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A general optimization framework for the annuity contracts with multiscale stochastic volatility

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Abstract: This paper develops the annuity contract model in the presence of multiscale stochastic volatility for studying the optimal investment strategy before and after retirement in a defined contribution pension plan where benefits are paid under the form of annuities with a guaranteed payment for a predefined fixed period of time. Besides, we consider two types of volatility, including a fast-moving one and a slowly-moving one. By applying the maximum principle, Legendre transformation and dual theory, we transform the complicated nonlinear partial differential equation (PDE) for the value function to a linear PDE.

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Keywords: annuity; multiscale stochastic volatility; stochastic optimal control; Hamilton–Jacobi–Bellman equation

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1. Introduction

Currently, there are two main types of pension funds: defined benefit (DB) pension fund and defined contribution (DC) pension fund. DB fund is a pension fund where the retirement benefits are calculated by a predetermined formula. Retirement benefits are usually calculated using average salary over the last few years before retire and the number of years one has worked in the company or public service. In general, market fluctuations have limited effect on the value of benefit, although in periods of prolonged economic downturn, DB could be affected. If the fund performance is poor, the trustee will generally ask an employer to help pay member benefits as required. Whereas, in a DC

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Yan Zhang has been undertaking research into optimal asset liability management problems with applications in the insurance industry, particularly those involving actuarial insights in life insurance, general insurance (non-life), re-insurance, pension plans, and annuity contracts. The first publication relating to consideration of state-dependent risk aversion was financially supported by a Curtin International Postgraduate Research Scholarship (CIPRS). The annuity research project focuses on the impact of multiscale stochastic volatilities on the optimal investment strategy. Other works by the author relate to re-insurance with stochastic interest rate, and stochastic volatility. In the future, the mathematical model built in these published works will be extended according to realistic contexts, and industrial collaborations will be expected to result in practical solutions of benefit to the insurance community.

PUBLIC INTEREST STATEMENT

As the population aging problem becomes more and more serious in most countries around the world, subscription to pension plans and annuity products draws greater attention from both government and the general public. This paper establishes the general optimization framework for annuity contracts with realistic constraints, comprising the first part of the research project, followed by further contributions to analytical optimal strategy outlined with numerical examples. Financial institutes and individuals may benefit from this research work as it outlines ways to optimally allocate net wealth over two stages, i.e. before and post-retirement. As many governments are considering pension reform plans, more valuable output could be expected by taking factors such as increased contribution rate, extended retirement age, and flexible benefits, into account.

plan, fixed contributions are paid into an individual account by both employers and employees. The contributions are then invested, for example in the stock market and the returns on the investment (which may be positive or negative) are credited to the individual's account. On retirement, the member's account is used to provide retirement benefits, sometimes through the purchase of an annuity which they provided a regular income. Hence the financial risk is borne by the contributors. DC plans have become widespread all over the world in recent years and are now the dominant form of plan in the private sector in many countries, due to frequent job-hopping within one field of industry or between different fields of industry. Besides, the employer contribution rate is usually used as an important motivation strategy. In conclusion, compared with DB plan, DC plan is fairer because the more contribution one has made in his or her career, the more retirement income he or she will get after retirement. In our paper, we study an annuity model in the context of a DC pension fund management and derive optimal investment strategy for both accumulation stage and distribution stage.

Nowadays, asset allocation optimization problem of annuity/DC pension fund management has drawn more and more attention of talented researchers, many of which have contributed high quality research output. For example, Haberman and Vigna (2002) studied optimal investment strategies and risk measures in defined contribution pension schemes. Devolder, Bosch, and Dominguez (2003) considered optimal investment problem for an annuity model using stochastic optimal control theory. Deelstra, Grasselli, and Koehl (2004) investigated optimal design of the guarantee for defined contribution funds. Cairns, Blake, and Dowd (2006) showed the advantages of stochastic lifestyle over the deterministic lifestyle in respect of DC pension plans. Hainaut and Devolder (2007) focused on the management of a pension fund under mortality risk and financial risk. Gao (2009a, 2009b, 2010) solved explicit solutions for annuity/DC pension model under constant elasticity of variance (CEV) model and asymptotic solutions for annuity under an extended CEV model. For more detailed discussion on annuity or pension fund management topics, the readers can refer to Albrecht and Maurer (2002), Blake, Cairns, and Dowd (2001, 2003), Booth and Yakoubov (2000), Cairns (2000), Charupat and Milevsky (2002), Gerrard, Haberman, and Vigna (2004), Haberman and Sung (1994) and Haberman and Vigna (2001), and Xiao, Zhai, and Qin (2007). Most of these research works assume that the volatility of the risky asset price is constant, time-dependent function, or follow the CEV model.

