sigma) d\sigma
\]

\[
= \sum_{i=1}^{n} \sum_{k(i)}^{n} \left( \int_{-h_j}^{t \to \infty} \psi(t - \sigma)((A_{0})_{j} + (\bar{A})_{j}) d\sigma \right) \varphi[j]
\]

\[
+ \sum_{i=1}^{n} \sum_{k(i)}^{n} \left( \int_{-h_j}^{t \to \infty} \psi(t - \sigma)((A_{0})_{j} + (\bar{A})_{j}) d\sigma \right) \varphi[j]
\]

\[
= \sum_{i=1}^{n} \sum_{k(i)}^{n} \left( \int_{-h_j}^{t \to \infty} \psi(t - \sigma)((A_{0})_{j} + (\bar{A})_{j}) d\sigma \right) \varphi[j]
\]

\[
= \sum_{i=1}^{n} \sum_{k(i)}^{n} \left( \int_{-h_j}^{t \to \infty} \psi(t - \sigma)((A_{0})_{j} + (\bar{A})_{j}) d\sigma \right) \varphi[j]
\]

\[
= \sum_{i=1}^{n} \sum_{k(i)}^{n} \left( \int_{-h_j}^{t \to \infty} \psi(t - \sigma)((A_{0})_{j} + (\bar{A})_{j}) d\sigma \right) \varphi[j]
\]

(18)

and the \( \alpha(\in \bar{A}) \)th component of \( x(t) \), \( x_A(t) = e_A x(t) \) is:

\[
x_A(t) = \int_{-h_j}^{0} \left[ \psi(t + \epsilon_i) \left( (A_{0})_{i} + (\bar{A})_{i} \right) - \psi(t + \epsilon_i) \right] \psi(t) \psi[j] \right] d\sigma \right)
\]

for some real constants \( \epsilon_i, \in (h_j, -\gamma_j, h_j) \) for \( \alpha, i \in \bar{A} \) and the given \( j \in q \cup \{0\}, i, \epsilon \in \bar{A} \), where \( (A_{0})_{i} \) (respectively \( \bar{A} \)) is the \( (\epsilon, i) \)th \( n \)x \( \bar{A} \) entry of the matrix \( A_{0} \) (respectively, \( \bar{A} \)) whose \( \epsilon \)th \( n \)th column is \( (A_{0})_{i} \) (respectively, \( \bar{A} \)) and \( \psi[j] \) for \( \epsilon \in \bar{A} \), \( \forall j \in q \) is the \( \ell \)th component of \( \varphi(t) = \psi \) for \( t \in [-h_j, -h_j + \gamma_j] \subseteq [-h_j, -h_j + \gamma_j] \) and \( \epsilon \in \bar{A} \). The expression (19) follows from the mean value theorem for integrals which is applicable here since the evolution operator is everywhere continuous on its definition domain. Since \( \psi(t) \) is also positive for \( t \in R_0 \), \( \psi(0) = 1 \) for each given \( \psi \in (0, 1) \), it always exists a sufficiently small \( \gamma_j \in (h_j, -h_j) \) such that \( 0 \leq \psi[t] \leq \psi \) for \( j, \ell(j) \in \bar{A} \) and \( 1 - \psi \leq \psi[t] \leq 1 + \psi \) for all \( t \in [0, \gamma_j] \) with \( \psi[0] = 1 \); \( \forall j \in \bar{A} \), so that:

\[
x_A(t) \leq \int_{-h_j}^{0} \left[ (n - 1) \left( \sum_{k(i)}^{n} \max_{i \neq \ell(j)} \left( (A_{0})_{i} + (\bar{A})_{i} \right) \right) - (1 - \psi) \left( (A_{0})_{i} + (\bar{A})_{i} \right) \right] \psi[j] \right] d\sigma \right)
\]

for \( t \in [-h_j, -h_j + \gamma_j] \) and any sufficiently small \( \gamma_j \in (h_j, -h_j, h_j) \) if the \( \ell \)th component of \( \psi(t) \) in \( [-h_j, -h_j + \gamma_j] \) is a constant value \( \varphi[j] = \psi \) and \( \psi \) is small enough satisfying:
\[(n-1)\psi \left( \sum_{k=0}^{n} \max_{\nu \in \mathbb{N}} \left( A_0 + (A_j)_\nu \right) \right) + \psi \left( \left( A_0 + (A_j) \right)_s \right) < \left( A_0 + (A_j) \right)_s \] \hspace{1cm} (21)

So, we have found a solution to (1) which is not positive if \( A_j = (A_0 + \tilde{A}_j) \geq 0 \) fails under admissible non-negative initial conditions. Note that this contradiction proof is directly extendable to the case when more than one entry of \((A_0 + \tilde{A}_j)\) is negative or when there are one or more negative entries in \((A_0 + \tilde{A}_j)\) for more than one \( i \in \tilde{q} \cup \{0\} \).

Now, assume that \((A_0 + \tilde{A}_0) \in M^n_0, (A_0 + \tilde{A}_j) \geq 0, \forall i \in \tilde{q}, B_j \geq 0, \forall i \neq j \in \tilde{q} \) and some \( j \in \tilde{q} \) for which \( B_j \geq 0 \) fails so that at least one entry \((B_j)_i < 0\) for some \( a \in n, s \in m \). Take \( \varphi(t) = 0 \) for \( t \in [-h, 0] \) and \( \nu_j(t) = 0; \forall j \in \tilde{q} \cup \{0\}, \forall t \in \mathbb{R}_{\nu_s} \). One gets from (16) that

\[
x_{\nu}(t) = \sum_{r=0}^{\nu} \sum_{(i,j) \neq 0} \int_{0}^{t} e^{r}(t - \sigma)(B)_{ij} u_{ij}(\sigma - h) d\sigma
\]

Next, take, furthermore \( u_{ij}(t) = 0 \) for \( k \neq t; \forall t \in \mathbb{R}, u(t) = 0 \) for \( t \in \mathbb{R}_{\nu} \) and \( u_{ij}(t') = u > 0 \) for \( t \in [0, \epsilon) \)

and \( \epsilon \in \left( 0, \max_{i \in \tilde{q} \cup \{0\}} \left| h_i - h_j \right| \right) \)

Thus, one gets that for sufficiently small \( t \in \mathbb{R}_{\nu} \) and \( a \in \tilde{n}, \)

\[
x_{\nu}(t) = \tilde{u} \int_{0}^{t} \sum_{(i,j) \neq 0} \left[ \sum_{r=0}^{\nu} \psi_{\nu}(t - \sigma)(B)_{ij} + \psi_{\nu}(t - \sigma)(B)_{ij} - \psi_{\nu}(t - \sigma)(B)_{ij} \right] d\sigma < 0
\] \hspace{1cm} (23)
\( (A_0 + \hat{A}_0) \succeq 0 \); \( \forall i \in \tilde{q} \) while \( B_i \succeq 0 \) fails for some \( i \in \tilde{q}' \cup \{0\} \), the system (1)–(2) is not positive. Thus, \( B_i \succeq 0 \); \( \forall i \in \tilde{q}' \cup \{0\} \) is necessary for positivity of (1)–(2). The extended proof under negativity of more than one entry of the set \( \{B_i : i \in \tilde{q}' \cup \{0\}\} \) could be obtained under a direct more cumbersome development via contradiction arguments.

To prove the necessity of \( E \succeq 0 \); \( \forall i \in \tilde{q}' \cup \{0\} \) for positivity of the system, take \( u(t) = 0 \); \( \forall t \in \mathbb{R}_{\text{pos}} \) and some constant \( v_i(t) = \bar{v}_i \succeq 0 \); \( \forall i \in \tilde{q}' \cup \{0\} \); \( \forall t \in \mathbb{R}_{\text{pos}} \). Then, one gets from (1):

\[
\dot{x}(t) = (A_0 + \bar{v}_0E_0^j)x(t) + \sum_{j=1}^q (A_j + \bar{v}_jE_j^i)x(t - h_i)
\]

(24)

if some component of \( E_0 \) is negative, say \((E_0)_i < 0\), it always exists an off-diagonal \((i, j)\) entry of \( A_0 + \bar{v}_0E_0^j \) which is negative for a sufficiently large \((\bar{v}_0)_j > 0\). Thus, \( A_0 + \bar{v}_0E_0^j \) is not Metzler so that the unique solution of (26) given, equivalently, by:

\[
x(t) = e^{(\bar{v}_0 + \bar{v}_0E_0^j)t}\left[x_0 + \sum_{j=1}^q \int_0^t e^{-(\bar{v}_0 + \bar{v}_0E_0^j)\sigma} (A_j + \bar{v}_jE_j^i)x(\sigma - h_j) \, d\sigma\right]
\]

(25)

