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On the energy of frenet vectors fields in R^n

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Abstract: In this paper, we compute the energy of Frenet vector fields for a given curve C in n-dimensional Euclidean space. We observe that the energy and angle may be expressed in terms of the curvature functions of C. If the first curvature function of the curve is the identity function then its first integral is the angle between the velocity vectors and its second integral gives the energy of the velocity vector of the curve.

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Keywords: energy; Frenet vector field; Sasaki metric; unit vector field

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1. Introduction

The volume of unit vector fields has been studied by Gluck and Ziller (1986), Johnson (1988), and Higuchi, Kay, and Wood (2001) among other scientists. They define the volume of unit vector field X as the volume of the submanifold of the unit tangent bundle defined by $X(M)$. In Wood (1997), the energy of a unit vector field on a Riemannian manifold M is defined as the energy of the mapping $X : M \rightarrow T^1 M$, where the unit tangent bundle $T^1 M$ is equipped with the restriction of the Sasaki metric.

Generally, every geometric problem about curves can be solved using the curves' Frenet vectors field. Therefore, in this paper, we focus on the curve C, instead of the manifold M. Let C be a curve with a pair (I, α) of parametric unit speed in R^n. Let us take an initial point $a \in I$ and the Frenet frames $\{V_1(\alpha(a)), \ldots, V_r(\alpha(a))\}$ and $\{V_1(\alpha(s)), \ldots, V_r(\alpha(s))\}$ at the points $\alpha(a)$ and $\alpha(s)$, respectively. We calculated the energy of a Frenet vector field and, in Altın (1999), the angle between each vector $V_i(\alpha(a))$ and $V_i(\alpha(s))$ where $1 \leq i \leq r$. Further, we observed that the energy and angle may be expressed in terms of the curvature functions of the given curve C. If the 1st curvature function of the curve is identity function its first integral is angle between vectors $V_1(\alpha(a))$ and $V_1(\alpha(s))$, its second integral is the energy, from a to s, of the velocity vector of curve. Then we defined the energy of the velocity and velocity field of the curve. This definition will give us a new approach to elastic curves (see Brander, Gravesen, & Nrbjerg, 2017; Guven , Valencia, & Vazquez-Montejo, 2014; Santiago, Chacón-Acosta, & Gonzalez-Gaxiola, 2013 for examples).

ABOUT THE AUTHOR

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PUBLIC INTEREST STATEMENT

In this study we calculated the energy of the Frenet vector fields of a curve and found the energy of the velocity vector field as half of the integral of the square of the first curvature function.

We expect that this result will give us a new approach to Classical Bernoulli-Euler Elastic Curves.

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Definition 1.1 Let I be an open interval in R and α be a differentiable map $\alpha: I \to R^n$. We call $\alpha(I) = C$ a curve C in R^n and (I, α) a parametric pair for C.

THEOREM 1.1 *Let (I, α) be a parametric pair for a curve C in a space R^n. There exists a parametric pair (J, β) of the curve C such that for each $r \in J$, $\|\beta'(r)\| = 1$, where J is an open interval in R. The pair (J, β) is called a parametric pair with unit speeds (O'Neill, 1966).*

Definition 1.2 Let (I, α) be a parametric pair of a curve C in a space R^n. Let the system $\Psi = \{\alpha', \alpha'', \ldots, \alpha^r\}$ be a maximal linearly independent set. The orthonormal system $\{V_1, V_2, \ldots, V_r\}$ obtained from Ψ is named as Frenet frame fields of C, and $\{V_1(\alpha(s)), V_2(\alpha(s)), \ldots, V_r(\alpha(s))\}$ at the point $\alpha(s) \in C$ as Frenet frames.

Definition 1.3 Let (I, α) be a parametric pair for a curve C in a space R^n and $\{V_1(\alpha(s)), V_2(\alpha(s)), \ldots, V_r(\alpha(s))\}$ be Frenet frames at the point $\alpha(s) \in C$. Let

$$\forall s \in I, \ k_i(\alpha(s)) = k_i(s) = <V_i^{'}(\alpha(s)), \ V_{i+1}(\alpha(s)) > = <V_i(s), V_{i+1}(s)>, \ 1 \le i < r,$$

be defined as curvature function on C and the real number $k_i(\alpha(s))$ be defined as ith curvature on C at the point $\alpha(s)$.

THEOREM 1.2 *Let (I, α) be a parametric pair for a curve C in a space R^n. If we take ith curvature $k_i(\alpha(s))$ and Frenet frames $\{V_1(\alpha(s)), V_2(\alpha(s)), \ldots, V_r(\alpha(s))\}$ at the point $\alpha(s)$, then, the following relations are hold:*

$$V_1^{'}(s) = k_1(s)V_2(s) \tag{1}$$

$$V_i^{'}(s) = -k_{i-1}(s)V_{i-1}(s) + k_i(s)V_{i+1}(s), \ 1 < i < r \tag{2}$$

$$V_r^{'}(s) = -k_{r-1}(s)V_{r-1}(s). \tag{3}$$

Proposition 1.1 The connection map $K: T(T^1M) \to T^1M$ verifies the following conditions.