However, in the existing actuarial literature, optimal investment problem for annuity contracts with multiscale stochastic volatility (MSSV) has not been well studied yet, either numerically or analytically. The goal of our model is to apply annuity model into the MSSV case, which is based on the model proposed in Fouque, Papanicolaou, Sircar, and Solna (2011). Two timescale factors will be applied in our stochastic volatility, which means we use $\sigma(Y_t, Z_t)$ instead of $\sigma(t)$. Y_t and Z_t satisfy the following two modified Wiener processes

$$dy(t) = \frac{1}{\xi} b(y(t))dt + \frac{1}{\sqrt{\xi}} a(y(t))dW(t),$$

$$dz(t) = \delta c(z(t))dt + \sqrt{\delta} g(z(t))dW(t),$$

where the first one denotes fast timescale, and the second one denotes slow timescale. The difference between slow and fast depends on the data whether it is low frequency or high frequency. And the process is still ergodic by introducing the two process, which means we still have the same distribution as before, as described in Fouque et al. (2011). Basically, we apply the methods used in Gao (2010) to transform original nonlinear Hamilton–Jacobi–Bellman PDE to a linear PDE so as to develop a general optimization framework for the annuity contract with MSSV. In addition, All of our analysis is based on two stages: before and after retirement.

The rest of the paper is organized as follows. In Section 2, we present the formulation of optimal investment problem for annuity contracts with MSSV. Sections 3 and 4 derive the general framework of the optimization problem for the time periods before retirement and after retirement, respectively. The simplified linear PDEs are obtained via maximum principle, Legendre transform and dual theory.

2. Model formulation

Given a filtered probability space $(\Omega, \mathbf{P}, \mathcal{F}, \{\mathfrak{F}_t\}_{t \geq 0})$, $\mathfrak{F}_t = \sigma\{W(s); 0 \leq s \leq t\}$ be augmented by all the \mathbf{P} -null sets in \mathcal{F} , where $\mathcal{F} = \mathfrak{F}_T$. Let $W(t)$ be a one-dimensional standard Brownian motion defined on $(\Omega, \mathbf{P}, \mathcal{F})$ over $t \geq 0$, T denotes the investment time horizon before retirement, and N denotes the number of periods of the guaranteed annuity payments. All random variables considered in this paper are continuously differentiable and bounded over $t \geq 0$. All stochastic processes introduced in this paper are assumed to be well-defined and adapted processes in this space.

2.1. The financial market

We consider a financial market with the standard assumptions: (i) continuous trading is allowed; (ii) no transaction cost or tax is involved in trading; and (iii) all assets are infinitely divisible. The financial market consists of one risk-free asset and one risky asset. The price of the risk-free asset is governed by the following ordinary differential equation

$$\begin{cases} dB(t) = rB(t)dt, \\ B(0) = b_0 > 0, \end{cases} \quad (2.1)$$

where b_0 is the initial price of the risk-free asset; r is a positive risk-free interest rate. The price of the risky asset satisfies the following stochastic differential equation system

$$\begin{cases} dS(t) = S(t-)\left(\alpha(y(t), z(t))dt + \beta(y(t), z(t))dW^0(t)\right), \\ dy(t) = \frac{1}{\xi}b(y(t))dt + \frac{1}{\sqrt{\xi}}a(y(t))dW^1(t), \\ dz(t) = \delta c(z(t))dt + \sqrt{\delta}g(z(t))dW^2(t), \\ S(0) = s_0 > 0, \quad y(0) = y_0 > 0, \quad z(0) = z_0, \end{cases} \quad (2.2)$$

