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## APPLIED & INTERDISCIPLINARY MATHEMATICS | RESEARCH ARTICLE

# Mathematical analysis of HIV/AIDS infection model with Caputo-Fabrizio fractional derivative

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**Abstract:** This manuscript is concerned to the existence and stability of HIV/AIDS infection model with fractional order derivative. The corresponding derivative is taken in Caputo-Fabrizio sense, which is a new approach for such type of biological models. With the help of Sumudu transform, some new results are handled. Further for the corresponding results, existence theory and uniqueness for the equilibrium solution are provided via using nonlinear functional analysis and fixed point theory due to Banach.

**Subjects:** Advanced Mathematics; Applied Mathematics; Physics

**Keywords:** HIV/AIDS model; Caputo-Fabrizio derivative; Sumudu transform; existence and uniqueness

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### 1. Introduction

Acquired immune deficiency syndrome (AIDS) has developed into a global pandemic since the first patients were identified in 1981. Human immune deficiency virus HIV is a lenti virus that causes

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### PUBLIC INTEREST STATEMENT

In most of the situations in mathematical modeling of physical, biological process and phenomenon, the hereditary materials and memory terms cannot be explained accurately by using classical derivatives. As mathematical models are used in physics, biology, Earth science, meteorology and engineering disciplines (such as computer science, artificial intelligence, as well as in the social sciences. Therefore by using arbitrary order derivative one can easily explain the aforesaid terms with excellent way. Further by adopting Caputo Febrizo fractional derivative, the aforementioned terms can be further excellently described. In recent time, the investigation of dynamical systems through fractional derivative of Caputo Febrizo type is a warm area of research.



Samia Bushnaq

AIDS. This serious disease destroys the immune system of human being which produce life-threatening opportunistic infections in the body. It is reported that 38.6 million people currently live with HIV/AIDS infection, 4.1 million people have been newly infected and 2.8 million AIDS deaths occurred in 2005. HIV/AIDS is an epidemic disease which spreading continuously all over the world and there have been few generator which continue it. In fact numbers of virus in the blood is a great indicator for the stages of the disease. In some situation, these stages are meant to correspond to CD4+ T-cell count ranges. In a normal healthy individual's peripheral blood, the level of CD4+ T-cells is between 800 and 1,200/mm<sup>3</sup> and once this number fall to 200 or below in an HIV infected patient, the person is classified as having AIDS. The aforesaid viruses transfer from a calamitous illness into a chronic conditions. The said infection led to dramatic change in mobility and mortality from illness. Furthermore, despite these improvement on the biomedical front, the spreading of this epidemic continue and treatment remains unavailable to the overwhelming majority of those who require it. It causes expenditure of a very large amount of money in health care and research and destroy millions of peoples. In the medical field there have been many achievement for health care, but there is still no vaccine available for HIV/AIDS. For the determination of the transmission dynamics of HIV/AIDS disease, a mathematical model is a useful tool which also provide techniques to control the spread of these disease Ali, Zaman, and Alshomrani (2017). In Wang and Li (2006), a simple mathematical model introduced by Perelson for the primary infection with HIV/AIDS. In the field of mathematical modeling for HIV/AIDS infection this model has great importance and also many other models with HIV/AIDS infection have been proposed, which consider this model as their inspiration. Arqub and El-Ajou (2013), made modification in this model to study the evaluation of drug resistance. Sadegh Zebai et al. also present a mathematical model of HIV/AIDS infected and evaluate the stability of the model. According to clinical symptoms or viral load and CD4 C T cell count, 2–6 stages of infection before AIDS can be classified.

Mathematical models are powerful tools in this approach which help us to optimize the use of finite sources or simply to goal (the incidence of infection) control measures more impressively. The classical order model of HIV/AIDS is provided in Cai, Li, Ghosh, and Guo (2009) and given by

$$\begin{cases} \frac{dS(t)}{dt} = \mu k - c\beta(I + bJ)S - \mu S, \\ \frac{dI(t)}{dt} = c\beta(I + bJ)S - (\mu + k_1)I + \alpha J, \\ \frac{dJ(t)}{dt} = k_1 I - (\mu + k_2 + \alpha)J, \\ \frac{dA(t)}{dt} = k_2 J - (\mu + d)A. \end{cases} \quad (1)$$

where  $S$  represent number of individuals who are uninfected/suseptible,  $I$  represents number of individuals who are in the asymptomatic phase,  $J$  represent number of individuals who are in the symptomatic phase and  $A$  represent a full-blown AIDS group. The recruitment rate of the population is denoted by  $\mu k$ ,  $\mu$  is natural death rate,  $c$  represents the average number of contacts of an individual per unit time,  $\beta$  and  $b\beta$  are probability of disease transmission per contact by an infective in the first stage and in the second stage, respectively.  $k_1$  represent transfer rate of individuals from the asymptomatic phase to the symptomatic phase,  $k_2$  represent transfer rate of individuals from the symptomatic phase to the AIDS cases,  $\alpha$  is treatment rate from the symptomatic phase to the asymptomatic phase and  $d$  denoted the disease related death rate. The parameter unite can be taken in numbers of cell per cubic millimeters. Recently, to study mathematical models by using fractional order derivative is an important area of research. Because, it has been found that the area involving fractional order differential equations have significant applications in various disciplines of science and technology, we refer few of them in Caputo and Fabrizio (2015, 2016) and El-Saka (2014). In recent years, the fractional-order models were given much attentions, because the biological models that involved fractional-order derivative are more realistic and accurate as compared to the classical order models, for detail see Toledo-Hernandez, Rico-Ramirez, Iglesias-Silva, and

