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Accepted Manuscript Version
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Publisher: Cogent OA
Journal: *Cogent Mathematics & Statistics*
DOI: http://doi.org/10.1080/23311835.2017.1389622

Common Fixed Point Theorems for Mappings Satisfying a Contractive Condition of Rational Expression on a Ordered Complex Partial Metric Space

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Abstract : By introducing a complex partial metric spaces, we obtain some common fixed point results for the mappings satisfying rational expressions in a complex partial metric spaces. The proved results generalize and extend some of the known results in the literature. Also we provide examples to illustrate our results.

Keywords : common fixed point; partial metric spaces; complex valued metric spaces; weakly increasing mapping; partially ordered set.

1 Introduction and Preliminaries

The most important Banach contraction principle is proved by Stefan Banach in 1922. His valuable work has been elaborated via generalizing the metric conditions or by imposing conditions on the metric spaces. As a consequence of those generalizations so many metric spaces introduced namely uniformly convex Banach spaces, strictly convex Banach spaces, cone metric spaces, pseudo metric spaces, B- metric spaces, fuzzy metric spaces etc. Huge work have been done in this direction, for example the recent works are see., [29],[26], [12],[28] and Schwarz lemma involving the boundary fixed point [25] is the very interesting topic in complex analysis. Also many authors weakening the contraction condition of Banach, [11], [13], [14], [20] these fixed point results are useful in establishing the unique-

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ness of the solution of nonlinear differential and integral equations. Recently H.M. Srivastava et al., [23] proved the hybrid fixed point theorems to fractional integral equations for proving the existence of solutions under certain monotonicity conditions blending with the existence of the upper or lower solution.

In the same way in 1994 Matthews introduced a partial metric spaces, which emphasis that the distance between the point to itself need not be equal to zero. The motivation and example for partial metric is given by Matthews et al., [2] as follows,

Let S^w be the set of all infinite sequences $x = (x_0, x_1, x_2, \cdots)$ over the set S. For all such sequences x and y let $p_s(x, y) = 2^{-k}$, where k is the largest number (possibly ∞) such that $x_i = y_i$ for each $i < k$. Thus $p_s(x, y)$ is defined to be 1 over 2 to the power of the length of the longest initial sequence common to both x and y. It can be shown that (S^w, p_s) is a metric space.

How might computer scientists view this metric space? for the simplicity they split up the infinite sequence to the finite sequences $(), (x_0), (x_0, x_1), (x_0, x_1, x_2)$ and so on. After each value x_k is printed, the finite sequences $(x_0, x_1, \cdots, x_k)$ represents that the part of the infinite sequence produced so far. Suppose now that the above definition of p_s is extended to S^* i.e., the set of all finite sequences over S. All the axioms of the metric is still hold except the self distance property i.e., $d(x, y) = 0$ if and only if $x = y$. However if x is finite sequence then $p_s(x, x) = 2^{-k}$ for some number $k < \infty$, which is not zero, since $x_i = x_j$ can only hold if x_j is defined. Thus the self distance property of the metric does not hold for any finite sequences.

After Matthews[1] enormous work done in partial metric spaces. Several authors proved the existence and uniqueness of fixed points also providing applications, see e.g., [2], [3], [4], [5],[6], [7], [8], [9]. In [7] Pragadeeswarar and Marudai established a fixed point theorem for a contraction map satisfying a rational expression in partial metric spaces.

Theorem 1.1. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a partial metric p in X such that (X, p) is a complete partial metric space. Let $T : X \to X$ be a continuous and nondecreasing mapping such that*

$$p(Tx, Ty) \leq \frac{\alpha p(x, Tx)p(y, Ty)}{p(x, y)} + \beta p(x, y),$$

for $x, y \in X, x \geq y, x \neq y$, with $\alpha \geq 0, \beta \geq 0, \alpha + \beta < 1$. If there exists $x_0 \in X$ with $x_0 \leq Tx_0$, then T has fixed point $z \in X$ and $p(z, z) = 0$.