where s_0 , y_0 and z_0 are respectively the initial price and initial volatilities of the risky asset. $\alpha(y(t), z(t))$ and $\beta(y(t), z(t))$ are respectively the appreciation rate and volatility rate of the risky asset's price. As described in the research work by Fouque, Pun, and Wong (2016) and Fouque, Sircar, and Zariphopoulou (2015), the dynamics of $y(t)$ and $z(t)$ respectively shows the fast and slow variation of volatility with very small values of parameters ξ and δ . Besides, we assume that the process $y(t) = y^1\left(\frac{t}{\xi}\right)$ in distribution, where $y^1(t)$ is an ergodic diffusion process with unique invariant distribution Φ , independent of ξ . As described in Fouque et al. (2011), we denote $\langle \cdot \rangle$ as the invariant expectation with respect to Φ :

$$\langle g \rangle = \int g(y)\Phi(dy). \quad (2.3)$$

Remark 1 The standard Brownian motions, $W^0(t)$, $W^1(t)$, and $W^2(t)$, are correlated with

$$\text{Cov}(W^0(t), W^1(t)) = \rho_1, \quad \text{Cov}(W^0(t), W^2(t)) = \rho_2, \quad \text{Cov}(W^1(t), W^2(t)) = \rho_{12}, \quad (2.4)$$

where $-1 < \rho_1 < 1$, $-1 < \rho_2 < 1$, $-1 < \rho_{12} < 1$, and $1 + 2\rho_1\rho_2\rho_{12} - \rho_1^2 - \rho_2^2 - \rho_{12}^2 > 0$, to ensure positive definiteness of the covariance matrix of the three Brownian motions.

2.2. The wealth process for the insurer

Following Devolder et al. (2003), Xiao et al. (2007), and Gao (2009b, 2010), we establish different models for two stages, i.e. before retirement and after retirement, according to our problem setting. Before retirement, the member of a defined contribution pension fund makes contribution regularly. The insurer manages the pension fund and accumulates wealth by investing in financial market. After retirement, the insurer will purchase a paid-up annuity for the member which provides a guaranteed fixed number of periods of payment.

Before retirement Assume that the insurer with an initial wealth x_0 at time 0 invests its wealth in the market dynamically before retirement time T . Denote by $x(t)$ the wealth of the insurer at time t , and $\pi(t)$ be the amount invested in the risky asset at time t . Then the capital amount invested in the risk-free asset at time t is $x(t) - \pi(t)$. We assume that the accumulated contribution function $K(t)$ satisfies the following ordinary differential equation

$$dK(t) = kdt, \quad 0 \leq t \leq T,$$

where k is a positive constant and it denotes the contribution rate. Taking the investment return into consideration, the insurer's wealth could be expressed in the following stochastic differential equation

$$\begin{aligned} dx(t) &= \pi(t) \left(\alpha(y(t), z(t))dt + \beta(y(t), z(t))dW^0(t) \right) \\ &\quad + (x(t) - \pi(t))rdt + kdt \\ &= (\pi(t)\alpha(y(t), z(t)) + (x(t) - \pi(t))r + k)dt \\ &\quad + \pi(t)\beta(y(t), z(t))dW^0(t), \quad 0 \leq t \leq T. \end{aligned} \tag{2.5}$$

After retirement Upon retirement, the insurer withdraws amount $D(\leq x(T))$ to purchase a paid-up annuity of N periods. The remaining part of the pension fund will be paid back to the participants after time $T + N$. The purchase rate is computed on a predetermined interest rate. The benefits to pay during $[T, T + N]$ is as follows

$$\begin{cases} V = \frac{D}{\bar{a}_{N|}}, \\ \bar{a}_{N|} = \frac{1 - \exp(-\mu N)}{\mu}, \end{cases}$$

where $\bar{a}_{N|}$ is a common actuarial notation to indicate the continuous annuity with duration of N time periods. μ is a continuous technical rate. During $[T, T + N]$, the insurer pays out benefits and invests its wealth in the financial market with one risk-free asset and one risky asset. So the dynamics of the wealth is as follows:

$$\begin{aligned} dx(t) &= \pi(t) \left(\alpha(y(t), z(t))dt + \beta(y(t), z(t))dW^0(t) \right) \\ &\quad + (x(t) - \pi(t))rdt - Vdt \\ &= (\pi(t)\alpha(y(t), z(t)) + (x(t) - \pi(t))r - V)dt \\ &\quad + \pi(t)\beta(y(t), z(t))dW^0(t), \quad T \leq t \leq T + N. \end{aligned} \tag{2.6}$$