Diwekar (2014) and Wang, Yang, Ma, and Sun (2014). By adapting fractional order derivative in mathematical modeling is a global operator as compared to classical derivative which is local. Motivated by the above work, in this manuscript, we considered the model discussed in Cai et al. (2009), by taking the Caputo-Fabrizio fractional derivative of the system.

The model that we study in this paper is a Caputo-Fabrizio derivative and fractional order  $\beta$  such that  $\beta \in (0, 1]$  as given below

$$\begin{cases} {}_0^{\text{CF}}D_t^\beta S(t) = \mu k - c\beta(I + bJ)S - \mu S, \\ {}_0^{\text{CF}}D_t^\beta I(t) = c\beta(I + bJ)S - (\mu + k_1)I + \alpha J, \\ {}_0^{\text{CF}}D_t^\beta J(t) = k_1 I - (\mu + k_2 + \alpha)J, \\ {}_0^{\text{CF}}D_t^\beta A(t) = k_2 J - (\mu + d)A. \end{cases} \quad (2)$$

Taking  $\beta = 1$ , one can get the classical model as given in (8). With the help of Sumudu transform some new results are handled which demonstrate existence and uniqueness of equilibrium solutions to the proposed model. By existence of equilibrium solutions, we concluding the wellposedness of the proposed model. Since the Caputo fractional derivative contains singular kernel which often cannot explain many phenomenons excellently. Therefore, to describe various process and phenomenons of biology, and physical science, the Caputo-Fabrizio derivative and fractional can excellent describes the aforesaid. For further characteristics and features of the aforesaid derivative, one can read the articles refer as Atangana1 and Torkhan (2015) and Baleanu, Mousalou, and Rezapour (2017).

## 2. Preliminaries

We first give the definitions of Caputo-Fabrizio derivative of fractional order. Caputo-Fabrizio derivative with fractional order has been considered with no singular kernel Caputo and Fabrizio (2015) and Losada and Nieto (2015).

*Definition 2.1* Let  $\phi \in H^1(a, b)$ ,  $b > a$ ,  $\beta \in (0, 1)$  then the new fractional order in Caputo derivative sense is recalled as

$${}^C D_t^\beta(\phi(t)) = \frac{M(\beta)}{1 - \beta} \int_a^t \phi'(x) \exp\left[-\beta \frac{t-x}{1-\beta}\right] dx,$$

where the normalization function is denoted by  $M(\beta)$  with  $M(0) = M(1) = 1$ . But, if the function does not belong to  $H^1(a, b)$ , then the derivative can be reformulated as

$${}^C D_t^\beta(\phi(t)) = \frac{\beta M(\beta)}{1 - \beta} \int_a^t (\phi(t) - \phi(x)) \exp\left[-\beta \frac{t-x}{1-\beta}\right] dx.$$

*Remark 2.2* If, we take  $\sigma = \frac{1-\beta}{\beta} \in [0, \infty)$ ,  $\beta = \frac{1}{1+\sigma} \in [0, 1]$ , then we will get the new Caputo derivative having fractional order as

$${}^C D_t^\beta(\phi(t)) = \frac{M(\sigma)}{\sigma} \int_a^t \phi'(x) \exp\left[-\frac{t-x}{\sigma}\right] dx, \quad M(0) = M(\infty) = 1.$$

In addition,

$$\lim_{\sigma \rightarrow 0} \frac{1}{\sigma} \exp\left[-\frac{t-x}{\sigma}\right] = \delta(x - t).$$

The connected anti-derivative turns out to be imperative at this instant following to the preface of the novel derivative, which was proposed by Losada and Nieto (2015).

**Definition 2.3** Let  $\beta \in (0, 1)$ . Then the fractional integral of order  $\beta$  of a function  $f$  is defined by

$${}^{CF}I_t^\beta(\phi(t)) = \frac{2(1-\beta)}{(2-\beta)M(\beta)} + \frac{2\beta}{(2-\beta)M(\beta)} \int_0^t \phi(s)ds, \quad t \geq 0.$$