Recently in 2011 Azam, Fisher, Khan introduced complex valued metric spaces[12] which is a special class of cone metric spaces. The idea of complex valued metric spaces can be exploited to define complex valued normed spaces and complex valued Hilbert spaces also it encourages numerous research activities in mathematical analysis. Azam et al.,[12] proved the following existence and uniqueness fixed point theorem for a pair of maps satisfying a contraction condition with rational expression. Although many results in analysis cannot be generalized to cone metric spaces, yet the idea is intended to define rational expression is not meaningful in cone metric spaces.

Theorem 1.2. *Let (X, d) be a complete complex valued metric space and let the mappings $S, T : X \to X$ satisfy:*

$$d(Sx, Ty) \leq \lambda d(x, y) + \frac{\mu d(x, Sx)d(y, Ty)}{1 + d(x, y)}$$

for $x, y \in X$, where λ, μ are non negative reals with $\lambda + \mu < 1$. Then S, T have a unique common fixed point.

Recently many authors have been done a wide range of research in complex valued metric spaces, see.,[10], [14], [15], [16], [17],[18], [19], [21], [22], etc.

The aim of this article to introduce the concept of a complex partial metric spaces and to study the fixed point and common fixed point results for two mappings satisfying rational inequalities. The results of Harjani [11], Pragadeeswarar and Marudai [7] are going to be the special case of our result for the real partial metric space. Also we provide examples to illustrate our results.

First we recollect some of the definitions of the complex valued metric spaces[12] and some of their properties.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\leq$ on $\mathbb{C}$ as follows: $z_1 \leq z_2$ if and only if $Re(z_1) \leq Re(z_2)$, $Im(z_1) \leq Im(z_2)$.

It follows that $z_1 \leq z_2$ if one the following condition is satisfied:

1. $Re(z_1) = Re(z_2)$, $Im(z_1) < Im(z_2)$
2. $Re(z_1) < Re(z_2)$, $Im(z_1) = Im(z_2)$
3. $Re(z_1) < Re(z_2)$, $Im(z_1) < Im(z_2)$
4. $Re(z_1) = Re(z_2)$, $Im(z_1) = Im(z_2)$

In particular we will write $z_1 \lneq z_2$ if one of the $(1), (2)$ and (4) is satisfied, we write $z_1 \lneq z_2$ if only (3) satisfied and

1. $z_1 \leq z_2 \Longrightarrow |z_1| \leq |z_1|$,
2. $0 \leq z_1 \leq z_2$ and $0 \leq c \in \mathbb{C} \Longrightarrow c + z_1 \leq c + z_2$

Here $\mathbb{C}^+$ denotes for all $0 \leq c \in \mathbb{C}$, we now give the definition for complex partial metric space.

Definition 1.1. *A complex partial metric on a non empty set X is a function $p_c : X \times X \to \mathbb{C}^+$ such that for all $x, y, z \in X$:*

1. $0 \leq p_c(x, x) \leq p_c(x, y)$ *(small self-distances)*
2. $p_c(x, y) = p_c(y, x)$ *(symmetry)*
3. $p_c(x, x) = p_c(x, y) = p_c(y, y)$ *if and only if* $x = y$ *(equality)*

4. $p_c(x,y) \le p_c(x,z) + p_c(z,y) - p_c(z,z)$ *(triangularity)*

For the complex partial metric p_c on X, the function $d_{p_c} : X \times X \to \mathbb{C}^+$ given by $d_{p_c} = 2p_c(x,y) - p_c(x,x) - p_c(y,y)$ is a (usual) metric on X. Each complex partial metric p_c on X generates a topology τ_{p_c} on X with the base family of open p_c-balls $\{B_{p_c}(x,\epsilon) : x \in X, \epsilon > 0\}$, where $B_{p_c}(x,\epsilon) = \{y \in X : p_c(x,y) < p_c(x,x) + \epsilon\}$ for all $x \in X$ and $0 < \epsilon \in \mathbb{C}^+$

A complex valued metric space is a complex partial metric space. But a complex partial metric space need not be a complex valued metric space. The following example illustrate such a complex partial metric space.