2.3. Optimization problem

Definition 1 (Admissible Strategy). Denote $\mathbf{L}_r^2(t, T + N; \mathbb{R}^n)$ the set of all \mathbb{R}^n -valued and measurable stochastic processes $f(s)$ adapted to $\{\mathfrak{F}_s\}_{s \geq t}$ on $[0, T + N]$ such that $E \left[\int_t^{T+N} |f(s)|^2 ds \right] < +\infty$. For the insurer, a strategy $\pi(\cdot) = \{\pi(t); t \in [0, T + N]\}$ is called admissible if $\pi(\cdot) \in \mathbf{L}_r^2(t, T + N; \mathbb{R}^n)$, and $\pi(t)$ satisfies the stochastic differential equation (2.5) over $[0, T]$ and the stochastic differential equation (2.6) over $[T, T + N]$. In addition, we denote $\Pi(0, T + N)$ the set of all such admissible solutions over $[0, T + N]$.

The optimization management problem for the annuity contract with MSSV refers to the problem of finding the optimal investment strategy such that the utility function for the insurer is maximized. We express the optimization problems as follows:

Before retirement

$$\begin{aligned} \max_{\pi(t)} \quad & E[U(x(T))], \\ \text{subject to} \quad & (2.5). \end{aligned}$$

After retirement

$$\begin{aligned} & \max_{\pi(t)} E[U(x(T+N))], \\ & \text{subject to} \quad (2.6). \end{aligned}$$

In the next two sections, we define the value function, derive the associated Hamilton–Jacobi–Bellman equation, and convert the nonlinear partial differential equation (PDE) to a linear PDE by applying the maximum principle, Legendre transform, and dual theory, for the time periods before retirement and after retirement, respectively.

3. General framework before retirement

We define the value function as follows

$$H(t, x, y, z) = \sup_{\pi(t)} E[U(x(T)) | x(t) = x, y(t) = r, z(t) = z], \quad 0 \leq t \leq T, \quad (3.1)$$

with boundary condition $H(T, x, y, z) = U(x(T))$. The Hamilton–Jacobi–Bellman equation associated to the optimization problem before retirement is

$$\begin{aligned} H_t + \sup_{\pi(t)} \left\{ (\pi(t)\alpha(y, z) + (x - \pi(t))r + k)H_x + \frac{b(y)}{\xi}H_y \right. \\ \left. + \delta c(z)H_z + \frac{1}{2}\pi^2(t)\beta^2(y, z)H_{xx} + \frac{1}{2\xi}a^2(y)H_{yy} + \frac{1}{2}g^2(z)\delta H_{zz} \right. \\ \left. + \frac{\pi(t)\beta(y, z)\alpha(y)\rho_1}{\sqrt{\xi}}H_{xy} + \pi(t)\beta(y, z)g(z)\rho_2\sqrt{\delta}H_{xz} + \sqrt{\frac{\delta}{\xi}}a(y)g(z)\rho_{12}H_{yz} \right\} = 0, \end{aligned} \quad (3.2)$$

where $H_t, H_x, H_y, H_z, H_{xx}, H_{yy}, H_{zz}, H_{xy}, H_{xz}$, and H_{yz} denote partial derivatives of first-order and second-order with respect to time t , wealth $x(t)$, fast-varying stochastic volatility $y(t)$, and slowly-varying stochastic volatility $z(t)$. We apply the same kind of symbol to denote partial derivatives in the rest of this paper. By the first-order maximizing condition with respect to $\pi(t)$ we have

$$\pi^*(t) = - \frac{(\alpha(y, z) - r)H_x + \frac{\alpha(y)\beta(y, z)\rho_1}{\sqrt{\xi}}H_{xy} + \sqrt{\delta}g(z)\beta(y, z)\rho_2H_{xz}}{\beta^2(y, z)H_{xx}}. \quad (3.3)$$