(1) $\pi \circ K = \pi \circ d\pi$ and $\pi \circ K = \pi \circ \widetilde{\pi}$, where $\widetilde{\pi}: T(T^1M) \to T^1M$ is the tangent bundle projection.

(2) For $\omega \in T_xM$ and a section $\xi: M \to T^1M$, we have:

$$K(d\xi(\omega)) = \nabla_\omega \xi$$

where ∇ is the Levi–Civita covariant derivative (Chacón, Naveira, & Weston, 2001).

Definition 1.4 For $\eta_1, \eta_2 \in T_\xi(T^1M)$, we define

$$g_s(\eta_1, \eta_2) = <d\pi(\eta_1), d\pi(\eta_2)> + <K(\eta_1), K(\eta_2)>. \tag{4}$$

This gives a Riemannian metric on TM. As mentioned, g_s is called the Sasaki metric. The metric g_s makes the projection $\pi: T^1M \to M$ a Riemannian submersion (Chacón et al., 2001).

Definition 1.5 The energy of a differentiable map

$f: (M, <, >) \to (N, h)$ between Riemannian manifolds is given by

$$\mathcal{E}(f) = \frac{1}{2} \int_M \left(\sum_{a=1}^{n} h(df(e_a), df(e_a))v \right)$$

(5)

where v is the canonical volume form in M and $\{e_a\}$ is a local basis of the tangent space (see Chacón & Naveira, 2004; Wood, 1997).

The energy of a unit vector field X is defined to be the energy of the section $X: M \to T^1 M$, where $T^1 M$ is the unit tangent bundle equipped with the restriction of the Sasaki metric on TM. Now let $\pi: T^1 M \to M$ be the bundle projection, and let $T(T^1 M) = \mathcal{V} \oplus \mathcal{H}$ denote the vertical/horizontal splitting induced by the Levi–Civita connection. Further, define $TM = \mathcal{F} \oplus \mathcal{G}$ where $\mathcal{F}$ denotes the line bundle generated by X, and $\mathcal{G}$ is the orthogonal complement (Chacón et al., 2001). Furthermore, the energy of the velocity vector fields of the curve is related to the elastic curves (elastica) (see Brander et al., 2017; Guven et al., 2014; Santiago et al., 2013 for examples).

2. The energy of Frenet vectors fields

Now, we are in a position to prove our main result which was pointed out before.

THEOREM 2.1   *Let C be a curve with a pair (I, α) of parametric unit speeds in R^n. Let us take an initial point $a \in I$. Further, let*

$\{V_1(\alpha(a)), \ldots, V_r(\alpha(a))\}$ and $\{V_1(\alpha(s)), \ldots, V_r(\alpha(s))\}$

be the Frenet frames at the points $\alpha(a)$ and $\alpha(s)$, respectively. Then we have the following conditions:

(i) If the energy of V_i is $\mathcal{E}(V_i(s))$, i.e. if the function $\mathcal{E}(V_i)$ is defined as

$\mathcal{E}(V_i): I \to R,\ 1 \le i \le r.$

then the following relations are valid.

$$\mathcal{E}(V_1)(s) = \frac{1}{2} \int_a^s k_1^2(u)\, du + \frac{1}{2}(s-a)$$

$$\mathcal{E}(V_i)(s) = \frac{1}{2} \int_a^s \left(k_{i-1}^2(u) + k_i^2(u) \right) du + \frac{1}{2}(s-a), \quad 2 \le i \le r-1$$

$$\mathcal{E}(V_r)(s) = \frac{1}{2} \int_a^s k_{r-1}^2(u)\, du + \frac{1}{2}(s-a).$$

(ii) If the angle between vectors $V_i(\alpha(a))$ and $V_i(\alpha(s))$ is $\theta_i(s)$ i.e. if the function θ_i is defined as $\theta_i: I \to R,\ 1 \le i \le r$, then the following relations are valid (Altın, 1999).

$$\theta_1(s) = \int_a^s k_1(u)\, du$$

$$\theta_i(s) = \int_a^s \sqrt{k_{i-1}^2(u) + k_i^2(u)}\, du, \quad 2 \le i \le r-1$$

$$\theta_r(s) = \int_a^s |k_{r-1}(u)| du.$$

Proof (i) Let TC be the tangent bundle and let $\{V_1, V_2, \ldots, V_r\}$ be Frenet vector fields of the curve C. So we have $V_1 : C \to TC = \bigcup_{t \in I} T_{\alpha(t)} C$. Let $\pi : TC \to C$ be the bundle projection. The Levi–Civita connection map $K : T(TC) \to TC$. By using Equation (5) we obtain the energy of V_1 as

$$\mathcal{E}(V_1)(s) = \frac{1}{2} \int_a^s g_S(dV_1(V_1(u)), dV_1(V_1(u))du \tag{6}$$

where du is the element of arc length. From (4) we have

$$g_S(dV_1(V_1), dV_1(V_1)) = <d\pi(dV_1(V_1)), d\pi(dV_1(V_1))> + <K(dV_1(V_1)), K(dV_1(V_1))>$$

Since V_1 is a section, we obtain $d(\pi) \circ d(V_1) = d(\pi \circ V_1) = d(id_C) = id_{TC}$.