\( \forall t \in \mathbb{R}_+ \) with \( x(t) = \varphi(t) \geq 0 \) for \( t \in [-h, 0) \) is not positive for some \( t \in \mathbb{R}_+ \) and \( x_0 = x(0) = \varphi(0) > 0 \) for some given admissible vector function of initial conditions is not positive for some such a function if the first delay \( h_i \) is sufficiently large. In the same way, if \( E \) is not positive for some \( j \in \tilde{q} \), it also follows that \( A_j + \bar{v}_jE_j^i \) is not positive for sufficiently large acting bilinear input \( \bar{v}_j \), so that the solution is not positive for some \( t \in \mathbb{R}_+ \). So, a necessary condition for positivity of the time-invariant version of (1)–(2) is that \( E_j > 0 \); \( \forall j \in \tilde{q}' \cup \{0\} \). Also, it turns out from (2) that if either \( C > 0 \) or \( D \succeq 0 \) fails then there are \( x_j > 0 \) with \( u(0) = 0 \) or \( u(0) > 0 \) with \( x(0) = 0 \) such that \( y(0) \) is not positive. \( \square \)

Remark 1. Note that, more generally, and without essential changes in the proof of Corollary 2, the bilinear action input could be chosen constant belonging to the admissibility class

\[
C_{\text{ad}} = \left\{ v : \mathbb{R}_{\text{pos}} \rightarrow \mathbb{R}^{n \times n} : (v_j) \geq -\min_{i \in \tilde{q}} \frac{\bar{A}_i}{\bar{E}_i} \right\}, \quad \forall i \in \tilde{q}, \quad \forall j \in \tilde{q}' \cup \{0\}, \forall t \in \mathbb{R}_{\text{pos}}
\]

with \( \bar{A}_i = \tilde{A}_0 + \bar{v}_0E_0^j, \bar{A}_j = A_j + \bar{v}_jE_j^i, \forall j \in \tilde{q} \) and the condition \((A_{un} + \bar{A}_0) \in M_{\text{es}}^\text{pos} \) replaced with \( A_{un} \in M_{\text{es}}^\text{pos} \). The following result states that if the positive bilinear system converges to a time-invariant one and the control input converges to a value having at least a positive component then there is a unique strict equilibrium point if and only if \( \sum_{j=0}^q (A_j + \bar{v}_jE_j^i) \) is a Metzler matrix.

Theorem 4. Assume that (1) is positive and time-invariant (or that it converges to a time-invariant one as \( t \to \infty \)). Assume, furthermore, that \( v_j(t) \to (v_j) \geq 0 \); \( \forall j \in \tilde{q}' \cup \{0\} \); \( u(t) \to u \succeq 0 \) as \( t \to \infty \), and \( \sum_{j=0}^q (B_j) \succeq 0 \) and \((\sum_{j=0}^q B_j)_i > 0 \) for at least one \( i \in \tilde{q}' \cup \{0\} \).

Thus, there is a unique positive equilibrium point \( x_e = x_e(v_{0e}, \ldots, v_{qe}, u_e) \) given by:

\[
x_e = -\left[\sum_{j=0}^q (A_j + \bar{v}_jE_j^i)^{-1} \left(\sum_{j=0}^q B_j\right)u_e\right]^+ \sum_{j=0}^q (A_j + \bar{v}_jE_j^i)u_e
\]

(26)

if and only if \( \sum_{j=0}^q (A_j + \bar{v}_jE_j^i) \) is a Metzler stability matrix. Furthermore, \( x_e > > 0 \).

Proof. One gets for the time-invariant case that (1) becomes

\[
\dot{x}(t) = (A_0 + v_0E_0^j)x(t) + \sum_{j=1}^q (A_j + v_jE_j^i)x(t - h_i) + \sum_{j=0}^q B_ju(t - h_i)
\]

(27)

At rest, one gets

\[
0 = \sum_{j=0}^q (A_j + v_jE_j^i)x + \sum_{j=0}^q B_ju = (A_0 + v_0E_0^j)x + \left[\sum_{j=1}^q (A_j + v_jE_j^i)x + \sum_{j=0}^q B_ju\right];
\]

(28)
∀x ∈ X_p > 0 and X_p ⊃ {x_p} being the equilibrium set. Since

\[ \sum_{i=0}^{q} B_i u_e = \sum_{i=1}^{m} \sum_{j=0}^{q} \left( B_j (u_e) \right) \geq \sum_{i=0}^{q} \left( B_j (u_e) \right) \geq \sum_{j=0}^{q} B_j u_e > 0 \]  

(29)

\[ \sum_{i=1}^{q} \left( A_j + v_j E_j^T \right) x + \sum_{j=0}^{q} B_j u_e \geq \sum_{j=0}^{q} B_j u_e > 0 \]  

(30)

it follows that \( X_p \subseteq R^n \) (De la Sen, 2007; Kao, 2014; Mailleret, 2004). On the other hand, since the system is positive \( (A_0 + v_0 E_0^T) \in M^n_p \), \( A_i > 0 \), \( E_i > 0 \), \( \forall i \in q, B_j \geq 0; \forall j \in q \cup \{0\} \) from Corollary 2 what implies trivially as well that \( \sum_{i=0}^{q} \left( A_j + v_j E_j^T \right) \in M^n_p \). Since \( \bar{x} \in (X_p) > 0 \), \( \sum_{i=0}^{q} \left( A_j + v_j E_j^T \right) \in M^n_p \) and \( \sum_{i=1}^{q} \left( A_j + v_j E_j^T \right) x + \sum_{j=0}^{q} B_j u_e > 0 \), the constraints (27) holds if and only if \( \sum_{i=0}^{q} \left( A_j + v_j E_j^T \right) \) is a stability matrix, which, since Metzler, implies also the existence of \( -\left[ \sum_{i=0}^{q} \left( A_j + v_j E_j^T \right) \right] > 0 \). Since such an inverse and \( u_e \) are unique then \( X_p = \{x_p\} \) so that the equilibrium point is unique. Furthermore, note that since \( -\left[ \sum_{i=0}^{q} \left( A_j + v_j E_j^T \right) \right] \) is non-singular and positive it has at least one non-zero entry per row so that \( x_p = -\left[ \sum_{i=0}^{q} \left( A_j + v_j E_j^T \right) \right]^{-1} \left( \sum_{j=0}^{q} B_j u_e \right) > 0 \) since \( \sum_{j=0}^{q} B_j u_e > 0 \). \( \square \)

3. Excitability, strong excitability, external excitability and transparency

The following excitability definitions extend directly the one given in Farina and Rinaldi (2000) for the concept of excitability of a time-invariant linear system then extended in de la Sen (2008b) for a time-invariant system under point delays.

Definition 2 (excitability). A positive bilinear system (1)–(2) is said to be excitable (respectively, externally excitable) if each state variable (respectively, output variable) can be made positive by applying an appropriate non-negative control input and bilinear input action to the system being initially at rest, i.e. for the case \( \phi(t) = 0 \) for \( t \in [-h, 0] \).

Remark 2. It turns out that we could refer to excitability and external excitability of a particular state (respectively, output) variable it is individually excitable without the need for excitability of the complete state (respectively, output) vector. We could also refer in a natural way to asymptotic excitability if the excitability is an asymptotic property.

On the other hand, we can also refer to the above excitability properties with respect to any of the two inputs or, eventually, their components. Excitability, respectively, external excitability being achievable from any control input component or bilinear action component of some control input or bilinear input action will be referred to as strong excitability or strong external excitability, respectively. Note that in many applications the bilinear input action coincides with the control input and it can be of no free-choice.

Theorem 4 leads to a direct excitability consequent result as follows:

Corollary 3. Assume that all the assumptions of Theorem 4 hold. Then, the resulting time-invariant (1) is asymptotically excitable and also excitable (in a finite time).

Proof. The asymptotic excitability of is obvious from Theorem 4 by fixing \( u(t) = u_\omega, v(t) = v_\omega \) for \( j \in q \cup \{0\} \) and \( t \in R_0 \), since the state trajectory solution converges asymptotically to a strictly positive equilibrium point. On the other hand, (25) implies that that the strictly positive equilibrium point is strictly increasing with \( u_\omega \) for any constant fixed \( v(t) = v_\omega \) for \( j \in q \cup \{0\} \) and \( t \in R_0 \). So, for any such \( u_\omega \), it exists \( \epsilon, \lambda_\omega \in R^n \) and a finite time \( T = T(u_\omega, \epsilon, \lambda_\omega) \) such that the state trajectory solution is strictly positive, \( (x_p) > \epsilon \forall i \in h \) and the system is excitable if \( u_\omega(t) = u_\omega = \lambda u_\omega v(t) = v_\omega \) for \( j \in q \cup \{0\} \) and all \( t \in R_0 \) and any \( \lambda \in R^n_+ \) ≥ \( \lambda_\omega \) verifying furthermore that \( 0 < (x_p) - \epsilon \leq x(t) \leq (x_p) - \epsilon \forall i \in h \) and any \( T \geq T \) with \( x(t) \rightarrow x_p \) as \( t \rightarrow \infty \). Thus, the result is proved. \( \square \)
Remark 3. Note that the excitatibility of Corollary 3 is guaranteed without requiring the need for conditions for the existence of a strictly positive equilibrium, just from the positivity of the system and the constraint \( \sum_{j=0}^{q} B_j \) \( \gg 0 \), which is a particular case guaranteeing the necessary and sufficient condition of excitability of the time-invariant version of the positive system.

