Samraiz et al., *Cogent Mathematics & Statistics* (2018), 5: 1426205
https://doi.org/10.1080/23311835.2018.1426205

cogent mathematics & statistics

PURE MATHEMATICS | RESEARCH ARTICLE

Generalized integral inequalities for fractional calculus

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Received: 09 October 2017
Accepted: 06 January 2018
First Published: 25 January 2018

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Reviewing editor:
Lishan Liu, Qufu Normal University, China

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Abstract: In this paper, we present a variety of integral inequalities in L_p and $L_{p,r}$ spaces for the integral operator involving generalized Mittag-Leffler function in its kernel, Hilfer fractional derivative, generalized Riemann-Liouville and Riemann-Liouville k-fractional integral operators.

Subjects: Advanced Mathematics; Applied Mathematics; History & Philosophy of Mathematics

Keywords: Riemann-Liouville fractional integral; generalized integral inequality; Mittag-Leffler function; Hilfer fractional derivative

AMS subject classifications: 26D15; 26A24; 26D15; 26A33

1. Introduction

The importance of the fractional integral inequalities is enormous in establishing the uniqueness of solutions for certain fractional partial differential equations. This theory is also helpful in providing bounds for the solutions of fractional boundary value problems. In this era of progress and development, the theory of fractional integral inequalities catches the attention of many mathematicians and they provide plenty of applications of integral inequalities in fractional calculus. For more details see Anastassiou (2009), Ansari, Liu, and Mishra (2017), Mishra and Sen (2016), Iqbal, Pečarić, Samraiz, and Sultana (2015), Iqbal, Krulić, and Pečarić (2010), Niculescu and Persson (2006).

Mitrinović and Pečarić (1991) introduced an integral inequality which later generalized by Farid, Iqbal, and Pečarić (2015). In the present work, we have paid attention to provide applications of the generalized integral inequality presented in (Farid et al., 2015) for fractional calculus.

We start with the definition of $L_{p,r}$ space given in Mubeen and Iqbal (2016).

Sajid Iqbal

ABOUT THE AUTHOR

Sajid Iqbal is working as an assistant professor of Mathematics in University of Sargodha (Sub-Campus Bhakkar), Bhakkar, Pakistan. He is mainly known for works in Mathematical Inequalities involving convex functions and application in fractional calculus. He received his PhD degree from Abdus Salam School of Mathematical University, Government College University Lahore, Pakistan. He is a member of Pakistan Mathematical Society. He has published more than 30 research papers in high-quality international journals and works as a reviewer for in many international journals. He has supervised a PhD student and 13 MPhil students.

PUBLIC INTEREST STATEMENT

The importance of Mathematical inequalities is felt from the beginning and is now extensively known as one of the most important motivating forces behind the progress of current real analysis. This theory plays significant part in approximately all branches of Mathematics as well as in other areas of science. Here we have focused to present a variety of integral inequalities in generalized L_p spaces involving fractional integral operators. The involvement of generalize fractional integral operator makes our results more general.

Definition 1.1 A space $L_{p,r}[a,b]$ is defined as a space of continuous real valued function $h(y)$ on $[a,b]$, such that

$$\left(\int_a^b |h(y)|^p y^r dy\right)^{\frac{1}{p}} < \infty,$$

where $1 \leq p < \infty$, and $r \geq 0$.

THEOREM 1.2 *Let* $(\Omega_1, \Sigma_1, \mu_1)$, $(\Omega_2, \Sigma_2, \mu_2)$ *be measure spaces with* σ*-finite measures and* $f_i:\Omega_2 \to \mathbb{R}$, $i = 1,2,3,4$, *be non-negative measurable functions. Let g belongs to a particular class of functions U(f, k) which admits the representation*

$$g(x) = \int_{\Omega_1} k(x,t) f(t) d\mu_1(t),$$

where $k:\Omega_2 \times \Omega_1 \to \mathbb{R}$ is a general non-negative kernel and $f:\Omega_1 \to \mathbb{R}$ be a real valued function. If p, q are two real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, then the inequality