Example 1.2. *Let $X = [0,\infty)$ endowed with complex partial metric p_c is defined by $p_c : X \times X \to \mathbb{C}^+$ with*

$p_c(x,y) = max\{x,y\} + i\ max\ \{x,y\}$ *for all* $x,y \in X$,

It is easy to verify that (X,p_c) is a complex partial metric space and note that self distance need not be zero , for example $p_c(1,1) = 1 + i \neq 0$. Now the metric induced by p_c is follows, $d_{p_c} = 2p_c(x,y) - p_c(x,x) - p_c(y,y)$ without loss of generality suppose $x \ge y$ then $d_{p_c} = 2\{max\{x,y\} + imax\{x,y\}\} - \{x + ix\} - \{y + iy\}$. Therefore, $d_{p_c}(x,y) = |x-y| + i|x-y|$.

We can easily verify the following definitions and lemma.

Theorem 1.3. *Let (X,p_c) be a complex partial metric space, then (X,p_c) is T_0.*

Proof. Suppose $x,y \in X$ and $x \neq y$, from condition (1) and (3) in Definition 1.1, we get $p_c(x,x) < p_c(x,y)$ or $p_c(y,y) < p_c(x,y)$. We suppose that $p_c(x,x) < p_c(x,y)$, which implies that $0 < p_c(x,y) - p_c(x,x)$. Now let $c_x \in \mathbb{C}^+$ such that $0 < c_x < p_c(x,y) - p_c(x,x)$. So we find that $x \in B_{p_c}(x,c_x)$ and $y \notin B_{p_c}(x,c_x)$. Then we conclude that (X,p_c) is T_0. □

Definition 1.3. *Let (X,p_c) be a complex partial metric space (CPMS). A sequence (x_n) in a CPMS (X,p_c) is converges to $x \in X$, if for every $0 < \epsilon \in \mathbb{C}^+$ there is $N \in \mathbb{N}$ such that for all $n \ge N$ we get $x_n \in B_{p_c}(x,\epsilon)$.*

Then x said to be a limit of (x_n), which is denoted by $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$.

Lemma 1.4. *Let (X,p_c) be a complex partial metric space. A sequence (x_n) in a CPMS (X,p_c) is converges to $x \in X$ if and only if $p_c(x,x) = \lim\limits_{n\to\infty} p_c(x,x_n)$.*

Proof. Suppose that (x_n) converges to x, for a given real number $\epsilon > 0$, let

$$c = \frac{\epsilon}{\sqrt{2}} + i\frac{\epsilon}{\sqrt{2}}.$$

Then $0 < c \in \mathbb{C}$ and there is a natural number N, such that $x_n \in B_{p_c}(x,c)$ for all $n \ge N$ i.e., $p_c(x_n,x) < c + p_c(x,x)$. So that when $n \ge N$, $|p_c(x_n,x) - p_c(x,x)| < \epsilon$. This means that $p_c(x_n,x) \to p_c(x,x)$ $(n \to \infty)$.

Conversely, suppose that $p_c(x_n, x) \to p_c(x, x)$ $(n \to \infty)$. For each $0 < c \in \mathbb{C}$, there exists a real number $\delta > 0$ such that

$$|z| < \delta \Rightarrow z < c.$$

For this $\delta > 0$, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$ we have

$$|p_c(x_n, x) - p_c(x, x)| < \delta.$$

Which implies that $p_c(x_n, x) < c + p_c(x, x)$ for all $n \geq N$. Hence x_n converges to x. □

We note that let (X, p_c) be a complex partial metric space. If $p_c(x_n, x) \to p_c(x, x)$ $(n \to \infty)$ then $p_c(x_n, x_n) \to p_c(x, x)$ $(n \to \infty)$.