**Remark 2.4** According to above definition, it has to be noted that the fractional integral of Caputo type of a function of order  $0 < \beta \leq 1$  is an average between function  $\phi$  and its integral of order one. This therefore imposes

$$\frac{2(1-\beta)}{(2-\beta)M(\beta)} + \frac{2\beta}{(2-\beta)M(\beta)} = 1.$$

The above equation generate an explicit formula for

$$M(\beta) = \frac{2}{(2-\beta)}, \quad 0 \leq \beta \leq 1$$

Keep in view the above derivative, reformulated the new Caputo derivative of order  $0 < \beta < 1$  as

$${}^CD_t^\beta(\phi(t)) = \frac{1}{1-\beta} \int_a^t \phi'(x) \exp\left[-\beta \frac{t-x}{1-\beta}\right] dx.$$

**Definition 2.5** For any function  $\phi(t)$  over a set, the Sumudu transform will be given as

$$\mathbf{A} = \{\phi(t): \text{there exist } \Lambda, \tau_1, \tau_2 > 0, |\phi(t)| < \Lambda \exp\left(\frac{|t|}{\tau_i}\right), \text{ if } t \in (-1)^j \times [0, \infty)\}$$

is defined by

$$F(u) = S[\phi(t)] = \int_0^\infty \exp(-t)\phi(ut)dt, \quad u \in (-\tau_1, \tau_2).$$

**Definition 2.6** The Sumudu transform of ordinary Caputo fractional-order derivative of a function  $\phi(t)$  is given by

$$S[{}^CD^\beta \phi(t)] = u^{-\beta} \left[ F(u) - \sum_{i=0}^n u^{\beta-i} \left[ {}^CD^{\beta-i}(\phi(t)) \right]_{t=0} \right], \quad n-1 < \beta \leq n.$$

In view of Definition 2.6, we recall the Sumudu transform of a function  $\phi(t)$  with Caputo-Fabrizio fractional derivative as below:

**Definition 2.7** Atangana and Alkahtani (2015) Let  $\phi(t)$  be a function for which the Caputo-Fabrizio exists, then the Sumudu transform of the Caputo-Fabrizio fractional derivative of  $f(t)$  is given as

$$S({}_0^{CF}D_t^\beta)(\phi(t)) = M(\beta) \frac{S(\phi(t)) - \phi(0)}{1 - \beta + \beta u}.$$

### 3. Derivation of the special solution

The aim of this section is to provide a special solution of the system (2) by applying the Sumudu transform on both sides of all equations of (2) together with an iterative method. To get this, we proceed as:

Applying Sumudu transform on both sides of proposed model (2), we obtain

$$\begin{aligned}
 M(\beta) \frac{S(S(t)) - S(0)}{1 - \beta + \beta u} &= S[\mu k - c\beta(I + bJ)S - \mu S], \\
 M(\beta) \frac{S(I(t)) - I(0)}{1 - \beta + \beta u} &= S[c\beta(I + bJ)S - (\mu + k_1)I + \alpha J], \\
 M(\beta) \frac{S(J(t)) - J(0)}{1 - \beta + \beta u} &= S[k_1 I - (\mu + k_2 + \alpha)J], \\
 M(\beta) \frac{S(A(t)) - A(0)}{1 - \beta + \beta u} &= S[k_2 J - (\mu + d)A].
 \end{aligned}$$

Rearranging, we obtain

$$\begin{aligned}
 S(S(t)) &= S(0) + \frac{(1 - \beta + \beta u)}{M(\beta)} S[\mu k - c\beta(I + bJ)S - \mu S], \\
 S(I(t)) &= I(0) + \frac{(1 - \beta + \beta u)}{M(\beta)} S[c\beta(I + bJ)S - (\mu + k_1)I + \alpha J], \\
 S(J(t)) &= J(0) + \frac{(1 - \beta + \beta u)}{M(\beta)} S[k_1 I - (\mu + k_2 + \alpha)J], \\
 S(A(t)) &= A(0) + \frac{(1 - \beta + \beta u)}{M(\beta)} S[k_2 J - (\mu + d)A].
 \end{aligned} \tag{3}$$

Now applying the inverse Sumudu transform on both sides of Equation (3), we obtain

$$\begin{aligned}
 S(t) &= S(0) + S^{-1} \left[ \frac{(1 - \beta + \beta u)}{M(\beta)} S[\mu k - c\beta(I + bJ)S - \mu S] \right], \\
 I(t) &= I(0) + S^{-1} \left[ \frac{(1 - \beta + \beta u)}{M(\beta)} S[c\beta(I + bJ)S - (\mu + k_1)I + \alpha J] \right], \\
 J(t) &= J(0) + S^{-1} \left[ \frac{(1 - \beta + \beta u)}{M(\beta)} S[k_1 I - (\mu + k_2 + \alpha)J] \right], \\
 A(t) &= A(0) + S^{-1} \left[ \frac{(1 - \beta + \beta u)}{M(\beta)} S[k_2 J - (\mu + d)A] \right].
 \end{aligned}$$