Substituting (3.3) into (3.2), we obtain the PDE for the value function $H(t, x, y, z)$

$$\begin{aligned} H_t + (xr + k)H_x + \frac{b(y)}{\xi}H_y + \delta c(z)H_z + \frac{1}{2\xi}a^2(y)H_{yy} \\ + \frac{1}{2}g^2(z)\delta H_{zz} + \sqrt{\frac{\delta}{\xi}}a(y)g(z)\rho_{12}H_{yz} \\ - \frac{\left((\alpha(y, z) - r)H_x + \frac{\alpha(y)\beta(y, z)\rho_1}{\sqrt{\xi}}H_{xy} + \sqrt{\delta}g(z)\beta(y, z)\rho_2H_{xz} \right)^2}{2\beta^2(y, z)H_{xx}} = 0. \end{aligned} \quad (3.4)$$

Then we use the same technique as Jonsson and Sircar (2002), Xiao et al. (2007), Gao (2009b, 2010), we can transform the nonlinear PDE (3.4) into a dual linear PDE. Define a Legendre transform for H

$$\hat{H}(t, l, y, z) = \sup_{x>0} \{H(t, x, y, z) - lx | 0 < x < \infty\}, \quad 0 < t < T,$$

where $l > 0$ is the dual variable to x . The optimum is attained at $x = h(t, l, y, z)$, so we have

$$h(t, l, y, z) = \inf_{x>0} \{x | H(t, x, y, z) \geq lx + \hat{H}(t, l, y, z)\}, \quad 0 < t < T.$$

Hence,

$$\hat{H}(t, l, y, z) = H(t, x, y, z) - lh, \quad h(t, l, y, z) = x. \tag{3.5}$$

Next, we calculate derivatives of the value function H , the dual function \hat{H} , and the dual function h and obtain the following results

$$\begin{aligned} H_x &= l, \quad H_t = \hat{H}_t, \quad H_y = \hat{H}_y, \quad H_z = \hat{H}_z, \quad h = -\hat{H}_l, \\ H_{xx} &= -\frac{1}{\hat{H}_l}, \quad H_{yy} = \hat{H}_{yy} - \frac{\hat{H}_{yl}^2}{\hat{H}_l}, \quad H_{zz} = \hat{H}_{zz} - \frac{\hat{H}_{zl}^2}{\hat{H}_l}, \\ H_{xy} &= -\frac{\hat{H}_{yl}}{\hat{H}_l}, \quad H_{xz} = -\frac{\hat{H}_{zl}}{\hat{H}_l}, \quad H_{yz} = \hat{H}_{yz} - \frac{\hat{H}_{yl}\hat{H}_{zl}}{\hat{H}_l}. \end{aligned} \tag{3.6}$$

According to the boundary condition of the value function H , we define the boundary conditions of the dual functions \hat{H} and h

$$\begin{aligned} h(T, l, y, z) &:= \inf_{x>0} \{x \mid U(x) \geq lx + \hat{H}(T, l, y, z)\}, \\ \hat{H}(T, l, y, z) &:= \sup_{x>0} \{U(x) - lx\}, \\ h(T, l, y, z) &= (U')^{-1}(l). \end{aligned} \tag{3.7}$$

Substituting (3.5) and (3.6) into (3.4), we obtain the PDE for the dual function \hat{H}

$$\begin{aligned} \hat{H}_t + (xr + k)l + \frac{b(y)}{\xi} \hat{H}_y + \delta c(z) \hat{H}_z + \frac{1}{2\xi} \alpha^2(y) \left(\hat{H}_{yy} - \frac{\hat{H}_{yl}^2}{\hat{H}_l} \right) \\ + \frac{1}{2} g^2(z) \delta \left(\hat{H}_{zz} - \frac{\hat{H}_{zl}^2}{\hat{H}_l} \right) + \sqrt{\frac{\delta}{\xi}} \alpha(y) g(z) \rho_{12} \left(\hat{H}_{yz} - \frac{\hat{H}_{yl}\hat{H}_{zl}}{\hat{H}_l} \right) \\ - \frac{\left((\alpha(y, z) - r)l + \frac{\alpha(y)\beta(y, z)\rho_1}{\sqrt{\xi}} \left(-\frac{\hat{H}_{yl}}{\hat{H}_l} \right) + \sqrt{\delta} g(z) \beta(y, z) \rho_2 \left(-\frac{\hat{H}_{zl}}{\hat{H}_l} \right) \right)^2}{2\beta^2(y, z) \left(-\frac{1}{\hat{H}_l} \right)} = 0. \end{aligned} \tag{3.8}$$