Conditions of excitability of a positive system are now formalized in the subsequent result through the excitability of its associated system. For such purposes, one takes advantage of the fact that the successive powers of the matrix \( A_{\infty}^{(k)} \) are positive even if all those of the Metzler \( A_{\infty} \) are not positive matrices.

**Theorem 5.** Consider the system (1)–(2) with \( A_0 \in M_{n}^{+}, A_0 > 0, A_i(t) = (A_0 + \tilde{A}_i(t)) \geq 0, E_i(t) \geq 0, B_j(t) \geq 0; \forall t \in \mathbb{R}_{+,} \forall i \in \bar{q} \cup \{0\}, \forall j \in \bar{q'} \cup \{0\}, \sum_{j=0}^{q} B_j \geq 0, C > 0, D \geq 0, \) and \( v \in C_r \) with

\[
C_r = \left\{ v : \mathbb{R}_{+} \rightarrow \mathbb{R}^{q+1,n} : (v_j(t)) \geq -\min_{\text{ip}} \left( \frac{\tilde{A}_j(t)}{E_j(t)} \right); \forall k \in n, \forall j \in q \cup \{0\}, \forall t \in R_{+} \right\}
\]

Assume that each entry of each of the matrices \( \tilde{A}_i(t), E_i(t), B_j(t), C(t) \) and \( D(t); \forall i \in \bar{q} \cup \{0\}, \forall j \in \bar{q'} \cup \{0\}, \) and \( v \in \bar{q'} \cup \{0\} \) are either null or non-zero for all time so that their associated matrices are constant. Assume also that the maximum internal and external delays \( \tilde{h} \) and \( h' \) are sufficiently small such that some real \( t \) exists which satisfies the constraint \( \max(\tilde{h}, h') < t < h_1 + \varepsilon \) with \( \varepsilon \) defined in Lemma A.1 (Appendix A). Then, the following properties hold:

(i) A necessary and sufficient condition for the system to be strongly excitable, and then excitable, from the control input is:

\[
\sum_{k=0}^{n-1} A_{\infty}^{(k)} \left( I_n + \sum_{j=1}^{q} \sum_{i=0}^{n-1} A_0^{(i)} A_{\infty}^{(i)} \right) B_r \gg 0; \forall j \in \bar{m}
\]

A sufficient condition for the \( i \)th state component to be strongly excitable, and then excitable, from the control input is:

\[
\sum_{k=0}^{n-1} e^t A_{\infty}^{(k)} \left( I_n + \sum_{j=1}^{q} \sum_{i=0}^{n-1} A_0^{(i)} A_{\infty}^{(i)} \right) B_r \gg 0; \forall j \in \bar{m}
\]

(ii) Assume, in addition, that \( \sum_{j=0}^{q} B_j \) is monomial. A sufficient condition for the system to be excitable from the control input is that

\[
\sum_{k=0}^{n-1} A_{\infty}^{(k)} \left( I_n + \sum_{j=1}^{q} A_0^{(i)} \right) \gg 0
\]

A sufficient condition for the \( i \)th state component to be excitable from the control input is

\[
\sum_{k=0}^{n-1} e^t A_{\infty}^{(k)} \left( I_n + \sum_{j=1}^{q} A_0^{(i)} \right) \gg 0
\]

Weaker corresponding sufficient conditions if \( \sum_{j=0}^{q} B_j \gg 0 \) is monomial are, respectively,

\[
\sum_{k=0}^{n-1} A_{\infty}^{(k)} \left( I_n + \sum_{j=1}^{q} \sum_{i=0}^{n-1} A_0^{(i)} A_{\infty}^{(i)} \right) \gg 0
\]

\[
\sum_{k=0}^{n-1} e^t A_{\infty}^{(k)} \left( I_n + \sum_{j=1}^{q} \sum_{i=0}^{n-1} A_0^{(i)} A_{\infty}^{(i)} \right) \gg 0
\]

Further weaker corresponding sufficient conditions, which are also necessary, are, respectively,

\[
\sum_{k=0}^{q} \sum_{k=0}^{n-1} A_{\infty}^{(k)} \left( I_n + \sum_{j=1}^{q} \sum_{i=0}^{n-1} A_0^{(i)} A_{\infty}^{(i)} \right) B_r \gg 0
\]
\[ \sum_{r=0}^{q} \sum_{k=0}^{n-1} e^{k} A^{k}_{00} \left( I_{n} + \sum_{j=1}^{q} \sum_{i=0}^{n-1} A^{(g)} A^{(g)}_{j} \right) B^{(g)}_{r} \gg 0 \]

(iii) A necessary and sufficient condition for the system to be strongly externally excitable from the control input is:

\[ \sum_{r=0}^{q} \sum_{k=0}^{n-1} C^{(g)} A^{k}_{00} \left( I_{n} + \sum_{j=1}^{q} \sum_{i=0}^{n-1} A^{(g)} A^{(g)}_{j} \right) B^{(g)}_{r} + D^{(g)}_{j} \gg 0; \forall j \in \hat{m} \]

A sufficient condition for the \( i \)th output component to be strongly excitable from the control input is:

\[ \sum_{r=0}^{q} \sum_{k=0}^{n-1} e^{k} C^{(g)} A^{k}_{00} \left( I_{n} + \sum_{j=1}^{q} \sum_{i=0}^{n-1} A^{(g)} A^{(g)}_{j} \right) B^{(g)}_{r} + e^{l} D^{(g)} > 0 \]

(iv) A necessary and sufficient condition for the system to be externally excitable from the control input is:

\[ \sum_{r=0}^{q} \sum_{k=0}^{n-1} C^{(g)} A^{k}_{00} \left( I_{n} + \sum_{j=1}^{q} \sum_{i=0}^{n-1} A^{(g)} A^{(g)}_{j} \right) B^{(g)}_{r} + D^{(g)} > 0 \]

A necessary and sufficient condition for the \( i \)th output component to be excitable from the control input is:

Now, define

\[ C_{v} = \left\{ v : R_{0+} \rightarrow R^{q+1} : \left( v(t) \right)_{k} \geq \delta - \min_{i \in n} \left( \hat{A}_{i} \left( t \right) \right) \frac{\hat{A}_{i} \left( t \right)}{\hat{E}_{i} \left( t \right)} ; \forall k \in \hat{n}, \forall j \in \hat{q} \cup \{0\}, \forall t \in R_{0+} \right\} \]

for any \( \delta \in R_{0+} \) (note that \( C_{v} = C_{j} \)) and, in addition, assume that \( v \in C_{v} \). Then, the following additional properties hold:

(v) A sufficient condition for the system to be strongly excitable from the bilinear input action is

\[ \sum_{k=0}^{n-1} A^{k}_{00} \left( I_{n} + \sum_{j=1}^{q} \sum_{i=0}^{n-1} A^{(g)} A^{(g)}_{j} \right) \gg 0 \]

A sufficient condition for the \( i \)th state component to be strongly excitable from the bilinear input action is that

\[ \sum_{k=0}^{n-1} e^{k} A^{k}_{00} \left( I_{n} + \sum_{j=1}^{q} \sum_{i=0}^{n-1} A^{(g)} A^{(g)}_{j} \right) \gg 0 \]

(vi) A sufficient condition for the system to be externally excitable from the bilinear input action is:

\[ \sum_{k=0}^{n-1} C^{(g)} A^{k}_{00} \left( I_{n} + \sum_{j=1}^{q} \sum_{i=0}^{n-1} A^{(g)} A^{(g)}_{j} \right) \gg 0 \]

A sufficient condition for the \( i \)th output component to be excitable from the bilinear input action is:

\[ \sum_{k=0}^{n-1} e^{i} C^{(g)} A^{k}_{00} \left( I_{n} + \sum_{j=1}^{q} \sum_{i=0}^{n-1} A^{(g)} A^{(g)}_{j} \right) \gg 0 \]

(vii) Define \( e = (1, 1, \ldots, 1)^{T} = \sum_{i=1}^{n} e_{i} \in R^{n} \) and the subsets of \( \hat{n} \)
\( n_u = \left\{ i \in \mathcal{P} : \sum_{i=0}^{q} \sum_{k=0}^{\mu-1} e_i A_{00}^k \left( I_n + \sum_{i=0}^{q} \sum_{k=0}^{\mu-1} A_{00}^i A_{00}^k \right) B_{i}^k > 0 \right\} \)