On the other hand, by Proposition 1.1, we may write

$$K(dV_1(V_1)) = \nabla_{V_1} V_1 = V_1'.$$

Then we obtain

$$g_S(dV_1(V_1), dV_1(V_1)) = <V_1, V_1> + <V_1', V_1'>.$$

From (1) we get

$$g_S(dV_1(V_1), dV_1(V_1)) = 1 + k_1^2. \tag{7}$$

By putting (7) in (6), we get

$$\mathcal{E}(V_1(s)) = \frac{1}{2} \int_a^s k_1^2(u)du + \frac{1}{2}(s - a).$$

Let $N_i C$ be the ith normal bundle. Thus we have $V_i : C \to N_i C$ where $N_i C = \bigcup_{t \in I} N_{i\alpha(t)} C$ and here $N_{i\alpha(t)} C$ denotes generated by V_i. Now, let $\pi_i : N_i C \to C$ be the ith bundle projection. The Levi–Civita connection map $K_i : T(N_i C) \to N_i C$ By using Equation (5), we obtain the energy of V_i, $2 \leq i \leq r$ as

$$\mathcal{E}(V_i)(s) = \frac{1}{2} \int_a^s g_S(dV_i(V_1(u), dV_i(V_1(u))du. \tag{8}$$

From (4) we have

$$g_S(dV_i(V_1), dV_i(V_1)) = <d\pi_i(dV_i(V_1)), d\pi_i(dV_i(V_1))> + <K(dV_i(V_1)), K(dV_i(V_1))>$$
$$= <d(\pi_i \circ V_i)(V_1), d(\pi_i \circ V_i)(V_1)> + <\nabla_{V_1} V_i, \nabla_{V_1} V_i>$$
$$= <V_1, V_1> + <V_i', V_i'>.$$

By (2) we have

$$g_S(dV_i(V_1), dV_i(V_1)) = 1 + k_{i-1}^2 + k_i^2$$

by using (8), we obtain

$$\mathcal{E}(V_i)(s) = \frac{1}{2} \int_a^s (k_{i-1}^2(u) + k_i^2(u))du + \frac{1}{2}(s - a), \ 2 \leq i \leq r - 1.$$

So, (3) gives us

$$g_s(dV_r(V_1), dV_r(V_1)) = 1 + k_{r-1}^2,$$

and then (8) yields

$$\mathcal{E}(V_r)(s) = \frac{1}{2}\int_a^s k_{r-1}^2(u)du + \frac{1}{2}(s-a).$$

We may ignore the constant term of $\frac{1}{2}(s-a)$ and we can give the following definition.

Definition 2.1 The integral

$$\mathcal{E}(V_1)(s) = \frac{1}{2}\int_a^s k_1^2(u)du$$

is called the energy of the velocity vector field of curve C at a fixed point $a \in I$, and is denoted by $\mathcal{E}(V_1)$.

COROLLARY 2.1 *If the first curvature function of the curve is identity function, then the first integral of the curvature function gives the angle between the velocity vectors and the second integral gives the energy of the velocity vector of the curve.*

Proof According to Theorem 2.1, if $k_1(s) = s$ then

$$\theta_1(s) = \int_0^s k_1(s)ds = \frac{1}{2}s^2$$

and

$$\mathcal{E}(V_1)(s) = \frac{1}{2}\int_0^s k_1^2(s)ds = \frac{1}{2}\int_0^s s^2 ds.$$

3. Conclusion

In this work, we calculate the energy of the Frenet vectors fields and the angle between the vectors $V_i(a)$ and $V_i(s)$, where $1 \le i \le r$. So, we see that both energy and angle depend on the curvature functions of the curve C and the energy of velocity vector field is $\mathcal{E}(V_1(s)) = \frac{1}{2}\int_a^s k_1^2(u)du + \frac{1}{2}(s-a)$. We may ignore the constant term of $\frac{1}{2}(s-a)$ and we can give the definition. The integral $\mathcal{E}(V_1)(s) = \frac{1}{2}\int_a^s k_1^2(u)du$ is called the energy of the velocity vector field of curve C at a fixed point $a \in I$.

On the other hand, the classical curve known as the elastica is the solution to a variational problem proposed by Daniel Bernoulli to Leonhard Euler (1744), that of minimizing the bending energy of a thin inextensible wire (See, e.g. Love, 1927). The mathematical idealization of this problem is that of minimizing the integral of the squared curvature for curves of a fixed length satisfying given first-order boundary data (Singer, 2007). It is obvious, the energy of the velocity vector fields of the curve is related to the elastic curves.

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