Definition 1.5. *Let (X, p_c) be a complex partial metric space. A sequence (x_n) in a CPMS (X, p_c) is called Cauchy if there is $a \in \mathbb{C}^+$ such that for every $\epsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n, m \geq N$*

$$|p_c(x_n, x_m) - a| < \epsilon$$

Definition 1.6. *Let (X, p_c) be a complex partial metric space (CPMS).*

1. *A CPMS (X, p_c) is said to be complete if a Cauchy sequence (x_n) in X converges, with respect to τ_{p_c}, to a point $x \in X$ such that $p_c(x, x) = \lim_{n,m\to\infty} p_c(x_n, x_m)$.*
2. *A mapping $T : X \to X$ is said to be continuous at $x_0 \in X$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that $T(B_{p_c}(x_0, \delta)) \subset B_{p_c}(T(x_0), \epsilon)$.*

Lemma 1.7. *Let (X, p_c) be a complex partial metric space. A sequence (x_n) is Cauchy sequence in the CPMS (X, p_c) then (x_n) is Cauchy in a metric space (X, d_{p_c}).*

Proof. Let (x_n) be a Cauchy sequence in (X, p_c). There is $a \in \mathbb{C}^+$ such that for every real $\epsilon > 0$, there is $N \in \mathbb{N}$, for all $n, m \geq N$, $|p_c(x_n, x_m) - a| < \dfrac{\epsilon}{4}$. Hence

$$d_{p_c}(x_n, x_m) = 2(p_c(x_n, x_m) - a) - (p_c(x_n, x_n) - a) - (p_c(x_m, x_m) - a)$$

for $n, m \geq N$, we have $|d_{p_c}(x_n, x_m)| < \epsilon$. That is $d_{p_c}(x_n, x_m) \to 0$ $(n, m \to \infty)$. □

Let X be a complex partial metric space and $A \subseteq X$. A point $x \in X$ is called an interior point of set A, if there exists $0 < r \in \mathbb{C}$ such that $B_{p_c}(x, r) = \{y \in X : p_c(x, y) < p_c(x, x) + r\} \subseteq A$. A subset A is called open, if each point of A is an interior point of A. A point $x \in X$ is said to be a limit point of A, for every $0 < r \in \mathbb{C}$, $B_{p_c}(x, r) \cap (A - \{x\}) \neq \emptyset$. A subset $B \subseteq X$ is called closed, B contains all it's limit points.

The following definition is given by Radenovic[10].

Definition 1.8. *Let $(X, \preceq)$ be a partially ordered set. A pair (f, g) of self-maps of X is said to be weakly increasing if $fx \preceq fgx$ and $gx \preceq gfx$ for all $x \in X$. If $f = g$, then we have $fx \preceq f^2x$ for all $x \in X$ and in this case, we say that f is a weakly increasing mapping.*

A point $x \in X$ is said to be common fixed point for the pair of self mappings (f, g) on X is such that $x = fx = gx$.

2 Main Results

In this section we discussed the common fixed point results for weakly increasing maps on an ordered complex partial metric space.

Theorem 2.1. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a complex partial metric p_c in X such that (X, p_c) is a complete complex partial metric space. Let $T, S : X \to X$ be a pair of weakly increasing mapping and suppose that for every comparable $x, y \in X$ we have either*

$$p_c(Sx, Ty) \leq \frac{\alpha p_c(x, Sx)p_c(y, Ty)}{p_c(x, y)} + \beta p_c(x, y)$$

for $p_c(x, y) \neq 0$ with $\alpha \geq 0, \beta \geq 0, \alpha + \beta < 1$, or

$$p_c(Sx, Ty) = 0 \text{ if } p_c(x, y) = 0.$$

If S or T is continuous then S and T have a common fixed point $z \in X$ and $p_c(z, z) = 0$.