\( n_v = \left\{ i \in \mathcal{P} \cap \left( \bigcup_{k=0}^{\mu-1} n_{\text{null}} \right) : \sum_{i=0}^{q} \sum_{k=0}^{\mu-1} e_i A_{00}^k \left( I_n + \sum_{i=0}^{q} \sum_{k=0}^{\mu-1} A_{00}^i A_{00}^k \right) \left( \bar{A}_{00}^i + e_j e_j^T \right) B_{i}^k > 0 \right\}; \forall j \in \mathcal{N} \),

where \( \chi_j^k = \left( \chi_j^k(1), \chi_j^k(2), \ldots, \chi_j^k(q) \right)^T \in \mathbb{R}^v \) with \( \chi_j^k(0) = 0 \in \mathbb{R}^v \), \( \chi_j^k(1) = 1 \) if \( i \in \left( \bigcup_{k=0}^{\mu-1} n_{\text{null}} \right) \), \( \forall j \in \mathcal{N} \). Then, a sufficient condition for the system to be excitable from the combined control input and bilinear input action is that \( n_u = \bigcup_{k=0}^{\mu-1} n_{\text{null}} \). A sufficient condition for strong combined excitability is quite similar by redefining the sets \( n_u \) and \( n_v; \forall \theta \in Q \cup \{0\}, \forall s \in Q \cup \{0\} \), by removing from the definitions of \( n_u \) and \( n_v \), the summations \( \sum_{i=0}^{q} (\cdot) \) and \( \sum_{k=0}^{\mu-1} (\cdot) \). Sufficient conditions for strong output excitability and output excitability are also direct under small "ad hoc" modifications.

**Proof** Assume with no loss of generality that \( \bar{A}_{00}^i (t) \geq 0; \forall \theta \in Q \cup \{0\}, \forall t \in \mathcal{R}_+ \) and that the fundamental matrix function \( \psi(g, \cdot) \) is calculated from Theorem 1 by replacing the parameterization of the original system by that of its associated one. If this were not the case, since \( A_0 (t) \geq 0; \forall \theta \in Q \cup \{0\}, \forall t \in \mathcal{R}_+ \), it could be that there are (non-unique) additive decompositions of the forms \( A_0 (t) = \bar{A}_{00}^i + \bar{A}_0 (t) \geq 0 \) satisfying \( \bar{A}_{00}^i \cup \bar{A}_0 (t) = \bar{A}_0 = \bar{A}_0 + \bar{A}_1 (t) \geq 0; \forall \theta \in Q \cup \{0\}, \forall t \in \mathcal{R}_+ \), so that a valid fundamental matrix for the homogeneous associated system can be calculated via Theorem 1 with the replacements \( A_0 \to \bar{A}_{00}^i, \bar{A}_0 (t) \to \bar{A}_1 (t); \forall \theta \in Q \cup \{0\}, \forall t \in \mathcal{R}_+ \), to be used in the subsequent formulas of the proof.

One has from Theorem 1 for \( u: \mathcal{R}_+ \to \mathbb{R}^v \) and \( \nu \in C \), that the state of the associated system, under corresponding normalized non-negative initial conditions, is described by:

\[
\begin{align*}
\chi^0(t) &= \psi(g,t) \psi^0 + \sum_{i=0}^{q} \int_{h_i}^{t} \psi(g,t - \sigma) A_{00}^i \phi^g(\sigma - h_j) d\sigma \\
&\quad + \sum_{i=0}^{q} \left[ \sum_{j=0}^{q} B_j^i \left( \psi(g,t - \sigma) \right) \right] d\sigma \tag{31.a} \\
&\geq \int_{h_i}^{t} \psi^0(t - \sigma) \left[ \sum_{j=0}^{q} B_j^i \left( \psi(g,t - \sigma) \right) \right] d\sigma; \forall t \in \mathcal{R}_+ \tag{31.b}
\end{align*}
\]

Since each entry of each of the matrices \( \bar{A}_{00}^i, E(t), B(t), C(t) \) and \( D(t); \forall \theta \in Q \cup \{0\}, \forall \nu \in Q \cup \{0\} \) is either null or non-zero, their corresponding matrices of the associate system are constant for all time and since the conditions on the parameters guarantee the positivity which lead to the inequalities (31.a)–(31.c) under any non-negative initial conditions and any non-negative control input as well as the time-invariance of the associated system, since \( \nu \in C \), with

\[
\begin{align*}
\psi(g,t) &= \psi^0 \left( I_n + \sum_{i=0}^{q} \int_{h_i}^{t} e^{-A_{00}^i (t - \sigma)} A_{00}^i \psi^0(\sigma - h_j) d\sigma \right) \\
&= \sum_{k=0}^{\mu-1} a_k(t) A_{00}^k + \sum_{k=0}^{\mu-1} \sum_{i=0}^{q} \left( \int_{h_i}^{t} a_k(t - \sigma) A_{00}^i A_{00}^k \psi^0(\sigma - h_j) U(\sigma - h_j) d\sigma \right) \tag{32.a} \\
&\geq \sum_{k=0}^{\mu-1} a_k(t) A_{00}^k + \sum_{k=0}^{\mu-1} \sum_{i=0}^{q} \left( \int_{h_i}^{t} a_k(t - \sigma) A_{00}^i A_{00}^k e^{A_{00}^i (t - \sigma)} U(\sigma - h_j) d\sigma \right) \\
&= \sum_{k=0}^{\mu-1} a_k(t) A_{00}^k + \sum_{k=0}^{\mu-1} \sum_{i=0}^{q} \left( \int_{h_i}^{t} a_k(t - \sigma) a_i(\sigma - h_j) A_{00}^i A_{00}^k U(\sigma - h_j) d\sigma \right) \tag{32.b}
\end{align*}
\]
with \( \Psi^{(G)}(t) = 0 \) for \( t \in [-h, 0) \), since from (5), \( \Psi^{(G)}(t) \geq e^{\Lambda^{(G)} t} = \sum_{k=0}^{n-1} a_k(t) A^{(G)}_k \) since \( A^{(G)}_k \geq 0; \forall j \in q \cup \{0\} \), \( \forall t \in R_{a0} \) where \( m (\mu, n) \leq n \) is the degree of the minimal polynomial of \( A^{(G)}_0 \) and \( \{ a_k(t); k \in \mu - T \cup \{0\}; t \in R_{a0} \} \) is a linearly independent set of real functions of real domain which are everywhere infinitely time-differentiable with respect to time. The use of (32) in (31.c) leads to

\[
\Psi^{(G)}(t) \geq \sum_{k=0}^{n-1} a_k(t) A^{(G)}_k = e^{\Lambda^{(G)} t} \sum_{k=0}^{n-1} a_k(t) A^{(G)}_k \geq 0; \forall j \in q \cup \{0\} \), \( \forall t \in R_{a0} \) where \( m (\mu, n) \leq n \) is the degree of the minimal polynomial of \( A^{(G)}_0 \) and \( \{ a_k(t); k \in \mu - T \cup \{0\}; t \in R_{a0} \} \) is a linearly independent set of real functions of real domain which are everywhere infinitely time-differentiable with respect to time. The use of (32) in (31.c) leads to

\[
x^{(G)}(t) = \int_0^t \Psi^{(G)}(t - \sigma) \left( \sum_{k=0}^{n-1} a_k(t - \sigma) A^{(G)}_k B^{(G)}_r \right) d\sigma; \forall t \in R_{a0} \tag{33}
\]

If \( h \) and \( h' \) are sufficiently small such that some real \( t \) exists which satisfies the constraint

\[
\max (h, h') < h < h_1 + \epsilon \tag{34}
\]

for \( \sigma \in [0, t] \) and, since \( a_k(t) > 0; \forall k \in \mu - \{0\} \) for \( t \in (\max (h, h'), h_1 + \epsilon) \), (34) holds equivalently

\[
\sum_{q=0}^{r} \sum_{k=0}^{n-1} a_k(t - \sigma) A^{(G)}_k B^{(G)}_r \geq 0 \tag{35}
\]

for \( \sigma \in [0, t] \) and, since \( a_k(t) > 0; \forall k \in \mu - \{0\} \) for \( t \in (\max (h, h'), h_1 + \epsilon) \), (34) holds equivalently

The proof of the sufficiency part of Property (i) follows directly from those considerations and Property (i) leading to the three given pairs of sufficiency-type conditions from the strongest one to the weakest one since if \( \sum_{k=0}^{n-1} B_k > 0 \) and monomial then \( \sum_{k=0}^{n-1} B_k \) is monomial and positive, so that it is non-singular with just a non-zero positive entry per row and per column, then its
pre-multiplication by any strictly positive matrix, as those in the given conditions, yields a strictly positive matrix as a result. The necessity follows by a close argument as that used for the proof of necessity of Property (i) got from the further weakest sufficient conditions of excitability and component-wise excitability:

\[
\sum_{j=0}^{q} \sum_{h=0}^{n-1} A_{h0}^{(j)} \left( I_n + \sum_{j=1}^{q} \sum_{h=0}^{n-1} A_{h0}^{(j)} A_{00}^{(j)} \right) B_{0j}^{(j)} \gg 0
\]

\[
\sum_{j=0}^{q} \sum_{h=0}^{n-1} A_{h0}^{(j)} \left( I_n + \sum_{j=1}^{q} \sum_{h=0}^{n-1} A_{h0}^{(j)} A_{00}^{(j)} \right) B_{0j}^{(j)} \gg 0.
\]

The proofs of the sufficient parts of Properties (iii) and (iv) follow in a very close way as those of Properties (i)–(ii) via (2) and the corresponding output equation for its associated system since via (2) and the corresponding output equation for its associated system since since via (2) and the corresponding output equation for its associated system since the

Note from (36) and Properties [(i)–(ii)] that the given double condition guarantee the excitability of the component-wise excitability:

\[\delta(t) \geq \sum_{j=0}^{q} \int_{0}^{t} \psi^{(j)}(t - \sigma) \left( \tilde{A}_0^{(j)}(\sigma) x_{0}^{(j)}(\sigma - h_j) + E_j^{(j)} x_{0}^{(j)}(\sigma - h_j) \right) d\sigma \]  

(36)

for \( t \in (\max (h, h'), h_1 + \epsilon) \). Note that (36) does not leads to excitability since the system initially at rest implies that the state is identically zero for identically zero control input Comparing with the parametrical constraint specifying Property (i) which guarantees (33) via (35), the result is direct.