$$\int_{\Omega_2} f_1(x) f_2(x) g(x) d\mu_2(x) \leq C \left(\int_{\Omega_2} f_3(x) g(x) d\mu_2(x)\right)^{\frac{1}{p}} \left(\int_{\Omega_2} f_4(x) g(x) d\mu_2(x)\right)^{\frac{1}{q}} \tag{1.1}$$

holds true, where

$$C = \sup_{t \in \Omega_1} \left\{ \left(\int_{\Omega_2} k(x,t) f_1(x) f_2(x) d\mu_2(x)\right) \left(\int_{\Omega_2} k(x,t) f_3(x) d\mu_2(x)\right)^{\frac{-1}{p}} \right.$$
$$\left. \times \left(\int_{\Omega_2} k(x,t) f_4(x) d\mu_2(x)\right)^{\frac{-1}{q}} \right\}.$$

COROLLARY 1.3 *If we set* $f_3(x) = f_1^p(x)$, $f_4(x) = f_2^q(x)$, *then we get the following inequality*

$$\int_{\Omega_2} f_1(x) f_2(x) g(x) d\mu_2(x) \leq C \left(\int_{\Omega_2} f_1^p(x) g(x) d\mu_2(x)\right)^{\frac{1}{p}} \left(\int_{\Omega_2} f_2^q(x) g(x) d\mu_2(x)\right)^{\frac{1}{q}},$$

where

$$C = \sup_{t \in \Omega_1} \left\{ \left(\int_{\Omega_2} k(x,t) f_1(x) f_2(x) d\mu_2(x)\right) \left(\int_{\Omega_2} k(x,t) f_1^p(x) d\mu_2(x)\right)^{\frac{-1}{p}} \right.$$
$$\left. \times \left(\int_{\Omega_2} k(x,t) f_2^q(x) d\mu_2(x)\right)^{\frac{-1}{q}} \right\}.$$

The rest of the paper is organized as follows: In Section 2, we present the generalized integral inequality for six parameter fractional integral operator with Mittag-Leffler function in its kernel. Section 3 contains results for Hilfer fractional derivative. Section 4 consists of consequences for generalized Riemann-Liouville fractional integral operator. In the last section, we derive results for the Riemann-Liouville k—fractional integral.

2. Generalized integral inequality for fractional integral operator with six parameter Mittag-Leffler function in its kernel

First, we give the definition of the Mittag-Leffler function (see Mittag-Leffler, 1903) and fractional integral operator involving the generalized Mittag-Leffler function appearing in the kernel (see Salim & Faraj, 2012).

Definition 2.1 Let $\alpha,\ \beta,\ \gamma,\ \delta \in \mathbb{C}$; $\min\{\Re(\alpha),\ \Re(\beta),\ \Re(\gamma),\ \Re(\delta)\} > 0; p,\ q > 0$ and $q < \Re\alpha + p$. Then the generalized Mittag-Leffler function defined in Salim and Faraj (2012) is given by

$$E^{\gamma,\delta,q}_{\alpha,\beta,p}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\alpha n + \beta)} \frac{z^n}{(\delta)_{pn}}, \tag{2.1}$$

where $(\gamma)_n$ represents the Pochhammer symbol defined by $(\gamma)_n = \gamma(\gamma-1)(\gamma-2)\dots(\gamma-n+1)$ and Γ is the Euler gamma function i.e. $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$. The function (2.1) represents all the previous generalizations of the Mittag-Leffler function by setting the following values.

- $p = q = 1$, this reduces to $E^{\gamma,\delta}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta)} \frac{z^n}{(\delta)_n}$ defined by Salim in (2009).
- $\delta = p = 1$, this represents $E^{\gamma,q}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!}$ which was introduced by Shukla and Prajapati in (2007). In Srivastava and Tomovski (2009) investigated the properties of this function and its existence for a wider set of parameters.
- $\delta = p = q = 1$, the operator (2.1) is defined by Prabhakar in (1971) and is denoted as:- $E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!}$.
- $\gamma = \delta = p = q = 1$, it reduces to Wiman's function presented in Wiman (1905), and moreover, if $\beta = 1$, then the Mittag-Leffler function $E_\alpha(z)$ will be the result.