Proof. First we shall show that if S or T has a fixed point then it is a common fixed point of S and T .

Let z be a fixed point of S. Suppose $p_c(z, z) = 0$ then we have $p_c(Sz, Tz) = 0$ implies that $Sz = Tz$.

Suppose $p_c(z, z) \neq 0$ then

$$\begin{aligned} p_c(z, Tz) &= p_c(Sz, Tz) \\ &\leq \frac{\alpha p_c(z, Sz)p_c(z, Tz)}{p_c(z, z)} + \beta p_c(z, z). \end{aligned}$$

Since $p_c(z, z) \leq p_c(z, Tz)$ we have $|p_c(z, Tz)| \leq (\alpha + \beta)|p_c(z, Tz)|$. As $\alpha + \beta < 1$ so we have $p_c(z, Tz) = 0$ and z is a common fixed point of S and T. Similarly, if z is a fixed point of T, then it is a common fixed point of S. Now let x_0 be an arbitrary point in X and define

$$\begin{aligned} x_{2k+1} &= Sx_{2k} \\ x_{2k+2} &= Tx_{2k+1}, k = 0, 1, 2, \cdots . \end{aligned}$$

Since S and T are weakly increasing,

$$x_1 = Sx_0 \preceq TSx_0 = Tx_1 = x_2 \text{ and}$$
$$x_2 = Tx_1 \preceq STx_1 = Sx_2 = x_3.$$

Continuing this way, we have $x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_{n+1} \cdots$

Assume that $p_c(x_{2k}, x_{2k+1}) > 0$ for all $k \in \mathbb{N}$. If not, then $x_{2k} = x_{2k+1}$ for some k. For all those k, $x_{2k} = x_{2k+1} = Sx_{2k}$ and the proof is finished. Assume that $p_c(x_{2k}, x_{2k+1}) > 0$ for $k = 0, 1, 2, \cdots$ As x_{2k} and x_{2k+1} are comparable, so we have

$$\begin{aligned}
p_c(x_{2k+1}, x_{2k+2}) &= p_c(Sx_{2k}, Tx_{2k+1}) \\
&\leq \alpha \frac{p_c(x_{2k}, Sx_{2k}) p_c(x_{2k+1}, Tx_{2k+1})}{p_c(x_{2k}, x_{2k+1})} + \beta p_c(x_{2k}, x_{2k+1}) \\
&\leq \alpha p_c(x_{2k+1}, Tx_{2k+1}) + \beta p_c(x_{2k}, x_{2k+1}) \\
p_c(x_{2k+1}, x_{2k+2}) &\leq \frac{\beta}{1-\alpha} p_c(x_{2k}, x_{2k+1})
\end{aligned}$$

Now with $h = \dfrac{\beta}{1-\alpha}$, we have

$$\begin{aligned}
p_c(x_{2k+1}, x_{2k+2}) &\leq h p_c(x_{2k}, x_{2k+1}) \cdots \leq h^{2k+1} p_c(x_0, x_1) \\
p_c(x_n, x_m) &\leq p_c(x_m, x_{m-1}) + \cdots + p_c(x_{n+1}, x_n) - \sum_{i=1}^{m-n-1} p_c(x_{m-i}, x_{m-i}) \\
&\leq [h^{m-1} + \cdots + h^n] p_c(x_1, x_0) \\
&= h^n \frac{1 - h^{m-n}}{1-h} p_c(x_1, x_0) \\
|p_c(x_m, x_n)| &\leq \frac{h^n}{1-h} |p_c(x_1, x_0)| \to 0
\end{aligned}$$

as $m, n \to \infty$ which implies that $\lim_{n,m\to\infty} p_c(x_n, x_m) = 0$ such that x_n is a Cauchy sequence in X. Since (X, p_c) is complete there exists $z \in X$ such that $x_n \to z$ and

$$p_c(z, z) = \lim_{n\to\infty} p_c(z, x_n) = \lim_{n,m\to\infty} p_c(x_n, x_n) = 0.$$