Differentiating $h(t, l, y, z) = -\hat{H}_l$ with respect to $t, l, y,$ and $z,$ we can express the derivatives for \hat{H} in terms of h

$$\begin{aligned} \hat{H}_{tl} &= -h_t, \quad \hat{H}_{yl} = -h_y, \quad \hat{H}_{zl} = -h_z, \quad \hat{H}_{yyl} = -h_{yy}, \quad \hat{H}_{ll} = -h_l, \\ \hat{H}_{zzl} &= -h_{zz}, \quad \hat{H}_{yzl} = -h_{yz}, \quad \hat{H}_{lll} = -h_{ll}, \quad \hat{H}_{yyl} = -h_{yl}, \quad \hat{H}_{zll} = -h_{zl}, \\ \frac{\partial}{\partial l} \left(\frac{\hat{H}_{yl}^2}{\hat{H}_l} \right) &= -\frac{\partial}{\partial l} \left(\frac{h_y^2}{h_l} \right) = -\frac{2h_y h_{yl}}{h_l} + \frac{h_y^2 h_{ll}}{h_l^2}, \\ \frac{\partial}{\partial l} \left(\frac{\hat{H}_{zl}^2}{\hat{H}_l} \right) &= -\frac{\partial}{\partial l} \left(\frac{h_z^2}{h_l} \right) = -\frac{2h_z h_{zl}}{h_l} + \frac{h_z^2 h_{ll}}{h_l^2}, \\ \frac{\partial}{\partial l} \left(\frac{\hat{H}_{yl}\hat{H}_{zl}}{\hat{H}_l} \right) &= -\frac{\partial}{\partial l} \left(\frac{h_y h_z}{h_l} \right) = -\frac{h_{yl} h_z + h_y h_{zl}}{h_l} + \frac{h_y h_z h_{ll}}{h_l^2}. \end{aligned} \tag{3.9}$$

Differentiating (3.8) with respect to l and substituting (3.9) into the results yield the PDE for h as follows

$$\begin{aligned}
 & -h_t + hr + k - \frac{b(y)}{\xi} h_y - \delta c(z) h_z - \frac{1}{2\xi} a^2(y) h_{yy} - \frac{1}{2} g^2(z) \delta h_{zz} \\
 & - \sqrt{\frac{\delta}{\xi}} a(y) g(z) \rho_{12} h_{yz} - \frac{(\alpha(y, z) - r)^2 l^2}{2\beta^2(y, z)} h_{ll} - \frac{(\alpha(y, z) - r)^2 l}{\beta^2(y, z)} h_l \\
 & + \frac{(\alpha(y, z) - r) a(y) \rho_1}{\sqrt{\xi} \beta(y, z)} h_y + \frac{(\alpha(y, z) - r) l a(y) \rho_1}{\sqrt{\xi} \beta(y, z)} h_{yl} \\
 & + \frac{(\alpha(y, z) - r) \sqrt{\delta} g(z) \rho_2}{\beta(y, z)} h_z + \frac{(\alpha(y, z) - r) \sqrt{\delta} l g(z) \rho_2}{\beta(y, z)} h_{zl} \\
 & + \frac{1}{2\xi} (1 - \rho_1^2) a^2(y) \left(\frac{2h_y h_{yl}}{h_l} - \frac{h_y^2 h_{ll}}{h_l^2} \right) \\
 & + \frac{1}{2} \delta (1 - \rho_2^2) g^2(z) \left(\frac{2h_z h_{zl}}{h_l} - \frac{h_z^2 h_{ll}}{h_l^2} \right) \\
 & + \sqrt{\frac{\delta}{\xi}} a(y) g(z) (\rho_{12} - \rho_1 \rho_2) \left(\frac{h_{yl} h_z + h_y h_{zl}}{h_l} - \frac{h_y h_z h_{ll}}{h_l^2} \right) = 0.
 \end{aligned} \tag{3.10}$$