Note from (36) and Properties [(i)–(ii)] that the given double condition guarantee the excitability of the state components from the control input and that of the \((n - n_0)\) remaining state components from the bilinear input action through the activated links of the form \((\tilde{A}_0^{(j)} + E_j^{(j)} e_j)^{\Delta} \).

**Remark 4.** Consider the membership set of the bilinear input action defined by:

\[ C_{\text{vict}}(t_0) = \left\{ v : R_{0+} \to R^{q+1} : (v(t))_k \geq \delta(t) - \min_{i \text{ rel}} \left( \frac{\tilde{A}_i(t)}{(E_i(t))} \right) ; \forall k \in \bar{n}, \forall j \in q \cup \{0\}, \forall t \in R_{0+} \right\} \]

for any \( t_0 \in R_+ \) and some \( \delta : R_0+ \to R_{0+} \) such that \( \delta(t) > 0 \) within some, non-necessarily connected, interval \( T_0 \subset [t_0, \infty) \) of non-zero measure and, in addition, assume that \( v \in C_{\text{r}} \). Note that Theorem 5 [(iv)–(vii)] are also fulfilled under the weaker condition that \( v \in C_{\text{vict}}(t_0) \) for \( t_0 \in (\max (h, h'), h_1 + \epsilon) \).

Definition 3 (transparency). A positive bilinear system \((1)–(2)\) is said to be transparent if each output component of the homogeneous system can be made positive for some given non-negative appropriate function of initial conditions, i.e. for the case \( u(t) = v(t) = 0 \) for \( t \in R_{0+}, j \in q \cup \{0\} \).

Transparency being achievable from any function of initial conditions will be referred to as strong transparency.

The following result holds:

**Theorem 6.** Consider the system \((1)–(2)\) with \( A_0 \in M^0_{0}, A_0 > 0, A_i(t) = (A_0 + \tilde{A}_i(t)) \geq 0, E_i(t) \geq 0, B_j(t) \geq 0; \forall t \in R_{0+}, \forall i \in q \cup \{0\}, \forall j \in q \cup \{0\}, \forall \sum_{i=0}^{q} B_i > 0, C > 0, D > 0, and v \in C_{\text{r}} \) with

\[ C_{v} = \left\{ v : R_{0+} \to R^{q+1} : (v(t))_k \geq -\min_{i \text{ rel}} \left( \frac{\tilde{A}_i(t)}{(E_i(t))} \right) ; \forall k \in \bar{n}, \forall j \in q \cup \{0\}, \forall t \in R_{0+} \right\} \]
Assume that each entry of each of the matrices $\bar{A}_q(t)$, $E_i(t)$, $B_i(t)$, $C(t)$ and $D(t)$, $\forall t \in q \cup \{0\}$, $\forall j \in q' \cup \{0\}$ is either null or non-zero for all time. Assume also that the maximum internal and external delays $\bar{h}$ and $h'$ are sufficiently small such that some real $t$ exists which satisfies the constraint $\max(\bar{h}, h') < t < \bar{h} + \varepsilon$ with $\varepsilon$ defined in Lemma A.1 (Appendix A).

Then, the following properties hold:

(i) A sufficient condition for the system to be strongly transparent, and then transparent, is:

$$\sum_{k=0}^{n-1} C(t)A^{(G)}_{t0} \left( I_n + \sum_{j=1}^{q-1} \sum_{i=0}^{n-1} A^{(G)}_{0j} \left( A^{(G)}_{10} + \sum_{r \in \mathbb{R}} A^{(G)}_{0r'} \right) \right) \gg 0; \; \forall s \in n$$

A necessary and sufficient condition for the $i$th output component to be strongly transparent, and then transparent, is:

$$\sum_{k=0}^{n-1} C(t)A^{(G)}_{0i} \left( I_n + \sum_{j=1}^{q-1} \sum_{i=0}^{n-1} A^{(G)}_{0j} \left( A^{(G)}_{10} + \sum_{r \in \mathbb{R}} A^{(G)}_{0r'} \right) \right) \gg 0; \; \forall s \in n$$

(ii) A sufficient condition for the system to be transparent is:

$$\sum_{k=0}^{n-1} C(t)A^{(G)}_{t0} \left( I_n + \sum_{j=1}^{q-1} \sum_{i=0}^{n-1} A^{(G)}_{0j} \left( A^{(G)}_{10} + \sum_{r \in \mathbb{R}} A^{(G)}_{0r'} \right) \right) \gg 0$$

(iii) A necessary and sufficient condition for the system to be transparent is that

$$\sum_{k=0}^{n-1} C(t)A^{(G)}_{0i} \left( I_n + \sum_{j=1}^{q-1} \sum_{i=0}^{n-1} A^{(G)}_{0j} \left( A^{(G)}_{10} + \sum_{r \in \mathbb{R}} A^{(G)}_{0r'} \right) \right) \gg 0$$

Proof From (2), (31.a) and (32.b), one gets if $u(t) \in \mathbb{R}_0^m$, and $v \in C_{\infty}$; $\forall t \in \mathbb{R}_0^\infty$, $\forall v \in \mathbb{R}_0$, that

$$y^{(G)}(t) \geq C^{(G)}y^{(G)}(t)x^{(G)}_0 + C^{(G)} \sum_{j=1}^{\infty} \int_{-\bar{h}}^{0} y^{(G)}(t - \sigma)A^{(G)}_{0j} z^{(G)}(\sigma - h_x) d\sigma$$

$$+ C^{(G)} \sum_{r \in \mathcal{R}} \int_{0}^{\infty} y^{(G)}(t - \sigma)A^{(G)}_{0r'} (s - h_r) d\sigma$$

$$\geq \left[ \sum_{k=0}^{n-1} a_k(t)C^{(G)}A^{(G)}_{t0} \right] + \sum_{k=0}^{n-1} \sum_{i=0}^{n-1} \sum_{r \in \mathcal{R}} \left[ \int_{0}^{\infty} a_k(t - \sigma)A^{(G)}_{t0} A^{(G)}_{0j} A^{(G)}_{10} U(\sigma - h_x) d\sigma \right] x^{(G)}_0$$

$$+ \sum_{r \in \mathcal{R}} \int_{0}^{\infty} \sum_{j=1}^{\infty} \sum_{i=0}^{n-1} \sum_{r \in \mathcal{R}} \int_{0}^{\infty} \left[ \sum_{d=1}^{\infty} q_{d} (t - \sigma) A^{(G)}_{t0} A^{(G)}_{0j} A^{(G)}_{10} U(\sigma - h_x) d\sigma \right] \left[ A^{(G)}_{0j} \psi^{(G)}(\sigma - h_x) d\sigma \right]$$

$$= \sum_{k=0}^{n-1} \left[ \sum_{i=0}^{n-1} a_k(t)C^{(G)}A^{(G)}_{t0} \right] + \sum_{k=0}^{n-1} \sum_{i=0}^{n-1} \sum_{r \in \mathcal{R}} \left[ \int_{0}^{\infty} a_k(t - \sigma)A^{(G)}_{t0} A^{(G)}_{0j} A^{(G)}_{10} U(\sigma - h_x) d\sigma \right] x^{(G)}_0$$

$$+ \sum_{r \in \mathcal{R}} \int_{0}^{\infty} \sum_{j=1}^{\infty} \sum_{i=0}^{n-1} \sum_{r \in \mathcal{R}} \int_{0}^{\infty} \left[ \sum_{d=1}^{\infty} q_{d} (t - \sigma) A^{(G)}_{t0} A^{(G)}_{0j} A^{(G)}_{10} U(\sigma - h_x) d\sigma \right] \left[ A^{(G)}_{0j} \psi^{(G)}(\sigma - h_x) d\sigma \right]$$

$$+ \sum_{r \in \mathcal{R}} \int_{0}^{\infty} \sum_{j=1}^{\infty} \sum_{i=0}^{n-1} \sum_{r \in \mathcal{R}} \int_{0}^{\infty} \left[ \sum_{d=1}^{\infty} q_{d} (t - \sigma) A^{(G)}_{t0} A^{(G)}_{0j} A^{(G)}_{10} U(\sigma - h_x) d\sigma \right] \left[ A^{(G)}_{0j} \psi^{(G)}(\sigma - h_x) d\sigma \right]$$