Without loss of generality suppose T is continuous in (X, p_c). Therefore, $Tx_{2n+1} \to Tz$ in (X, p_c). i.e.,

$$p_c(Tz, Tz) = \lim_{n\to\infty} p_c(Tz, Tx_{2n+1}) = \lim_{n\to\infty} p_c(Tx_{2n+1}, Tx_{2n+1})$$

But

$$p_c(Tz, Tz) = \lim_{n\to\infty} p_c(Tx_{2n+1}, Tx_{2n+1}) = \lim_{n\to\infty} p_c(x_{2n+2}, x_{2n+2}) = 0.$$

Next we have to prove that z is a fixed point of T.

$$p_c(Tz, z) \leq p_c(Tz, Tx_{2n+1}) + p_c(Tx_{2n+1}, z) - p_c(Tx_{2n+1}, Tx_{2n+1}).$$

As $n \to \infty$, we obtain $p_c(Tz, z) \leq 0$. Thus, $p_c(Tz, z) = 0$. Hence $p_c(z, z) = p_c(z, Tz) = p_c(Tz, Tz) = 0$ and so $Tz = z$. Therefore $Sz = Tz = z$ and $p_c(z, z) = 0$. □

In the following Theorem we prove that Theorem 2.1 is still valid with out assuming continuity condition on T.

Theorem 2.2. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a complex partial metric p_c in X such that (X, p_c) is a complete complex partial metric space. Let $T, S : X \to X$ be a pair of weakly increasing mapping and suppose that for every comparable $x, y \in X$ we have either*

$$p_c(Sx, Ty) \leq \frac{\alpha p_c(x, Sx) p_c(y, Ty)}{p_c(x, y)} + \beta p_c(x, y)$$

for $p_c(x, y) \neq 0$ with $\alpha \geq 0, \beta \geq 0, \alpha + \beta < 1$, or $p_c(Sx, Ty) = 0$ if $p_c(x, y) = 0$.

Suppose X satisfying the condition that, for every increasing sequence (x_n) with $x_n \to z$ in X , we necessarily have $z = supx_n$ then S and T have a common fixed point $z \in X$ and $p_c(z, z) = 0$.

Proof. Given that $x_n \preceq z$ for all $n \in \mathbb{N}$. Following the proof of Theorem 2.1, It is enough to prove that z is a fixed point of S. suppose z is not a fixed point, then we have $p_c(z, Sz) = u > 0$ for some $u \in \mathbb{C}$, we obtain

$$\begin{aligned} u &\leq p_c(z, x_{2n+2}) + p_c(x_{2n+2}, Sz) - p_c(x_{2n+2}, x_{2n+2}) \\ &= p_c(z, x_{2n+2}) + p_c(Tx_{2n+1}, Sz) - p_c(x_{2n+2}, x_{2n+2}) \\ &\leq p_c(z, x_{2n+2}) + \alpha \frac{p_c(x_{2n+1}, Tx_{2n+1}) p_c(z, Sz)}{p_c(x_{2n+1}, z)} + \beta p_c(x_{2n+1}, z) - p_c(x_{2n+2}, x_{2n+2}). \end{aligned}$$

Suppose $p_c(z, z) = 0$, taking limit as $n \to \infty$ we have $u \leq 0$ which is a contradiction. Therefore z is a fixed point of S.