We next obtain the following recursive formula

$$\begin{aligned}
 S_{(n+1)}(t) &= S_n(0) + S^{-1} \left[ \frac{(1 - \beta + \beta u)}{M(\beta)} S[\mu k - c\beta(I_n + bJ_n)S_n - \mu S_n] \right], \\
 I_{(n+1)}(t) &= I_n(0) + S^{-1} \left[ \frac{(1 - \beta + \beta u)}{M(\beta)} S[c\beta(I_n + bJ_n)S - (\mu + k_1)I_n + \alpha J_n] \right], \\
 J_{(n+1)}(t) &= J_n(0) + S^{-1} \left[ \frac{(1 - \beta + \beta u)}{M(\beta)} S[k_1 I_n - (\mu + k_2 + \alpha)J_n] \right], \\
 A_{(n+1)}(t) &= A_n(0) + S^{-1} \left[ \frac{(1 - \beta + \beta u)}{M(\beta)} S[k_2 J_n - (\mu + d)A_n] \right].
 \end{aligned} \tag{4}$$

And the solution of (4) is provided by

$$\begin{aligned}
 S(t) &= \lim_{n \rightarrow \infty} S_{(n)}(t), \\
 I(t) &= \lim_{n \rightarrow \infty} I_{(n)}(t), \\
 J(t) &= \lim_{n \rightarrow \infty} J_{(n)}(t), \\
 A(t) &= \lim_{n \rightarrow \infty} A_{(n)}(t).
 \end{aligned}$$

#### 4. Application of fixed-point theorem for stability analysis of iteration method

Let us suppose  $(X_1, \|\cdot\|)$  as a Banach space and  $P$  as a self-map of  $X_1$ . Let  $y_{n+1} = g(P, y_n)$  be particular recursive procedure. Suppose that,  $F(P)$  the fixed-point set of  $P$  has at least one element and that  $y_n$  converges to a point  $p \in F(P)$ . Let  $\{x_n \subseteq X_1\}$  and define  $e_n = \|x_{n+1} - g(P, x_n)\|$ . If  $\lim_{n \rightarrow \infty} e^n = 0$  implies that  $\lim_{n \rightarrow \infty} x^n = p$ , then the iteration method  $y_{n+1} = g(P, y_n)$  is said to be  $P$ -stable. Analogously, we, therefore, consider that, our sequence  $\{x_n\}$  has an upper bound, otherwise there is no possibility of convergence. The iteration will be  $P$ -stable, if all these conditions are satisfied for  $y_{n+1} = Py_n$  which is also known as Picard's iteration.

**THEOREM 4.1** Let  $(X_1, \|\cdot\|)$  be a Banach space and  $P$  be a self-map of  $X_1$  satisfying

$$\|P_x - P_y\| \leq C\|x - P_x\| + c\|x - y\|$$

for all  $x, y \in X_1$  where  $0 \leq C, 0 \leq c < 1$ . Suppose that  $P$  is Picard  $P$ -stable. Let us take into account the following recursive formula from (4) connected to (2).

$$\begin{aligned} S_{(n+1)}(t) &= S_n(0) + S^{-1} \left[ \frac{(1-\beta+\beta u)}{M(\beta)} S[\mu k - c\beta(I_n + bJ_n)S_n - \mu S_n] \right], \\ I_{(n+1)}(t) &= I_n(0) + SL^{-1} \left[ \frac{(1-\beta+\beta u)}{M(\beta)} S[c\beta(I_n + bJ_n)S - (\mu + k_1)I_n + \alpha J_n] \right], \\ J_{(n+1)}(t) &= J_n(0) + S^{-1} \left[ \frac{(1-\beta+\beta u)}{M(\beta)} S[k_1 I_n - (\mu + k_2 + \alpha)J_n] \right], \\ A_{(n+1)}(t) &= A_n(0) + S^{-1} \left[ \frac{(1-\beta+\beta u)}{M(\beta)} S[k_2 J_n - (\mu + d)A_n] \right]. \end{aligned}$$

where  $\frac{(1-\beta+\beta u)}{M(\beta)}$  is the fractional Lagrange multiplier.

**THEOREM 4.2** Let us defined a self-map  $P$  as

$$\begin{aligned} P(S_n(t)) &= S_{(n+1)}(t) = S_n(0) + S^{-1} \left[ \frac{(1-\beta+\beta u)}{M(\beta)} S[\mu k - c\beta(I_n + bJ_n)S_n - \mu S_n] \right], \\ P(I_n(t)) &= I_{(n+1)}(t) = I_n(0) + S^{-1} \left[ \frac{(1-\beta+\beta u)}{M(\beta)} S[c\beta(I_n + bJ_n)S - (\mu + k_1)I_n + \alpha J_n] \right], \\ P(J_n(t)) &= J_{(n+1)}(t) = J_n(0) + S^{-1} \left[ \frac{(1-\beta+\beta u)}{M(\beta)} S[k_1 I_n - (\mu + k_2 + \alpha)J_n] \right], \\ P(A_n(t)) &= A_{(n+1)}(t) = A_n(0) + S^{-1} \left[ \frac{(1-\beta+\beta u)}{M(\beta)} S[k_2 J_n - (\mu + d)A_n] \right]. \end{aligned}$$

is  $P$ -stable in  $L^1(a, b)$  if

$$\begin{cases} \{1 - \mu f(\gamma) - c\beta(M + L)g(\gamma) - c\beta b(L + L_1)h(\gamma)\} < 1, \\ \{1 - (\mu + k_1)f_1(\gamma) + c\beta(M + L)g(\gamma) + c\beta b(L + L_1)h(\gamma)\} < 1, \\ \{1 + k_1 f_2(\gamma) - (\mu + k_2 + \alpha)g_2(\gamma)\} < 1, \\ \{1 + k_2 f_3(\gamma) - (\mu + d)g_3(\gamma)\} < 1. \end{cases} \quad (5)$$