4. General framework after retirement

We define the value function as follows

$$H(t, x, y, z) = \sup_{\pi(t)} E[U(x(T + N)) | x(t) = x, y(t) = r, z(t) = z], \quad T \leq t \leq T + N, \tag{4.1}$$

with boundary condition $H(T + N, x, y, z) = U(x(T + N))$. The Hamilton–Jacobi–Bellman equation associated to the optimization problem after retirement is

$$\begin{aligned}
 H_t + \sup_{\pi(t)} \left\{ \left(\pi(t) \alpha(y, z) + (x - \pi(t)) r - V \right) H_x + \frac{b(y)}{\xi} H_y \right. \\
 + \delta c(z) H_z + \frac{1}{2} \pi^2(t) \beta^2(y, z) H_{xx} + \frac{1}{2\xi} a^2(y) H_{yy} + \frac{1}{2} g^2(z) \delta H_{zz} \\
 \left. + \frac{\pi(t) \beta(y, z) a(y) \rho_1}{\sqrt{\xi}} H_{xy} + \pi(t) \beta(y, z) g(z) \rho_2 \sqrt{\delta} H_{xz} + \sqrt{\frac{\delta}{\xi}} a(y) g(z) \rho_{12} H_{yz} \right\} = 0,
 \end{aligned} \tag{4.2}$$

By the first-order maximizing condition with respect to $\pi(t)$, we have

$$\pi^*(t) = - \frac{(\alpha(y, z) - r) H_x + \frac{\alpha(y) \beta(y, z) \rho_1}{\sqrt{\xi}} H_{xy} + \sqrt{\delta} g(z) \beta(y, z) \rho_2 H_{xz}}{\beta^2(y, z) H_{xx}}. \tag{4.3}$$

Substituting (4.3) into (4.2), we obtain the PDE for the value function $H(t, x, y, z)$

$$\begin{aligned}
 H_t + (xr - V) H_x + \frac{b(y)}{\xi} H_y + \delta c(z) H_z + \frac{1}{2\xi} a^2(y) H_{yy} \\
 + \frac{1}{2} g^2(z) \delta H_{zz} + \sqrt{\frac{\delta}{\xi}} a(y) g(z) \rho_{12} H_{yz} \\
 - \frac{\left((\alpha(y, z) - r) H_x + \frac{\alpha(y) \beta(y, z) \rho_1}{\sqrt{\xi}} H_{xy} + \sqrt{\delta} g(z) \beta(y, z) \rho_2 H_{xz} \right)^2}{2\beta^2(y, z) H_{xx}} = 0.
 \end{aligned} \tag{4.4}$$

Then we use the same technique as Jonsson and Sircar (2002), Xiao et al. (2007), Gao (2009b, 2010), we can transform the nonlinear PDE (4.4) into a dual linear PDE. Define a Legendre transform for H

$$\hat{H}(t, l, y, z) = \sup_{x>0} \{ H(t, x, y, z) - lx | 0 < x < \infty \}, \quad T < t < T + N,$$

where $l > 0$ is the dual variable to x . The optimum is attained at $x = h(t, l, y, z)$, so we have

$$h(t, l, y, z) = \inf_{x>0} \{x|H(t, x, y, z) \geq lx + \hat{H}(t, l, y, z)\}, \quad T < t < T + N.$$

Hence,

$$\hat{H}(t, l, y, z) = H(t, x, y, z) - lh, \quad h(t, l, y, z) = x. \tag{4.5}$$

Next, we calculate derivatives of the value function H , the dual functions \hat{H} , and the dual function h and obtain the same results as in Section .

According to the boundary condition of the value function H , we define the boundary conditions of the dual functions \hat{H} and h

$$\begin{aligned} h(T + N, l, y, z) &= \inf_{x>0} \{x|U(x) \geq lx + \hat{H}(T + N, l, y, z)\}, \\ \hat{H}(T + N, l, y, z) &= \sup_{x>0} \{U(x) - lx\}, \end{aligned} \tag{4.6}$$

$$h(T + N, l, y, z) = (U')^{-1}(l).$$

Substituting (4.5) and (3.6) into (4.4), we obtain the PDE for the dual function \hat{H}