For $p_c(z, z) \neq 0$, Taking limit as $n \to \infty$ we have $u \leq \alpha p_c(z, Sz) + \beta p_c(z, z)$ and so, $|u| \leq (\alpha + \beta)|u|$ since $\alpha + \beta < 1$ we get a contradiction, which implies that $z = Sz$. Therefore by Theorem 2.1 we get $Sz = Tz = z$ and $p_c(z, z) = 0$. □

Theorem 2.3. *In addition to the hypothesis of the theorem2.1(or theorem2.2), suppose that the set of common fixed points of S and T is totally ordered if and only if S and T have a unique common fixed point.*

Proof. Now suppose that the common fixed points of S and T are totally ordered. We have to prove that common fixed points of S and T are unique. Assume on the contrary that z and w are distinct common fixed points of S and T. By supposition, we can replace x by z and y by w in 2.1 to obtain for $p_c(z, w) \neq 0$

$$\begin{aligned} p_c(z, w) &= p_c(Sz, Tw) \\ &\leq \alpha \frac{p_c(z, Sz) p_c(w, Tw)}{p_c(z, w)} + \beta p_c(z, w) \\ p_c(z, w) &\leq \beta p_c(z, w), \end{aligned}$$

a contradiction. Hence $z = w$. Conversely, if S and T have only one common fixed point then the set of common fixed point of S and T being singleton is totally ordered. □

Corollary 2.4. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a complex partial metric p_c in X such that (X, p_c) is a complete complex partial metric space. Let $T : X \to X$ be a weakly increasing mapping and suppose that for every comparable $x, y \in X$ we have either*

$$p_c(Tx, Ty) \le \frac{\alpha p_c(x, Tx) p_c(y, Ty)}{p_c(x, y)} + \beta p_c(x, y)$$

for $p_c(x, y) \neq 0$ with $\alpha \ge 0, \beta \ge 0, \alpha + \beta < 1,$ or

$$p_c(Tx, Ty) = 0$$

if $p_c(x, y) = 0$. Suppose T is continuous or for every increasing sequence (x_n) with $x_n \to z$ in X , we necessarily have $z = sup x_n$, then T has a fixed point $z \in X$ and $p_c(z, z) = 0$. Moreover, the set of fixed points of T is totally ordered if and only if T has a unique fixed point.

Corollary 2.5. *Let $(X, \preceq)$ be a partially ordered set such that (X, p_c) is a complete complex partial metric space and $T : X \to X$ satisfy:*

$$p_c(T^n x, T^n y) \le \frac{\alpha p_c(x, T^n x) p_c(y, T^n y)}{p_c(x, y)} + \beta p_c(x, y)$$

for $p_c(x, y) \neq 0$ with $\alpha \ge 0, \beta \ge 0, \alpha + \beta < 1,$ or

$$p_c(Tx, Ty) = 0$$

if $p_c(x, y) = 0$.Then T has a unique fixed point.

Proof. By Corollary 2.4 we obtain $z \in X$ such that $T^n z = z$.

Suppose $p_c(Tz, z) = 0$ there is nothing to prove, for $p_c(Tz, z) \neq 0$ we have

$$\begin{aligned} p_c(Tz, z) &= p_c(TT^n z, T^n z) = p_c(T^n Tz, T^n z) \\ &\le \alpha \frac{p_c(Tz, T^n z) p_c(z, T^n z)}{p_c(Tz, z)} + \beta p_c(z, Tz) \\ &\le (\alpha + \beta) p_c(z, Tz) \end{aligned}$$

a contradiction. Therefore $Tz = z$. □

Example 2.6. *Let $X = \{1, 2, 3, 4\}$ be endowed with the order $x \preceq y$ if and only if $y \le x$. Then $\preceq$ is a partial order in X. Define the complex partial metric $p_c : X \times X \to \mathbb{C}$ as follows:*

(x,y)	$p_c(x,y)$
$(1,1),(2,2)$	0
$(1,2),(2,1),(1,3),(3,1),(2,3),(3,2),(3,3)$	$e^{i\theta}$
$(1,4),(4,1),(2,4),(4,2),(3,4),(4,3),(4,4)$	$3e^{i\theta}$

It is easy to verify that (X,p_c) is a complex partial metric space for $\theta \in [0,\frac{\pi}{2}]$.