Definition 2.2 Let $\alpha,\ \beta,\ \gamma,\ \delta \in \mathbb{C}$; $\min\{\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\delta)\} > 0; p, q > 0$, and $q < \Re\alpha + p$. The integral operator which contains the Mittag-Leffler function (2.1) in the kernel is given by

$$\left(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega;a^+} f\right)(x) = \int_a^x (x-t)^{\beta-1} E^{\gamma,\delta,q}_{\alpha,\beta,p}(\omega(x-t)^\alpha) f(t)dt. \tag{2.2}$$

Our first result is a direct consequence of the integral inequality (1.1) for the generalized integral operator (2.2).

THEOREM 2.3 *Let p, q be two real numbers such that $\frac{1}{p} + \frac{1}{q} = 1, p > 1$. Let $\alpha,\ \beta,\ \gamma,\ \delta \in \mathbb{C}$; $\min\{\Re(\alpha),\ \Re(\beta),\ \Re(\gamma),\ \Re(\delta)\} > 0; p,\ q > 0$ and $q < \Re\alpha + p$, then the following inequality holds*

$$\int_a^x f_1(x)f_2(x)\left(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega;a^+} f\right)(x)dx \le C\left(\int_a^x f_3(x)\left(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega;a^+} f\right)(x)dx\right)^{\frac{1}{p}} \times \left(\int_a^x f_4(x)\left(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega;a^+} f\right)(x)dx\right)^{\frac{1}{q}}, \tag{2.3}$$

where

$$C = \sup_{t\in[a,b]}\left\{\left(\int_a^x (x-t)^{\beta-1}E^{\gamma,\delta,q}_{\alpha,\beta,p}(\omega(x-t)^{\alpha})f_1(x)f_2(x)dx\right)\right.$$
$$\times\left(\int_a^x (x-t)^{\beta-1}E^{\gamma,\delta,q}_{\alpha,\beta,p}(\omega(x-t)^{\alpha})f_3(x)dx\right)^{\frac{-1}{p}}$$
$$\left.\times\left(\int_a^x (x-t)^{\beta-1}E^{\gamma,\delta,q}_{\alpha,\beta,p}(\omega(x-t)^{\alpha})f_4(x)dx\right)^{\frac{-1}{q}}\right\}.$$

Proof Applying Theorem 1.2 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(t) = dt$, $d\mu_2(x) = dx$,

$$k(x,t) = \begin{cases} (x-t)^{\beta-1}E^{\gamma,\delta,q}_{\alpha,\beta,p}(\omega(x-t)^{\alpha}), & a\le t\le x;\\ 0, & x<t\le b\end{cases}$$

and $g(x) = \left(\epsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega;a^+}f\right)(x)$, we get the inequality (2.3). □

Corollary 2.4 *If we set $f_3(x) = f_1^p(x)$, $f_4(x) = f_2^q(x)$, then the inequality*

$$\int_a^x f_1(x)f_2(x)\left(\epsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega;a^+}f\right)(x)dx$$
$$\le C\left(\int_a^x f_1^p(x)\left(\epsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega;a^+}f\right)(x)dx\right)^{\frac{1}{p}}\left(\int_a^x f_2^q(x)\left(\epsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega;a^+}f\right)(x)dx\right)^{\frac{1}{q}}$$

holds true, where

$$C = \sup_{t\in[a,b]}\left\{\left(\int_a^x (x-t)^{\beta-1}E^{\gamma,\delta,q}_{\alpha,\beta,p}(\omega(x-t)^{\alpha})f_1(x)f_2(x)dx\right)\right.$$
$$\times\left(\int_a^x (x-t)^{\beta-1}E^{\gamma,\delta,q}_{\alpha,\beta,p}(\omega(x-t)^{\alpha})f_1^p(x)dx\right)^{\frac{-1}{p}}$$
$$\left.\times\left(\int_a^x (x-t)^{\beta-1}E^{\gamma,\delta,q}_{\alpha,\beta,p}(\omega(x-t)^{\alpha})f_2^q(x)dx\right)^{\frac{-1}{q}}\right\}.$$

By similar process we can also obtain results for Mittag-Leffler function of Shukla and Prajapati (2007) and Prabhakar (1971).