*Proof* In the first step of the proof we will show that  $P$  has a fixed point. For this, we evaluate the followings for all

$(m, n) \in \mathbb{N} \times \mathbb{N}$ .

$$\begin{aligned}
 P(S_{(n)}(t)) - P(S_{(m)}(t)) &= S_{(n)}(t) - S_{(m)}(t) \\
 &+ S^{-1} \left[ \frac{(1 - \beta + \beta u)}{M(\beta)} S[\mu k - c\beta(I_n + bJ_n)S_n - \mu S_n] \right] \\
 &- S^{-1} \left[ \frac{(1 - \beta + \beta u)}{M(\beta)} S[\mu k - c\beta(I_m + bJ_m)S_m - \mu S_m] \right], \\
 P(I_{(n)}(t)) - P(I_{(m)}(t)) &= I_{(n)}(t) - I_{(m)}(t) \\
 &+ S^{-1} \left[ \frac{(1 - \beta + \beta u)}{M(\beta)} S[c\beta(I_n + bJ_n)S_n - (\mu + k_1)I_n + \alpha J_n] \right] \\
 &- S^{-1} \left[ \frac{(1 - \beta + \beta u)}{M(\beta)} S[c\beta(I_m + bJ_m)S_m - (\mu + k_1)I_m + \alpha J_m] \right], \\
 P(J_{(n)}(t)) - P(J_{(m)}(t)) &= J_{(n)}(t) - J_{(m)}(t) \\
 &+ S^{-1} \left[ \frac{(1 - \beta + \beta u)}{M(\beta)} S[k_1 I_n - (\mu + k_2 + \alpha)J_n] \right] \\
 &- S^{-1} \left[ \frac{(1 - \beta + \beta u)}{M(\beta)} S[k_1 I_m - (\mu + k_2 + \alpha)J_m] \right], \\
 P(A_{(n)}(t)) - P(A_{(m)}(t)) &= A_{(n)}(t) - A_{(m)}(t) \\
 &+ S^{-1} \left[ \frac{(1 - \beta + \beta u)}{M(\beta)} S[k_2 J_n - (\mu + d)A_n] \right] \\
 &- S^{-1} \left[ \frac{(1 - \beta + \beta u)}{M(\beta)} S[k_2 J_m - (\mu + d)A_m] \right].
 \end{aligned} \tag{6}$$

Let consider the first equation of (6) and taking norm of both hand sides, then without loss of generality, we have

$$\begin{aligned}
 \|P(S_{(n)}(t)) - P(S_{(m)}(t))\| &= \left\| S_{(n)}(t) - S_{(m)}(t) \right. \\
 &+ S^{-1} \left[ \frac{(1 - \beta + \beta u)}{M(\beta)} S[\mu k - c\beta(I_n + bJ_n)S_n - \mu S_n] \right] \\
 &\left. - S^{-1} \left[ \frac{(1 - \beta + \beta u)}{M(\beta)} S[\mu k - c\beta(I_m + bJ_m)S_m - \mu S_m] \right] \right\|
 \end{aligned} \tag{7}$$

Thanks to the triangular inequality, the right-hand side of the Equation (7) becomes

$$\begin{aligned}
 \|P(S_{(n)}(t)) - P(S_{(m)}(t))\| &\leq \|S_{(n)}(t) - S_{(m)}(t)\| \\
 &+ \left\| S^{-1} \left[ \frac{(1 - \beta + \beta u)}{M(\beta)} S[\mu k - c\beta(I_n + bJ_n)S_n - \mu S_n] \right] \right. \\
 &\left. - S^{-1} \left[ \frac{(1 - \beta + \beta u)}{M(\beta)} S[\mu k - c\beta(I_m + bJ_m)S_m - \mu S_m] \right] \right\|.
 \end{aligned} \tag{8}$$

Upon further simplification, (8) yields that

$$\begin{aligned}
 \|P(S_{(n)}(t)) - P(S_{(m)}(t))\| &\leq \|S_{(n)}(t) - S_{(m)}(t)\| \\
 &+ S^{-1} \left[ S \frac{(1 - \beta + \beta u)}{M(\beta)} [\| -c\beta I_n(S_n - S_m) \| + \| -c\beta S_m(I_n - I_m) \| + \| -\mu(S_n - S_m) \| \right. \\
 &\left. + \| -c\beta bJ_n(S_n - S_m) \| + \| -c\beta bS_m(J_n - J_m) \|] \right].
 \end{aligned} \tag{9}$$