Note that $p_c(3,3) \neq 0$, so (X,p_c) is not a complex valued metric space. Define $S,T : X \to X$ by $Sx = 1$, $Tx = \begin{cases} 1 & \text{if } x \in \{1,2,3\} \\ 2 & \text{if } x = 4 \end{cases}$

Note that $Sx \preceq TSx$ and $Tx \preceq STx$ for all $x \in X$. Now for $\alpha = \beta = \frac{1}{3}$, we consider the following cases:

1. *If $x = 1$ and $y \in X - \{4\}$, then $Sx = Ty = 1$ and $p_c(Sx,Ty) = 0$ and the conditions of Theorem2.1 satisfied.*
2. *For $x = 1$,$y = 4$, $Sx = 1$ $Ty = 2$,*
$$\begin{aligned} p_c(Sx,Ty) &= e^{i\theta} \leq 3\beta e^{i\theta} \\ &= \alpha\frac{(0)(e^{i\theta})}{3e^{i\theta}} + \beta(3e^{i\theta}) \\ &= \alpha\frac{p_c(x,Sx)p_c(y,Ty)}{p_c(x,y)} + \beta p_c(x,y) \end{aligned}$$
3. *When $x = 2$,$y = 4$, $Sx = 1$ $Ty = 2$,*
$$\begin{aligned} p_c(Sx,Ty) &= e^{i\theta} \leq (\alpha + 3\beta)e^{i\theta} \\ &= \alpha\frac{(e^{i\theta})(3e^{i\theta})}{3e^{i\theta}} + \beta(3e^{i\theta}) \\ &= \alpha\frac{p_c(x,Sx)p_c(y,Ty)}{p_c(x,y)} + \beta p_c(x,y) \end{aligned}$$
4. *Suppose $x = 3$,$y = 4$, $Sx = 1$ $Ty = 2$,*
$$\begin{aligned} p_c(Sx,Ty) &= e^{i\theta} \leq (\alpha + 3\beta)e^{i\theta} \\ &= \alpha\frac{(e^{i\theta})(3e^{i\theta})}{3e^{i\theta}} + \beta(3e^{i\theta}) \\ &= \alpha\frac{p_c(x,Sx)p_c(y,Ty)}{p_c(x,y)} + \beta p_c(x,y) \end{aligned}$$
5. *For $x = 4$,$y = 4$, $Sx = 1$ $Ty = 2$,*
$$\begin{aligned} p_c(Sx,Ty) &= e^{i\theta} \leq 3(\alpha + \beta)e^{i\theta} \\ &= \alpha\frac{(3e^{i\theta})(3e^{i\theta})}{3e^{i\theta}} + \beta(3e^{i\theta}) \\ &= \alpha\frac{p_c(x,Sx)p_c(y,Ty)}{p_c(x,y)} + \beta p_c(x,y) \end{aligned}$$

Moreover for $\alpha = \beta = \dfrac{1}{3}$, *with* $\alpha + \beta = \dfrac{2}{3} < 1$ *the conditions of Theorem2.1 are satisfied. Therefore,* 1 *is the unique common fixed point of* S *and* T*.*

Acknowledgement(s) : I would like to thank the referee(s) for his comments and suggestions on the manuscript.

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PUBLIC INTEREST STATEMENT

Recently in 2011 Azam, Fisher, Khan introduced a complex valued metric spaces which is a special class of cone metric spaces. The idea of complex valued metric spaces can be exploited to define complex valued normed spaces and complex valued Hilbert spaces also it encourages numerous research activities in mathematical analysis. Usually in a metric space self distance is zero (i.e., $d(x, x) = 0$), but in partial metric space the self distance need not be equal to zero. In this paper we introduced a complex partial metric spaces which is a generalization of complex valued metric space.