3. Generalized integral inequality for Hilfer fractional derivative

In this section, we present the integral inequality (1.1) for the Hilfer fractional derivative. Let us recall the definition of Hilfer fractional derivative which is presented in Tomovski and Rudolf Hilfer Srivastava (2010).

Definition 3.1 Let $f\in L_1[a,b], f * K_{(1-\nu)(1-\mu)}\in AC^1[a,b]$. The fractional derivative operator $D^{\mu,\nu}_{a+}$ of order $0<\mu<1$ and type $0<\nu\le 1$ with respect to $x\in[a,b]$ is defined by

$$\left(D^{\mu,\nu}_{a+}f\right)(x) := I^{\nu(1-\mu)}_{a+}\frac{d}{dx}\left(I^{(1-\nu)(1-\mu)}_{a+}f(x)\right), \tag{3.1}$$

whenever the right hand side exists. The derivative (3.1) is usually called Hilfer fractional derivative.

The more general integral representation of equation (3.1) is given in Hilfer, Luchko, and Tomovski (2009) and is defined as: Let $f\in L_1\left[a,b\right]$, $f * K_{(1-\nu)(n-\mu)}\in AC^n\left[a,b\right]$, $n-1<\mu<n, 0<\nu\le 1$, $n\in\mathbb{N}$, then the following equation holds true:

$$\left(D_{a+}^{\mu,\nu}f\right)(x) = \left(I_{a+}^{\nu(n-\mu)}\frac{d^n}{dx^n}\left(I_{a+}^{(1-\nu)(n-\mu)}f(x)\right)\right). \tag{3.2}$$

Specially for $\nu = 0, D_{a+}^{\mu,0}f = D_{a+}^{\mu}f$ is a Riemann- Liouvile fractional derivative of order μ and for $\nu = 1$ it is a Caputo fractional derivative $D_{a+}^{\mu,1}f = ^C D_{a+}^{\mu}f$ of order μ. Applying the properties of Riemann-Liouvile fractional integral the relation (3.2) can be rewritten in the form:

$$\begin{aligned}\left(D_{a+}^{\mu,\nu}f\right)(x) &= \left(I_{a+}^{\nu(n-\mu)}\left(\left(D_{a+}^{n-(1-\nu)(n-\mu)}f\right)(x)\right)\right)\\ &= \frac{1}{\Gamma(\nu(n-\mu))}\int_a^x (x-t)^{\nu(n-\mu)-1}\left(\left(D_{a+}^{\mu+\nu(n-\mu)}f\right)(t)\right)dt.\end{aligned}$$

THEOREM 3.2 *Let $D_{a+}^{\mu+\nu(n-\mu)}f \in L_1[a,b]$ and the fractional derivative operator $D_{a+}^{\mu,\nu}$ of order $n-1<\mu<n$ and type $0<\nu\le 1$, p, q be two real numbers such that $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, then the inequality*

$$\begin{aligned}&\int_a^x \left(\left(D_{a+}^{\mu+\nu(n-\mu)}f_1\right)(x)\right)\left(\left(D_{a+}^{\mu+\nu(n-\mu)}f_2\right)(x)\right)\left(D_{a+}^{\mu,\nu}f\right)(x)dx\\ &\le C\left(\int_a^x \left(\left(D_{a+}^{\mu+\nu(n-\mu)}f_3\right)(x)\right)\left(D_{a+}^{\mu,\nu}f\right)(x)dx\right)^{\frac{1}{p}}\\ &\quad\times\left(\int_a^x \left(\left(D_{a+}^{\mu+\nu(n-\mu)}f_4\right)(x)\right)\left(D_{a+}^{\mu,\nu}f\right)(x)dx\right)^{\frac{1}{q}}\end{aligned} \tag{3.3}$$