(37)
Since \( h \) and \( h' \) are sufficiently small such that some real \( t \) exists which satisfies the constraint 
\[
\max \left( h, h' \right) = \frac{\mu - 1}{1 + \epsilon} \text{ for } \epsilon \text{ defined in Lemma } A.
\]
Thus, the real functions of the set 
\[
\left\{ t_k(t), k \in \mathbb{N} - \{0\}, t \in \mathbb{R}_{0+} \right\}
\]
are positive on the real interval \((0, \infty)\). The proofs of the sufficiency part of Properties ([i]–[iii]) follow directly from (38)–(39). The proof of necessity of Property (i) is close to its counterpart of Theorem 5(i). On the other hand, one has from (37) that 
\[
y^{(0)}(t) \geq C^{(0)} \psi^{(0)}(t) x_0^{(0)} + C^{(0)} \sum_{i=1}^{q-1} \int_{h_i}^{t} \psi^{(0)}(t - \sigma) A^{(0)} \varphi^{(0)}(\sigma - h_j) d\sigma
\]
\[
+ C^{(0)} \sum_{r=0}^{q-1} \int_{t}^{	au} \psi^{(0)}(t - \sigma) \tilde{A}^{(0)} \left[ \psi^{(0)}(\sigma - h_j) x_0 + \sum_{i=1}^{q} \int_{h_i}^{\sigma} \psi^{(0)}(\sigma - h_j - \theta) A^{(0)} \varphi^{(0)}(\theta - h_j) d\theta \right] d\sigma
\]
and, using (32.b) for the lower bound of the fundamental matrix, the sufficiency part of Property (iii) follows. The proof of necessity follows under close arguments to those used in the proof of the necessary part of Theorem 5(i). □

Note that, in the discrete-time case, the excitability and transparency conditions of a positive system can be equivalently got from the system itself, instead of from its associated one, since the matrix of dynamics has to be positive instead of simply Metzler. The subsequent example addresses these facts.

Example 1. Consider the following discrete bilinear system with a single one-step delay:
\[
x_{k+1} = \Phi_0 x_k + \Phi_1 x_{k-1} + B_0 x_k v_k + B_1 x_{k-1} v_{k-1} + B_2 u_k; v_k = C x_k; \quad \forall k \in \mathbb{Z}_{0+},
\]
under initial conditions \( x_i = 0 \) for \( i = -1, 0 \), with the non-negative scalar bilinear action sequence \( \{ v_k \}_{k \in \mathbb{Z}_{0+}} \subseteq \mathbb{R}_{0+} \), where \( C \in \mathbb{R}^{m \times n} \) and \( \Phi_0, \Phi_1, B_0, B_1 \in \mathbb{R}^{m \times m}, \quad B_2 \in \mathbb{R}^{m \times 1}, \quad i = 0, 1 \) are non-zero. It follows that the system is not excitable from any bilinear action sequence since \( x_i = 0; \forall k \in \mathbb{Z}_{0+} \), if \( u_i = 0; \forall k \in \mathbb{Z}_{0+}. \)

Equation (41) maybe described equivalently through an extended system of dimension \( 2n \) and state \( \bar{x}_k = (x_k', x_{k-1}')^T \) given by:
\[
\bar{x}_{k+1} = \Phi \bar{x}_k + \bar{B} u_k; \quad \forall k \in \mathbb{Z}_{0+},
\]
where
\[
\Phi = \Phi_1 (\{ v_k \}) = \begin{bmatrix} \Phi_0 + B_0 v_k & \Phi_1 + B_1 v_{k-1} \\ I_n & 0_{m \times n} \end{bmatrix}; \quad \bar{B} = \begin{bmatrix} B_2 \\ 0 \end{bmatrix}; \quad \forall k \in \mathbb{Z}_{0+}.
\]
so that
\[
\bar{x}_{k+2n} = \prod_{j=0}^{k-2} \tilde{\Phi}_j \bar{x}_k + \tilde{C}_k \bar{u}_k; \quad \forall k \in \mathbb{Z}_{0+},
\]
where \( \bar{u}_k' = (u_{k+2n-1}, u_{k+2n-2}, \ldots, u_k) \in \mathbb{R}^{2m}. \)

Equation (44) establishes the following result:

THEOREM 7 The following properties hold:

(i) Assume that, in addition, \( \text{rank} (\tilde{C}_k) = 2n \) and that is monomial for a given bilinear action sequence \( \{ v_k \} \subseteq \mathbb{R}_{0+} \) for some \( k \in \mathbb{Z}_{0+} \), where \( \tilde{C}_k \) is the controllability matrix with respect to the control input sequence \( \{ u_k \} \) defined by:
\[
\tilde{C}_k = \tilde{C}_k (\{ v_k \}) = \begin{bmatrix} \bar{B} & \Phi_{k+2n-1} \bar{B} & \Phi_{k+2n-2} \bar{B} & \cdots & \Phi_{k+1} \bar{B} \end{bmatrix} \prod_{j=0}^{k-2n-1} \tilde{\Phi}_j \bar{B}
\]
Then, the bilinear system (41) is reachable from the control input in the interval \([k, k+2n]\) for the
given bilinear action sequence \(\{v_i\}\), in the sense that there is some control input sequence
\(\{u_i\}_{k}^{k+2n-1} \subset \mathbb{R}_{o+}^{m}\), such that, if \(x_k = 0\) then \(x_{k+2n} = x^r\) for any given prefixed \(x^r > 0\). The conditions that
\(C_k\) is full rank and \(\hat{C}_k \hat{C}_k^T\) is monomial are also necessary for controllability on \([k, k+2n]\).

Under these conditions, the system is also excitable by some control input for each given \(x^r > 0\).

(ii) If \(\hat{C}_k \hat{C}_k^T\) is monomial and \(\text{rank}(\hat{C}_k) = 2n\) \(\forall k \in \mathbb{Z}_{o+}\), then the system (41) is uniformly reachable \(\forall k \in \mathbb{Z}_{o+}\).

(iii) Assume that, for some given admissible set \(\Omega = \{\Omega_k, \Omega_{k+1}, \ldots, \Omega_{k+2n}\}\) of finite dimension \(m\), the
bilinear input action \(v_i \in \Omega \subset \mathbb{R}_{o+}^{m}\); \(\forall k \in \mathbb{Z}_{o+}\). Then, the system is reachable in the discrete-time interval \([k, k+2n]\) from a combined control input and admissible bilinear input action if and only if there is a full rank controllability matrix in-between \(m^{\text{new}}\) possible ones of the form:

\[
\hat{C}_k = \hat{C}_k(\{v_i\}) = \begin{bmatrix} B \Phi_{k+1} & B \Phi_{k+2} & \cdots & B \Phi_{k+2n-1} \end{bmatrix} \begin{bmatrix} \Phi_{k+1} & \Phi_{k+2} & \cdots & \Phi_{k+2n-1} \end{bmatrix},
\]

(46)

with \(\Phi_i = \{v_i, v_{i+1}, \ldots, v_k\}, v_i \in \Omega; i = k, k+1, \ldots, k+2n - 1\).

**Proof** Since \(\text{rank}(\hat{C}_k) = 2n\) then \(\hat{C}_k \hat{C}_k^T\) is non-singular. One gets from (41) that, if \(\check{u}_k = \hat{C}_k^T g\) for some \(g\neq 0 \in \mathbb{R}_{o+}^{m}\), then

\[
\check{x}_{k+2n} = \sum_{i=k}^{k+2n-1} \hat{C}_k \check{u}_i + x_k \check{u}_i \hat{C}_k^T g; \ k \in \mathbb{Z}_{o+}.
\]

(47)

and \(\check{x}_{k+2n} = \check{x}^r\) if \(g = (\hat{C}_k \hat{C}_k^T)^{-1}\check{x}^r\) with \(\check{x}_k = 0\) so that \(\check{u}_k = \hat{C}_k^T (\hat{C}_k \hat{C}_k^T)^{-1} \check{x}^r\). Since \(\Phi_i, B_i \in \mathbb{R}_{o+}^{m}; i = 1, 2\) then \(\hat{C}_k > 0\) and \(\hat{C}_k \hat{C}_k^T\) is monomial then such a matrix and its inverse are both non-singular with one positive entry per row and column. Also, \(\check{u}_k > 0\) and \(\check{u}_k \neq 0\). Therefore, the system is controllable on \([k, k+2n]\). Sufficiency has been proved. The necessity follows from the un-solvability in the control input from the Rouche-Frobenius theorem, since being an incompatible algebraic system, of the state-targeting choice \(\check{x}^r = \check{x}_{k+2n} = \check{C}_k \check{u}_k\) if \(x^r > 0\) then \(\check{u}_k = \hat{C}_k^T (\hat{C}_k \hat{C}_k^T)^{-1} \check{x}^r\). Since the inverse of a positive monomial matrix is a positive monomial matrix) makes all the state components to be positive in finite time for the system initially at rest, i.e. if \(\check{x}_k = 0\). Property (i) has been fully proved and Property (ii) is a direct consequence of Property (i). Property (iii) is also a consequence of Property (i) since there are \(m^{2n}\) variations with repetition admissible bilinear action sequences, with at least one of them being required to generate a full rank controllability matrix.