Since both the solutions play the same role, we shall assume in this case that

$$\begin{aligned}
 \|S_{(n)}(t) - S_{(m)}(t)\| &\cong \|I_{(n)}(t) - I_{(m)}(t)\| \\
 \|S_{(n)}(t) - S_{(m)}(t)\| &\cong \|J_{(n)}(t) - J_{(m)}(t)\|, \\
 \|S_{(n)}(t) - S_{(m)}(t)\| &\cong \|A_{(n)}(t) - A_{(m)}(t)\|.
 \end{aligned}$$

Replacing this in Equation (9), we obtain the following relation

$$\begin{aligned} & \|P(S_{(n)}(t)) - P(S_{(m)}(t))\| \leq \|S_{(n)}(t) - S_{(m)}(t)\| \\ & + S^{-1} \left[ S^{\frac{1-\beta+\beta u}{M(\beta)}} [\| -c\beta I_n(S_n - S_m) \| + \| -c\beta S_m(S_n - S_m) \| + \| -\mu(S_n - S_m) \| \right. \\ & \left. + \| -c\beta b J_n(S_n - S_m) \| + \| -c\beta b S_m(S_n - S_m) \|] \right]. \end{aligned} \tag{10}$$

Since  $I_n, S_m, I_m$  and  $J_n$  are bounded as they are convergent sequence, therefore, we can find four different positive constants,  $M, L, M_1$  and  $L_1$  for all  $t$  such that

$$\|I_n\| < M, \|S_m\| < L, \|V_m\| < M_1, \|J_n\| < L_1, (m, n) \in \mathbb{N} \times \mathbb{N}. \tag{11}$$

Now considering Equation (10) with (11), we obtain the following

$$\|P(S_{(n)}(t)) - P(S_{(m)}(t))\| \leq \{1 - \mu f(\gamma) - c\beta(M + L)g(\gamma) - c\beta b(L + L_1)h(\gamma)\} \|S_{(n)} - S_{(m)}\|. \tag{12}$$

where  $f, g$  and  $h$  are functions from  $S^{-1} \left[ S^{\frac{1-\beta+\beta u}{M(\beta)}} \right]$ .

In the same way, we will get

$$\|P(I_{(n)}(t)) - P(I_{(m)}(t))\| \leq \{1 - (\mu + k_1)f_1(\gamma) + c\beta(M + L)g(\gamma) + c\beta b(L + L_1)h(\gamma)\} \|I_{(n)}(t) - I_{(m)}(t)\|, \tag{13}$$

$$\|P(J_{(n)}(t)) - P(J_{(m)}(t))\| \leq \{1 + k_1f_2(\gamma) - (\mu + k_2 + \alpha)g_2(\gamma)\} \|J_{(n)}(t) - J_{(m)}(t)\|, \tag{14}$$

$$\|P(A_{(n)}(t)) - P(A_{(m)}(t))\| \leq \{1 + k_2f_3(\gamma) - (\mu + d)g_3(\gamma)\} \|A_{(n)}(t) - A_{(m)}(t)\|. \tag{15}$$

where

$$\begin{aligned} & \{1 - \mu f(\gamma) - c\beta(M + L)g(\gamma) - c\beta b(L + L_1)h(\gamma)\} < 1, \\ & \{1 - (\mu + k_1)f_1(\gamma) + c\beta(M + L)g(\gamma) + c\beta b(L + L_1)h(\gamma)\} < 1, \\ & \{1 + k_1f_2(\gamma) - (\mu + k_2 + \alpha)g_2(\gamma)\} < 1, \\ & \{1 + k_2f_3(\gamma) - (\mu + d)g_3(\gamma)\} < 1. \end{aligned}$$

Thus the nonlinear  $P$ -self mapping has a fixed point. We next show that,  $P$  satisfies the conditions in Theorem 4.1. Let (12)–(15) hold and therefore using

$$c = (0, 0, 0, 0), \quad C = \begin{cases} \{1 - \mu f(\gamma) - c\beta(M + L)g(\gamma) - c\beta b(L + L_1)h(\gamma)\}, \\ \{1 - (\mu + k_1)f_1(\gamma) + c\beta(M + L)g(\gamma) + c\beta b(L + L_1)h(\gamma)\}, \\ \{1 + k_1f_2(\gamma) - (\mu + k_2 + \alpha)g_2(\gamma)\}, \\ \{1 + k_2f_3(\gamma) - (\mu + d)g_3(\gamma)\}. \end{cases}$$

Then the above shows that condition of Theorem 4.1 exist for the nonlinear mapping  $P$ . Thus all the conditions in Theorem 4.2 are satisfied for the defined non-linear mapping  $P$ . Hence,  $P$  is Picard  $P$ -stable.  $\square$

### 5. Uniqueness of the special solution

In this section, we show that the special solution of Equation (2) using the iteration method is unique. We shall first assume that, Equation (2) has an exact solution via which, the special solution converges for a large number  $m$ . We consider the following Hilbert space  $H = L((a, b) \times (0, T))$  which can be defined as

$$y: (a, b) \times (0, T) \longrightarrow \mathbb{R}, \quad \text{such that} \quad \iint uydydu < \infty.$$