holds true, where

$$\begin{aligned}C = \sup_{t\in[a,b]}\Bigg\{&\left(\int_a^x (x-t)^{\nu(n-\mu)-1}\left(\left(D_{a+}^{\mu+\nu(n-\mu)}f_1\right)(x)\right)\left(\left(D_{a+}^{\mu+\nu(n-\mu)}f_2\right)(x)\right)dx\right)\\ &\times\left(\int_a^x (x-t)^{\nu(n-\mu)-1}\left(\left(D_{a+}^{\mu+\nu(n-\mu)}f_3\right)(x)\right)dx\right)^{\frac{-1}{p}}\\ &\times\left(\int_a^x (x-t)^{\nu(n-\mu)-1}\left(\left(D_{a+}^{\mu+\nu(n-\mu)}f_4\right)(x)\right)dx\right)^{\frac{-1}{q}}\Bigg\}.\end{aligned}$$

Proof Applying Theorem 1.2 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(t) = dt$, $d\mu_2(x) = dx$,

$$k(x,t) = \begin{cases}\frac{(x-t)^{\nu(n-\mu)-1}}{\Gamma(\nu(n-\mu))}, & a\le t\le x;\\ 0, & x<t\le b\end{cases}$$

and $g(x) = \left(D_{a+}^{\mu,\nu}f\right)(x)$, we get the inequality (3.3). □

COROLLARY 3.3 *If we set*

$$\begin{aligned}\left(D_{a+}^{\mu+\nu(n-\mu)}f_3\right)(x) &= \left(\left(D_{a+}^{\mu+\nu(n-\mu)}f_1\right)(x)\right)^p,\\ \left(D_{a+}^{\mu+\nu(n-\mu)}f_4\right)(x) &= \left(\left(D_{a+}^{\mu+\nu(n-\mu)}f_2\right)(x)\right)^q,\end{aligned}$$

then the inequality

$$
\begin{aligned}
&\int_a^x \left(\left(D_{a+}^{\mu+\nu(n-\mu)} f_1\right)(x)\right)\left(\left(D_{a+}^{\mu+\nu(n-\mu)} f_2\right)(x)\right)\left(D_{a+}^{\mu,\nu} f\right)(x)dx \\
&\leq C\left(\int_a^x \left(\left(D_{a+}^{\mu+\nu(n-\mu)} f_1\right)(x)\right)^p \left(D_{a+}^{\mu,\nu} f\right)(x)dx\right)^{\frac{1}{p}} \\
&\quad\times\left(\int_a^x \left(\left(D_{a+}^{\mu+\nu(n-\mu)} f_2\right)(x)\right)^q \left(D_{a+}^{\mu,\nu} f\right)(x)dx\right)^{\frac{1}{q}}
\end{aligned}
$$

holds true, where

$$
\begin{aligned}
C = \sup_{t\in[a,b]} &\left\{\left(\int_a^x (x-t)^{\nu(n-\mu)-1}\left(\left(D_{a+}^{\mu+\nu(n-\mu)} f_1\right)(x)\right)\left(\left(D_{a+}^{\mu+\nu(n-\mu)} f_2\right)(x)\right)dx\right)\right. \\
&\times\left(\int_a^x (x-t)^{\nu(n-\mu)-1}\left(\left(D_{a+}^{\mu+\nu(n-\mu)} f_1\right)(x)\right)^p dx\right)^{\frac{-1}{p}} \\
&\left.\times\left(\int_a^x (x-t)^{\nu(n-\mu)-1}\left(\left(D_{a+}^{\mu+\nu(n-\mu)} f_1\right)(x)\right)^q dx\right)^{\frac{-1}{q}}\right\}.
\end{aligned}
$$