$$\begin{aligned} \hat{H}_t + (xr - V)l + \frac{b(y)}{\xi} \hat{H}_y + \delta c(z) \hat{H}_z + \frac{1}{2\xi} a^2(y) \left(\hat{H}_{yy} - \frac{\hat{H}_{yl}^2}{\hat{H}_{ll}} \right) \\ + \frac{1}{2} g^2(z) \delta \left(\hat{H}_{zz} - \frac{\hat{H}_{zl}^2}{\hat{H}_{ll}} \right) + \sqrt{\frac{\delta}{\xi}} a(y) g(z) \rho_{12} \left(\hat{H}_{yz} - \frac{\hat{H}_{yl} \hat{H}_{zl}}{\hat{H}_{ll}} \right) \\ - \frac{\left((\alpha(y, z) - r)l + \frac{\alpha(y)\beta(y, z)\rho_1}{\sqrt{\xi}} \left(-\frac{\hat{H}_{yl}}{\hat{H}_{ll}} \right) + \sqrt{\delta} g(z) \beta(y, z) \rho_2 \left(-\frac{\hat{H}_{zl}}{\hat{H}_{ll}} \right) \right)^2}{2\beta^2(y, z) \left(-\frac{1}{\hat{H}_{ll}} \right)} = 0. \end{aligned} \tag{4.7}$$

Differentiating $h(t, l, y, z) = -\hat{H}_l$ with respect to $t, l, y,$ and z , we can express the derivatives for \hat{H} in terms of h , the same as (3.9). Differentiating (4.7) with respect to l and substituting (3.9) into the results yield the PDE for h as follows

$$\begin{aligned} -h_t + hr - V - \frac{b(y)}{\xi} h_y - \delta c(z) h_z - \frac{1}{2\xi} a^2(y) h_{yy} - \frac{1}{2} g^2(z) \delta h_{zz} \\ - \sqrt{\frac{\delta}{\xi}} a(y) g(z) \rho_{12} h_{yz} - \frac{(\alpha(y, z) - r)^2 l^2}{2\beta^2(y, z)} h_{ll} - \frac{(\alpha(y, z) - r)^2 l}{\beta^2(y, z)} h_l \\ + \frac{(\alpha(y, z) - r) a(y) \rho_1}{\sqrt{\xi} \beta(y, z)} h_y + \frac{(\alpha(y, z) - r) l a(y) \rho_1}{\sqrt{\xi} \beta(y, z)} h_{yl} \\ + \frac{(\alpha(y, z) - r) \sqrt{\delta} g(z) \rho_2}{\beta(y, z)} h_z + \frac{(\alpha(y, z) - r) \sqrt{\delta} l g(z) \rho_2}{\beta(y, z)} h_{zl} \\ + \frac{1}{2\xi} (1 - \rho_1^2) a^2(y) \left(\frac{2h_y h_{yl}}{h_l} - \frac{h_y^2 h_{ll}}{h_l^2} \right) \\ + \frac{1}{2} \delta (1 - \rho_2^2) g^2(z) \left(\frac{2h_z h_{zl}}{h_l} - \frac{h_z^2 h_{ll}}{h_l^2} \right) \\ + \sqrt{\frac{\delta}{\xi}} a(y) g(z) (\rho_{12} - \rho_1 \rho_2) \left(\frac{h_{yl} h_z + h_y h_{zl}}{h_l} - \frac{h_y h_z h_{ll}}{h_l^2} \right) = 0. \end{aligned} \tag{4.8}$$

Remark 2 Comparing (3.10) and (4.8), we find that the only difference lies in the continuous contribution rate before retirement and the continuous benefit payment rate after retirement.

5. Conclusions

In this paper, we have investigated an investment problem for an annuity contract before and after retirement with consideration of MSSV. By applying the maximum principle, Legendre transform and dual theory, we convert the original nonlinear Hamilton–Jacobi–Bellman PDE to a linear one.

In the future, we could derive the optimal investment strategies for some specific form of utility functions by the asymptotic approximation method. Furthermore, we may study the solutions for a special case of MSSV, and thereafter, a numerical illustration is provided to examine the dynamic behavior of the optimal investment strategy and the sensitivity of key parameters accordingly.

In addition, we could extend our annuity model to more complex one with focus on stochastic interest rate, special assets which are driven by fractional Brownian motion, and/or regime switching. However, we need to apply some advanced numerical methodology to analyze our further research problems.

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