We now, consider the following operator

$$P(S, I, J, A) = \begin{cases} \mu k - c\beta(I + bJ)S - \mu S, \\ c\beta(I + bJ)S - (\mu + k_1)I + \alpha J, \\ \delta k_1 I - (\mu + k_2 + \alpha)J, \\ K_2 J - (\mu + d)A. \end{cases}$$

The aim of this part is to prove that the inner product of

$$P\left((X_{11} - X_{12}, X_{21} - X_{22}, X_{31} - X_{32}, X_{41} - X_{42}), (w_1, w_2, w_3, w_4)\right),$$

where  $(X_{11} - X_{12}), (X_{21} - X_{22}), (X_{31} - X_{32})$  and  $(X_{41} - X_{42})$ , are special solution of system. However,

$$\begin{aligned} &P\left((X_{11} - X_{12}, X_{21} - X_{22}, X_{31} - X_{32}, X_{41} - X_{42}), (w_1, w_2, w_3, w_4)\right) \\ &= \begin{cases} (-c\beta(X_{21} - X_{22}) + b(X_{31} - X_{32}))(X_{11} - X_{12}) - \mu(X_{11} - X_{12}), w_1, \\ (c\beta(X_{21} - X_{22}) + b(X_{31} - X_{32}))(X_{11} - X_{12}) - (\mu + k_1)(X_{21} - X_{22}) + \alpha(X_{31} - X_{32}), w_2 \\ (k_1(X_{21} - X_{22}) - (\mu + k_2 + \alpha)(X_{31} - X_{32})), w_3, \\ (K_2(X_{31} - X_{32}) - (\mu + d)(X_{41} - X_{42})), w_4. \end{cases} \quad (16) \end{aligned}$$

We shall evaluate the first equation in the system without loss of generality

$$\begin{aligned} &(-c\beta(X_{21} - X_{22}) + b(X_{31} - X_{32}))(X_{11} - X_{12}) - \mu(X_{11} - X_{12}), w_1 \\ &= (-c\beta(X_{21} - X_{22})(X_{11} - X_{12}) + b(X_{31} - X_{32})(X_{11} - X_{12}) - \mu(X_{11} - X_{12})), w_1 \\ &\cong (-c\beta(X_{21} - X_{22})(X_{11} - X_{12}), w_1) + (b(X_{31} - X_{32})(X_{11} - X_{12}), w_1) + (-\mu(X_{11} - X_{12})), w_1. \end{aligned} \quad (17)$$

Since both solutions play almost the same role, we can assume that,

$$(X_{11} - X_{12}) \cong (X_{21} - X_{22}) \cong (X_{31} - X_{32}) \cong (X_{41} - X_{42}).$$

Then the Equation (17) becomes

$$(-c\beta(X_{11} - X_{12})^2 + b(X_{11} - X_{12})^2 - \mu(X_{11} - X_{12}), w_1)$$

Based on the relationship between norm and the inner product, we obtain the following

$$\begin{aligned} &(-c\beta(X_{11} - X_{12})^2 + b(X_{11} - X_{12})^2 - \mu(X_{11} - X_{12}), w_1) \\ &\cong (-c\beta(X_{11} - X_{12})^2, w_1) + (b(X_{11} - X_{12})^2, w_1) + (-\mu(X_{11} - X_{12})), w_1 \\ &\leq c\beta\|(X_{11} - X_{12})^2\| \|w_1\| + b\|(X_{11} - X_{12})^2\| \|w_1\| + \mu\|(X_{11} - X_{12})\| \|w_1\| \\ &= (c\beta\bar{\omega}_1 + b\bar{\omega}_1 + \mu)\|(X_{11} - X_{12})\| \|w_1\|. \end{aligned} \quad (18)$$

Repeating the same fashion, from the second, third and fourth equations of the system (16), we can obtain as follow

$$\begin{aligned}
 & (c\beta(X_{21} - X_{22}) + b(X_{31} - X_{32}))(X_{11} - X_{12}) - (\mu + k_1)(X_{21} - X_{22}) + \alpha(X_{31} - X_{32}), w_2 \\
 & \leq (c\beta\bar{\omega}_2 + b\bar{\omega}_2 + \mu + k_1 + \alpha)\|(X_{21} - X_{22})\|\|w_2\|, \\
 & (k_1(X_{21} - X_{22}) - (\mu + k_2 + \alpha)(X_{31} - X_{32})), w_3 \leq (k_1 + \mu + k_2 + \alpha)\|(X_{31} - X_{32})\|\|w_3\|, \\
 & (K_2(X_{31} - X_{32}) - (\mu + d)(X_{41} - X_{42})), w_4 \leq (K_2 + \mu + d)\|(X_{41} - X_{42})\|\|w_4\|.
 \end{aligned} \tag{19}$$