4. Consequences for generalized Riemann-Liouville fractional integral operator

In this section, we find the applications of integral inequality (1.1) for generalized Riemann-Liouville fractional integral operator and extract the results of Farid et al. (2015) as special case. The generalized Riemann-Liouville fractional integral is defined as follows:

Definition 4.1 If $f \in L_{1,r}[a,b]$, then the left and right sided generalized Riemann-Liouville fractional integrals of order $\alpha \geq 0$ and $r \geq 0$ are given by

$$
I_{a+}^{\alpha,r} f(x) = \frac{(r+1)^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \left(x^{r+1}-t^{r+1}\right)^{\alpha-1} t^r f(t)dt, \quad t \in [a,b],
$$

$$
I_{a-}^{\alpha,r} f(x) = \frac{(r+1)^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \left(t^{r+1}-x^{r+1}\right)^{\alpha-1} t^r f(t)dt, \quad t \in [a,b],
$$

where Γ is the Euler gamma function.

THEOREM 4.2 *Let $f \in L_{1,r}[a,b]$ and the fractional integral operator $I_{a+}^{\alpha,r}$ of order $\alpha \geq 0$ and type $r \geq 0$. Moreover p, q be two real numbers such that $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, then the inequality*

$$
\begin{aligned}
&\int_a^x f_1(x)f_2(x)\left(I_{a+}^{\alpha,r} f\right)(x)dx \\
&\leq C\left(\int_a^x f_3(x)\left(I_{a+}^{\alpha,r} f\right)(x)dx\right)^{\frac{1}{p}}\left(\int_a^x f_4(x)\left(I_{a+}^{\alpha,r} f\right)(x)dx\right)^{\frac{1}{q}}
\end{aligned}
\tag{4.1}
$$

holds true, where

$$
\begin{aligned}
C = \sup_{t\in[a,b]} &\left\{\left(\int_a^x \left(x^{r+1}-t^{r+1}\right)^{\alpha-1} t^r f_1(x)f_2(x)dx\right)\right. \\
&\times\left(\int_a^x \left(x^{r+1}-t^{r+1}\right)^{\alpha-1} t^r f_3(x)dx\right)^{\frac{-1}{p}} \\
&\left.\times\left(\int_a^x \left(x^{r+1}-t^{r+1}\right)^{\alpha-1} t^r f_4(x)dx\right)^{\frac{-1}{q}}\right\}.
\end{aligned}
$$

Proof Applying Theorem 1.2 with $\Omega_1 = \Omega_2 = (a, b)$, $d\mu_1(t) = dt$, $d\mu_2(x) = dx$, kernel

$$k(x,t) = \begin{cases} \frac{(r+1)^{1-\alpha}\left(x^{r+1}-t^{r+1}\right)^{\alpha-1}t^r}{\Gamma(\alpha)}, & a \le t \le x; \\ 0, & x < t \le b \end{cases}$$

and $g(x) = \left(I^{\alpha,r}_{a+}f\right)(x)$, we get the inequality (4.1). □

Corollary 4.3 *If we set*

$$f_3(x) = f_1^p(x),\ f_4(x) = f_2^q(x),$$

then we get the following inequality:

$$\begin{aligned} &\int_a^x f_1(x)f_2(x)\left(I^{\alpha,r}_{a+}f\right)(x)dx \\ &\le C\left(\int_a^x f_1^p(x)\left(I^{\alpha,r}_{a+}f\right)(x)dx\right)^{\frac{1}{p}}\left(\int_a^x f_2^q(x)\left(I^{\alpha,r}_{a+}f\right)(x)dx\right)^{\frac{1}{q}}, \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} C = \sup_{t\in[a,b]} &\left\{\left(\int_a^x \left(x^{r+1}-t^{r+1}\right)^{\alpha-1}t^r f_1(x)f_2(x)dx\right)\right. \\ &\times \left(\int_a^x \left(x^{r+1}-t^{r+1}\right)^{\alpha-1}t^r f_1^p(x)dx\right)^{\frac{-1}{p}} \\ &\left.\times \left(\int_a^x \left(x^{r+1}-t^{r+1}\right)^{\alpha-1}t^r f_2^q(x)dx\right)^{\frac{-1}{q}}\right\}. \end{aligned}$$