Now, assume that

\[
B_0 = \begin{bmatrix} 0_{m \times n} \\ b_0^T \end{bmatrix}; \ B_1 = \begin{bmatrix} 0_{m \times n} \\ b_1^T \end{bmatrix}
\]

(48)

where \(b_i = (b_{i1}, b_{i2}, \ldots, b_{im})^T; i = 1, 2\). The particular non-necessarily positive delay-free case was
discussed in Tarn (1972) in the context of non-positive system generating an identically zero solution under zero initial state. Note that the system can be also described by the extended system of state \(\check{x}_k = (\check{x}_k^T, \check{x}_{k-1}^T)^T\) as follows:

\[
\check{x}_{k+1} = \check{F} \check{x}_k + \check{F} \check{x}_{k-1}\; \check{u}_k; \ k \in \mathbb{Z}_{o+}
\]

(49)

where
\[
\Phi = \begin{bmatrix} \Phi_0 & \Phi_1 \\ I_n & 0_{n \times n} \end{bmatrix}; \quad \tilde{f} = \begin{bmatrix} B_0 & B_1 \\ 0_{n \times n} & 0_{n \times n} \end{bmatrix} = \begin{bmatrix} 0_{(n-1) \times n} & 0_{(n-1) \times n} \\ b_0' & b_1' \\ 0_{n \times n} & 0_{n \times n} \end{bmatrix} \tilde{X}_k = \text{Diag}(x_k, x_{k-1})
\] (50)

and (43) yields for any \(k, j \in \mathbb{Z}_{0+}\) proceeding recursively that

\[
\tilde{x}_{kj} = \tilde{\Phi}_k \tilde{x}_k + \sum_{i=k}^{j-1} \tilde{\Phi}_{kj-i-1} \tilde{f} \tilde{x}_i \tilde{v}_j
\] (51)

and then

\[
\tilde{x}_{2n+k} = \tilde{\Phi}_{2n+k} x_0 = \sum_{i=0}^{2n+k-1} \tilde{\Phi}_{2n+k-i} \tilde{f} \tilde{x}_i \tilde{v}_j; \quad k \in \mathbb{Z}_{0+}
\] (52)

The following result holds:

**THEOREM 8.** Assume that \(\ker \left[ \tilde{f}, \tilde{\Phi} \tilde{f}, \ldots, \tilde{\Phi}^{2n-1} \tilde{f} \right] \cap \mathbb{R}^{2n} \neq \{0\}\), i.e. the pair \((\tilde{\Phi}, \tilde{f})\) is uncontrollable. Then, there is a subclass of bilinear control actions belonging to the admissibility class such that the system (41) is not excitable under zero control input for the bilinear input actions in such a subclass.

**Proof** Note that, equivalently, it is assumed that \(\text{rank} \left[ \tilde{f}, \tilde{\Phi} \tilde{f}, \ldots, \tilde{\Phi}^{2n-1} \tilde{f} \right] < 2n\), i.e. there are infinitely many non-zero sequences \((\tilde{v}_j) \in \mathbb{R}^n\), at least, infinitely many real non-negative sequences \((u_k)\) such that \(\tilde{x}_{2n+k} \neq \tilde{\Phi}_{2n+k} x_0\) for any given \(x_{k-1} \in \mathbb{R}^n\). If \(x_0 = x_{k-1} = 0\) then \(x_k = 0\) from (45), and \(x_k = 0\), \(\forall k \in \mathbb{Z}_{0+}\), for the class defined by such bilinear input action sequences. It is obvious that infinitely many of them are in the admissibility class for positivity while they do not excite the state. The result has been proved. \(\square\)

Note that if the control input is identically zero then one gets from (47) that

\[
y_{k+2n} = \prod_{k=0}^{k+2n-1} \tilde{\Phi}_k \tilde{x}_k; \quad k \in \mathbb{Z}_{0+}
\] (53)

where \(\tilde{C} = \begin{bmatrix} C & 0_{p \times n} \end{bmatrix}\) and

\[
\Omega_k = \begin{bmatrix} \tilde{C} & \tilde{\Phi}_k T \tilde{C}^T \tilde{\Phi}_k T \tilde{C}^T \tilde{\Phi}_k T \tilde{C}^T \\
\end{bmatrix} \prod_{k=0}^{k+2n-1} \tilde{\Phi}_k T \tilde{C}^T
\] (54)

Thus, the following holds:

**THEOREM 9.** The system (41) is observable on \([k, k+2n]\) if and only if \(\text{rank} \Omega_k = 2n\). This condition also guarantees that the system is transparent although it is not a necessary condition for it.

**Proof** The necessity and sufficiency for observability is a standard condition for the extended system of dimension \(2n\). Since the observability matrix is non-singular and positive, there is some positive \(x_1 > 0\) such that \(y_1 > 0\) (it suffices to take any \(x_1 > 0\) but, since \(\Omega_k > 0\), the observability is not needed to guarantee transparency. \(\square\)

Note that the system is not strongly transparent on \([k, k+2n]\) since \(\text{rank} \Omega_k \gg 0\) fails since \(\tilde{C} \gg 0\) fails because of its structure.

**Example 2.** Now, consider the following hybrid-continuous-time system under a discrete-time feedback controller

\[
\dot{x}(t) = Ax(t) + B_1 u_k + B_0 v x_k + B_1 v x_{k-1}; \quad t \in [kT, (k+1)T]; \quad \forall k \in \mathbb{Z}_{0+}
\]

\[
u_k = u(kT) = K x_k + K_0 x_{k-1}
\]
\[ \nu_k = \nu(kT) = \partial \nu_{k-1}; \quad \forall k \in \mathbb{Z}_{0+} \]

subject to initial conditions \( x_i \in \mathbb{R}_{0+} \) for \( i = -1, 0 \) for some given \( T \in \mathbb{R}_+ \) and some given control gains \( K, K_0 \in \mathbb{R}^n_{0+} \) and \( \vartheta \in \mathbb{R}_{0+} \) or \( \vartheta \in \mathbb{R}^n_{0+} \), where \( A \in \mathbb{M}^n_{\text{non}} \), \( B_2 \in \mathbb{R}^n_{0+} \) and \( B_2 \in \mathbb{R}^n_{0+} \). Then,

\[ x_{k+1} = \begin{bmatrix} (B, K) & \eta^T \vartheta v_0 \end{bmatrix} x_k + \begin{bmatrix} (B, K_0, K_0) & \eta^T \vartheta v_0 \end{bmatrix} x_{k-1} \]

\[ \tilde{x}_{k+1} = \phi_k x_k \]

\[ y_{k+2n} = \tilde{C} \tilde{x}_{k+2n} = \tilde{C} \tilde{O} x_k \]

where

\[ \phi_k = \begin{bmatrix} \phi_0 + B_2 \Gamma K + B_2 \vartheta^T v_0 & \Phi_1 + B_1 \Gamma K_0 + B_1 \vartheta^T v_0 \end{bmatrix} I_n + 0_{n \times n} \]

and \( I' = \int_0^1 e^{AT - M} d\vartheta. \) Since \( A \in \mathbb{M}^n_{\text{non}} \), \( e^A > 0 \) and also non-singular, since it is a fundamental matrix, and then \( I' > 0. \) Since, furthermore, \( B > 0; \quad i = 0, 1, 2 \) and \( K_0, K, \vartheta > 0; \quad C > 0; \phi_0 > 0 \) and \( \tilde{C} \tilde{O}_k > 0 \) for all \( k \in \mathbb{Z}_{0+} \), so the system is transparent for any given \( x_k >> 0. \) The discussion of potential extensions of the above examples to the positivity of dynamic systems subject to switching in-between several parameterizations and to the discretization of continuous-time systems under non-periodic sampling can be addressed directly being supported by some technical results proved in De la Sen (1983), De la Sen, Paz, and Luo (1998) and Ibeas, De La Sen, and Alonso-Quesada (2004). In particular, note that if the continuous-time matrix of dynamics of a given parameterization is a Meltzer one, then its associate state-transition (or fundamental) matrix is positive for each transition in-between any two consecutive samples and non-singular irrespective of the used sampling period sequence.
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Appendix A

Auxiliary Lemmas

Lemma A.1. There exists some $t_1 \in \mathbb{R}$, such that the functions $\{a_k(t): k \in \mathbb{Z} \cup \{0\}, t \in \mathbb{R}_+\}$ are positive on $(0, t_1)$ with $a_k(0) = 0$ for $k \in \mathbb{R}$ and $a_1(0) = 1$.