Putting Equations (18) and (19) in Equation (16), we get

$$\begin{aligned}
 & P\left((X_{11} - X_{12}, X_{21} - X_{22}, X_{31} - X_{32}, X_{41} - X_{42}), (w_1, w_2, w_3, w_4)\right) \\
 & \leq \begin{cases} (c\beta\bar{\omega}_1 + b\bar{\omega}_1 + \mu)\|(X_{11} - X_{12})\|\|w_1\|, \\ (c\beta\bar{\omega}_2 + b\bar{\omega}_2 + \mu + k_1 + \alpha)\|(X_{21} - X_{22})\|\|w_2\|, \\ (k_1 + \mu + k_2 + \alpha)\|(X_{31} - X_{32})\|\|w_3\|, \\ (K_2 + \mu + d)\|(X_{41} - X_{42})\|\|w_4\|. \end{cases} \tag{20}
 \end{aligned}$$

But, for sufficiently large values of  $m_i$ , with  $i = 1, 2, 3, 4$  both the solutions converge to the exact solution, using the topological concept, there exist three very small positive parameters  $l_{m_1}, l_{m_2}, l_{m_3}$  and  $l_{m_4}$  such that

$$\begin{aligned}
 \|S - X_{11}\|, \|S - X_{12}\| & < \frac{l_{m_1}}{3(c\beta\bar{\omega}_1 + b\bar{\omega}_1 + \mu)\|(X_{11} - X_{12})\|\|w_1\|}, \\
 \|I - X_{21}\|, \|I - X_{22}\| & < \frac{l_{m_2}}{3(c\beta\bar{\omega}_2 + b\bar{\omega}_2 + \mu + k_1 + \alpha)\|(X_{21} - X_{22})\|\|w_2\|}, \\
 \|J - X_{31}\|, \|J - X_{32}\| & < \frac{l_{m_3}}{3(k_1 + \mu + k_2 + \alpha)\|(X_{31} - X_{32})\|\|w_3\|}, \\
 \|A - X_{41}\|, \|A - X_{42}\| & < \frac{l_{m_4}}{3(K_2 + \mu + d)\|(X_{41} - X_{42})\|\|w_4\|}.
 \end{aligned}$$

Thus, plugging the exact solution in the right-hand side of Equation (20) and applying the triangular inequality by taking  $M = \max(m_1, m_2, m_3, m_4), l = \max(l_{m_1}, l_{m_2}, l_{m_3}, l_{m_4})$ . We obtain

$$\begin{cases} (c\beta\bar{\omega}_1 + b\bar{\omega}_1 + \mu)\|(X_{11} - X_{12})\|\|w_1\|, \\ (c\beta\bar{\omega}_2 + b\bar{\omega}_2 + \mu + k_1 + \alpha)\|(X_{21} - X_{22})\|\|w_2\|, \\ (k_1 + \mu + k_2 + \alpha)\|(X_{31} - X_{32})\|\|w_3\|, \\ (K_2 + \mu + d)\|(X_{41} - X_{42})\|\|w_4\|. \end{cases} < \begin{cases} l, \\ l, \\ l, \\ l. \end{cases}$$

As  $l$  is a small positive parameter, therefore, on the basis of topological idea, we have

$$\begin{cases} (c\beta\bar{\omega}_1 + b\bar{\omega}_1 + \mu)\|(X_{11} - X_{12})\|\|w_1\|, \\ (c\beta\bar{\omega}_2 + b\bar{\omega}_2 + \mu + k_1 + \alpha)\|(X_{21} - X_{22})\|\|w_2\|, \\ (k_1 + \mu + k_2 + \alpha)\|(X_{31} - X_{32})\|\|w_3\|, \\ (K_2 + \mu + d)\|(X_{41} - X_{42})\|\|w_4\|. \end{cases} < \begin{cases} 0, \\ 0, \\ 0, \\ 0. \end{cases}$$

But, it is obvious that

$$(c\beta\bar{\omega}_1 + b\bar{\omega}_1 + \mu) \neq 0, (c\beta\bar{\omega}_2 + b\bar{\omega}_2 + \mu + k_1 + \alpha) \neq 0, (k_1 + \mu + k_2 + \alpha) \neq 0, (K_2 + \mu + d) \neq 0.$$

Therefore, we have

$$\|X_{11} - X_{12}\| = 0, \|X_{21} - X_{22}\| = 0, \|X_{31} - X_{32}\| = 0, \|X_{41} - X_{42}\| = 0.$$

Which yields that

$$X_{11} = X_{12}, X_{21} = X_{22}, X_{31} = X_{32}, X_{41} = X_{42}.$$

In this way, we have completed the proof.

## 6. Conclusion

With the help of Picard successive approximation technique and Banach contraction theorem, we have investigated HIV/AIDS model with fractional order derivative. The arbitrary derivative of fractional order  $\beta$  has been taken in the Caputo-Febrizo sense which does not contain singular kernel. The concerned results have been handled by coupling Sumudu transform with mentioned iterative techniques. Further, the existence and uniqueness results for equilibrium solutions have been proved by applying Banach theorem.

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All authors equally contributed this paper and approved the final version.

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