Theorem 4.4 *Let p, q be two real numbers with* $\frac{1}{p}+\frac{1}{q}=1$, $p>1$ *and the fractional integral operator* $I^{\alpha,r}_{a-}$ *of order* $\alpha \ge 0$ *and type* $r \ge 0$*, then the inequality*

$$\begin{aligned} &\int_x^b f_1(x)f_2(x)\left(I^{\alpha,r}_{a-}f\right)(x)dx \\ &\le C\left(\int_x^b f_3(x)\left(I^{\alpha,r}_{a-}f\right)(x)dx\right)^{\frac{1}{p}}\left(\int_x^b f_4(x)\left(I^{\alpha,r}_{a-}f\right)(x)dx\right)^{\frac{1}{q}} \end{aligned} \tag{4.3}$$

holds true, where

$$\begin{aligned} C = \sup_{t\in[a,b]} &\left\{\left(\int_x^b \left(t^{r+1}-x^{r+1}\right)^{\alpha-1}t^r f_1(x)f_2(x)dx\right)\right. \\ &\left.\times \left(\int_x^b \left(t^{r+1}-x^{r+1}\right)^{\alpha-1}t^r f_3(x)dx\right)^{\frac{-1}{p}}\left(\int_x^b \left(t^{r+1}-x^{r+1}\right)^{\alpha-1}t^r f_4(x)dx\right)^{\frac{-1}{q}}\right\}. \end{aligned}$$

Proof Applying Theorem 1.2 with $\Omega_1 = \Omega_2 = (a, b)$, $d\mu_1(t) = dt$, $d\mu_2(x) = dx$,

$$k(x,t) = \begin{cases} 0, & a \le t < x; \\ \frac{(r+1)^{1-\alpha}\left(t^{r+1}-x^{r+1}\right)^{\alpha-1}t^r}{\Gamma(\alpha)}, & x \le t \le b, \end{cases}$$

and $g(x) = \left(I_{a-}^{\alpha,r}f\right)(x)$, we get the inequality (4.3). □

COROLLARY 4.5 *If we set $f_3(x) = f_1^p(x)$, $f_4(x) = f_2^q(x)$, then the inequality*

$$\begin{aligned}&\int_x^b f_1(x)f_2(x)\left(I_{a-}^{\alpha,r}f\right)(x)dx\\&\le C\left(\int_x^b f_1^p(x)\left(I_{a-}^{\alpha,r}f\right)(x)dx\right)^{\frac{1}{p}}\left(\int_x^b f_2^q(x)\left(I_{a-}^{\alpha,r}f\right)(x)dx\right)^{\frac{1}{q}}\end{aligned} \tag{4.4}$$

holds true, where

$$\begin{aligned}C = \sup_{t\in[a,b]}&\left\{\left(\left(\int_x^b \left(t^{r+1}-x^{r+1}\right)^{\alpha-1}t^r f_1(x)f_2(x)dx\right)\right.\right.\\&\times\left(\int_x^b \left(t^{r+1}-x^{r+1}\right)^{\alpha-1}t^r f_1^p(x)dx\right)^{\frac{-1}{p}}\\&\left.\times\left(\int_x^b \left(t^{r+1}-x^{r+1}\right)^{\alpha-1}t^r f_2^q(x)dx\right)^{\frac{-1}{q}}\right\}.\end{aligned}$$

Remark 4.6 If we set $r = 0$ in inequalities (4.1), (4.2), (4.3) and (4.4), then we get results for the Riemann-Liouville fractional integrals presented in Farid et al. (2015).