Proof. Note that

$$A_0^{(g)} e^{A_0^{(g)} t} = \sum_{k=0}^{\mu-1} a_k(t) A_0^{(g)} e^{A_0^{(g)} t} = \sum_{k=0}^{\mu-1} a_k(t) A_0^{(g)} e^{A_0^{(g)} t}$$

$$= \sum_{k=1}^{\mu-1} \left( a_{k-1}(t) + \mu_{k-1}(t) a_k(t) \right) A_0^{(g)} e^{A_0^{(g)} t} + a_{\mu-1}(t) a^{(\mu-1)}(t)$$

for all $t \in \mathbb{R}_+$ since $e^{A_0^{(g)} t} = \sum_{k=0}^{\mu-1} a_k(t) A_0^{(g)} e^{A_0^{(g)} t}$ with $A_0^{(g)} = \sum_{k=0}^{\mu-1} a_k A_0^{(g)}$ from Cayley-Hamilton theorem. The set $\{a_k(t): k \in \mathbb{Z} \cup \{0\}, t \in \mathbb{R}_+\}$ for the associated system is calculated from the subsequent linear algebraic system (see, for instance, De La Sen, 2002; De la Sen et al., 1998):

$$\frac{d}{dt} \begin{bmatrix} 1, \gamma, \ldots, \gamma^{\mu-1} \end{bmatrix} = \begin{bmatrix} a_0(t), a_1(t), \ldots, a_{\mu-1}(t) \end{bmatrix}^T$$

where $\{a_k(t) \in \mathbb{C} \in \mathbb{R}\}$ is the spectrum of the delay-free matrix of dynamics of the associated system $A_0^{(g)}$, and $\mu$ is the multiplicity of $\lambda_i$ in the minimal polynomial of $A_0^{(g)}$, of degree $\mu = \sum_{k=0}^{\mu-1} \mu_k$ for all $i \in \mathbb{R}$. If the minimal polynomial of $A_0^{(g)}$ is $p_\mu(s) = \det \left( sI - A_0^{(g)} \right) = s^\mu - \sum_{k=0}^{\mu} \mu_k s^k$ where $\mu_k$ are real constants which depend on the degree $\mu_k$ of such a minimal polynomial. The set $\{a_k(t): k \in \mathbb{Z} \cup \{0\}, t \in \mathbb{R}_+\}$ is a linearly independent set of functions on $\mathbb{R}_+$ which are everywhere continuously (and infinitely) differentiable with continuous-time derivatives. The identity $e^{A_0^{(g)} t} = \sum_{k=0}^{\mu-1} a_k(t) A_0^{(g)} t$ yields $a_0(t) = 1$, $a_1(t) = 0$ for $k \in \mathbb{Z} \cup \{0\}$. By comparing the above identities, one has due to the linear independence that

$$\dot{a}_k(t) = a_{k-1}(t) + \mu_k a_{k-1}(t); \quad \forall k \in \mathbb{R}_+$$

with initial conditions $a_0(0) = 1$, $a_k(0) = 0; \forall k \in \mathbb{R}_+$ Since $a_0(t)$ is everywhere continuous and $a_k(t)$ there is some $\varepsilon_0 \in \mathbb{R}$ such that $a_0(t) > 0$ for $t \in [0, \varepsilon_0)$ even in the event that $a_0(t) < 0$. Also, $a_k(0) = a_k(t)$ with $a_k(t) > 0$ for $t \in [0, \varepsilon_k)$. Thus, since the linearly independent set $\{a_k(t): k \in \mathbb{Z} \cup \{0\}, t \in \mathbb{R}_+\}$ is everywhere continuously differentiable with respect to time with everywhere continuous-time-derivatives, that there is some $\varepsilon_k \in \mathbb{R}_+$ such that $a_k(t) > 0$ for $t \in [0, \varepsilon_k)$. It has been noted that $a_k(t) > 1, a_k(t) = 0; \forall t \in \mathbb{R}_+$ Proceed by complete induction by assuming that for some given $k < \mu, a_k(t) > 0$ for $t \in [0, \varepsilon_k)$. Since $a_{k+1}(0) = 0, a_{k+1}(t) > 0$ for $t \in [0, \varepsilon_k)$. Proceed by complete induction by assuming that for any $A \in M_0^+$

Lemma A.2.

The following properties hold for any $A \in M_0^+$:
(i) $e^{At} >> 0$ for some $t \in R_+$ $\Rightarrow$ $e^{A't} >> 0; \forall t' \in [t, \infty)$.

(ii) If $e^{At} >> 0$ fails for some $t \in R_+$ then $e^{A't} > 0$ with non-strict positive for $\theta \in [0, t]$.

(iii) If $e^{At} >> 0$ for some $t \in R_+$ then there is $\theta \in (0, t]$ such that $e^{At} >> 0$ for $\sigma \in [\theta, \infty)$ and $e^{A'\sigma} >> 0$ for $\sigma \in [0, \theta]$.

(iv) If $\forall z_0 > 0, A(t) = (A_{00} + A(t)) \geq 0$ and $B_j(t) \geq 0; \forall t \in R_+$, $\forall \epsilon \in q \cup \{0\}, \forall j' \epsilon \in q \cup \{0\}$ then $\forall (t') >> 0; \forall t' \in (t, \infty)$ if $e^{At} >> 0$ for $t \in R_+$.

**Proof** Consider any $A \in M_n^R$ with obviously has a unique additive decomposition in a diagonal and an off-diagonal matrix as follows:

$$A = A_d + A_{od}$$

where $A_d = \text{Diag}(A)$ and $A_{od} = A - A_d$ has zero diagonal entries. Consider the auxiliary homogeneous system:

$$\dot{z}(t) = A_d z(t) + A_{od} z(t); \ z(0) = z_0$$

whose unique solution on $R_{od}$ is

$$z(t) = e^{(A_d + A_{od})t} z(0) = e^{A_d t} \left( x(0) + \int_0^t e^{-A_d \sigma} A_{od} z(\sigma) d\sigma \right) = e^{A_d t} \left( I_n + \int_0^t e^{-A_d \sigma} A_{od} e^{(A_d + A_{od})\sigma} d\sigma \right) z(0)$$

Since the above identity holds irrespective of the initial condition, the following identity is true

$$e^{(A_d + A_{od})t} = e^{A_d t} \left( I_n + \int_0^t e^{-A_d \sigma} A_{od} e^{(A_d + A_{od})\sigma} d\sigma \right); \ \forall t \in R_{od}$$

Note that

$$e^{At} = e^{A_d(t-t')} e^{A_{od} t'} \left( I_n + \int_0^t e^{-A_d \sigma} A_{od} e^{(A_d + A_{od})\sigma} d\sigma + \int_t^\infty e^{-A_d \sigma} A_{od} e^{(A_d + A_{od})\sigma} d\sigma \right)$$

$$\geq e^{A_d(t-t')} e^{A_{od} t'} \left( I_n + \int_0^t e^{-A_d \sigma} A_{od} e^{(A_d + A_{od})\sigma} d\sigma \right) > e^{A_d(t-t')} e^{A_{od} t'} \Rightarrow 0$$

for all $t' \in (t, \infty)$ and $\forall$ given $t \in R_+$ if $e^{At} >> 0$ for some $t \in R_+$ since $A_{od} \geq 0, e^{A_d(e^{-\theta^t})} > 0$ for $\theta' \in (0, t)$, $e^{A_{od}t} > 0$ for $\theta \in R_{od}$. Property (i) has been proved. Now, assume that $e^{At} > 0$ for some given $t \in R_+$ but non-strictly positive then $e^{At} > 0$ but non-strictly positive for $\sigma \in [0, t]$. Since $e^{A_d t} e^{A_{od} t'} e_j = 0, e^{A_d t} e^{A_{od} t'} e_i > 0$ for some $i, j \in [0, t]$. Property (ii) follows from Property (i) since $A_{od} \geq 0, A_d(t) = (A_{00} + A(t)) \geq 0$ and $B_j(t) \geq 0; \forall t \in R_+$, $\forall \epsilon \in q \cup \{0\}, \forall j' \epsilon \in q' \cup \{0\}$. Property (iii) is a direct consequence of Properties (i)-(ii) and the continuity of the entries of $e^{At}$ as functions of time on $R_{od}$. Property (iv) follows from Property (i) since $A_{od} > 0, A_d(t) = (A_{00} + A(t)) \geq 0$ and $B_j(t) \geq 0; \forall t \in R_+$, $\forall \epsilon \in q \cup \{0\}, \forall j' \epsilon \in q' \cup \{0\}$. □