5. Generalized integral inequality for Riemann-Liouville k-fractional integral operator

In this section, we present the integral inequalities for generalized Riemann-Liouville k–fractional integral operator presented in Mubeen and Habibullah (2012) and is defined as:

Definition 5.1 If $f \in L_{1,r}[a,b]$, then the generalized Riemann-Liouville k-fractional integral $I_{a,k}^{\alpha}$ of order $\alpha \ge 0$ and $k > 0$ is given by

$$I_{a,k}^{\alpha}f(x) = \frac{1}{k\Gamma_k(\alpha)}\int_a^x (x-t)^{\frac{\alpha}{k}-1}f(t)dt, \quad x \in [a,b], \tag{5.1}$$

where Γ_k is the gamma k function i.e. $\Gamma_k(y) = \int_0^{\infty} x^{y-1}e^{\frac{-x^k}{k}}dx$, $k > 0$.

THEOREM 5.2 *Let $f \in L_{1,r}[a,b]$ and the fractional derivative operator $I_{a,k}^{\alpha}$ of order $\alpha \ge 0$ and type $k > 0$. Moreover p, q be two real numbers such that $\frac{1}{p}+\frac{1}{q} = 1$, $p > 1$, then the inequality*

$$\begin{aligned}&\int_a^x f_1(x)f_2(x)\left(I_{a,k}^{\alpha}f\right)(x)dx\\&\le C\left(\int_a^x f_3(x)\left(I_{a,k}^{\alpha}f\right)(x)dx\right)^{\frac{1}{p}}\left(\int_a^x f_4(x)\left(I_{a,k}^{\alpha}f\right)(x)dx\right)^{\frac{1}{q}}\end{aligned} \tag{5.2}$$

holds true, where

$$\begin{aligned}C = \sup_{t\in[a,b]}&\left\{\left(\int_a^x (x-t)^{\frac{\alpha}{k}-1}f_1(x)f_2(x)dx\right)\left(\int_a^x (x-t)^{\frac{\alpha}{k}-1}f_3(x)dx\right)^{\frac{-1}{p}}\right.\\&\left.\times\left(\int_a^x (x-t)^{\frac{\alpha}{k}-1}f_4(x)dx\right)^{\frac{-1}{q}}\right\}.\end{aligned}$$

Proof Applying Theorem 1.2 with $\Omega_1 = \Omega_2 = (a,b), d\mu_2(x) = dx, d\mu_1(t) = dt$,

$$k(x,t) = \begin{cases} \frac{(x-t)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}, & a \le t \le x; \\ 0, & x < t \le b \end{cases}$$

and $g(x) = \left(I^{\alpha}_{a,k}f\right)(x)$, we get the inequality (5.2). □

COROLLARY 5.3 *If we set $f_3(x) = f_1^p(x)$, $f_4(x) = f_2^q(x)$, then the inequality*

$$\int_a^x f_1(x)f_2(x)\left(I^{\alpha}_{a,k}f\right)(x)dx \le C\left(\int_a^x f_1^p(x)\left(I^{\alpha}_{a,k}f\right)(x)dx\right)^{\frac{1}{p}}\left(\int_a^x f_2^q(x)\left(I^{\alpha}_{a,k}f\right)(x)dx\right)^{\frac{1}{q}} \tag{5.3}$$

holds true, where

$$C = \sup_{t\in[a,b]}\left\{\left(\int_a^x (x-t)^{\frac{\alpha}{k}-1}f_1(x)f_2(x)dx\right)\left(\int_a^x (x-t)^{\frac{\alpha}{k}-1}f_1^p(x)dx\right)^{\frac{-1}{p}} \times \left(\int_a^x (x-t)^{\frac{\alpha}{k}-1}f_2^q(x)dx\right)^{\frac{-1}{q}}\right\}.$$

Remark 5.4 If we set $k = 1$ in inequalities (5.2) and (5.3), then we get results for the Riemann-Liouville fractional integral presented in Farid et al. (2015).

Funding
The authors received no direct funding for this research.

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Citation information
Cite this article as: Generalized integral inequalities for fractional calculus, Muhammad Samraiz, Sajid Iqbal & Josip Pečarić, *Cogent Mathematics & Statistics* (2018), 5: 1426205.

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