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## APPLIED & INTERDISCIPLINARY MATHEMATICS | RESEARCH ARTICLE

# Existence and uniqueness of a stationary and ergodic solution to stochastic recurrence equations via Matkowski's FPT

Stelios Arvanitis<sup>1\*</sup>

**Abstract:** We establish the existence of a unique stationary and ergodic solution for systems of stochastic recurrence equations defined by stochastic self-maps on Polish metric spaces based on the fixed point theorem of Matkowski. The results can be useful in cases where the stochastic Lipschitz coefficients implied by the currently used method either do not exist, or lead to the imposition of unnecessarily strong conditions for the derivation of the solution.

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### 1. Introduction

Discrete time dynamics on a space  $E$  essentially involve some iterated homomorphism  $\Phi: E \rightarrow E$  interpreted as an action of  $\mathbb{Z}$  or  $\mathbb{N}$  on  $E$  (see for example Chapter 10 in Ghrist, 2014 for a category theory interpretation). Stochastic recurrence equations generalize this formulation in the sense that they allow for  $\Phi$  to measurably depend on appropriate sequences of  $E$ -valued random elements. Suchlike constructions play a central role in time series analysis, whereas many prominent examples can be

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Stelios Arvanitis current main research areas are: the study of preorders on spaces of probability distributions, with applications in Mathematical Finance, the design of statistical testing procedures concerning properties of such orders, and the establishment of their limit theory with applications in Financial Econometrics. The study of indirect estimators and their limit theory with applications in Econometric Theory. The study of martingale limit theorems with limits mixtures of stable distributions and the subsequent establishment of the asymptotic properties of econometric estimators and tests in the context of non-linear stationary and non-stationary time series models. The present note could be considered as indicative for the existence of analogous results involving richer classes of suchlike (stationary) models, and if this is true, the subsequent enlargement of the scope of the aforementioned limit theory via the lifting of unnecessary restrictions on the parameters of those models.

### PUBLIC INTEREST STATEMENT

We establish the uniqueness of a stationary and ergodic solution for systems of stochastic recurrence equations via a general fixed point theorem. The results can be useful in cases where currently used method either are inapplicable, or lead to the imposition of unnecessarily strong conditions. They could be considered as indicative in the following sense. The plethora of extensions and generalizations of the Banach Fixed Point Theorem could be relevant to the establishment of relatively weak conditions for results such as the above, concerning richer classes of such systems. Furthermore, appropriate reformulations of the results could also be of importance in Economic Theory.

perceived as solutions of such equations (see for example Chapter 7 in Kallenberg, 2006). Given such an equation, the question of the existence and uniqueness of a solution with appropriate probabilistic properties is of major importance in several fields (see for example Diaconis & Freedman, 1999 for several applications, as well as Goldie, 1991 for applications in probability theory, Straumann (2004) for applications in Statistics and Econometrics, Matkowski and Nowak (2011) for applications in Dynamical Economics). More specifically, the issue of the existence of a stationary solution (see for example Moyal, 2015 for some recent results), or more strongly the issue of the existence of a unique stationary and ergodic solution constructed as an appropriate limit of backward iterations can be of great importance in such applications.

The latter is usually handled via the use of some stochastic extension of a fixed point theorem (see Bharucha-Reid, 1976). The most prominent example is the relevant extension of the classical fixed point theorem of Banach (as established for example in Bougerol, 1993). It assumes a strongly contractive property for the form of the equation which in some cases may not exist or be quite restrictive. The following example provides with some relevant indication.

*Example* Suppose that  $E = \mathbb{R}_+$ , and for  $(z_t)_{t \in \mathbb{Z}}$  a random (double-) sequence of random variables,  $\theta \in \Theta$  a compact subset of  $E$ , and  $(g_{t,\theta})_{t \in \mathbb{Z}}$  a sequence of random real valued functions defined on  $\mathbb{R}_+$ , such that for any  $x, \theta \in E$ ,  $(z_t, g_{t,\theta}(x))_{t \in \mathbb{Z}}$  is jointly stationary and ergodic, and consider the stochastic recurrence equation

$$x_{t+1} = (z_t + g_{t,\theta}(x_t))^+, \quad t \in \mathbb{Z}. \tag{1}$$

Given the non-strongly contractive nature of  $(\cdot)^+$  and the possibility for something analogous for  $g_t$  the use of the aforementioned result of Bougerol (1993) is not immediate for the establishment of the existence of a unique stationary and ergodic solution to Equation (1) as an appropriate backward limit, or could be associated with unnecessarily strong restrictions on the properties of  $g_{t,\theta}$ . For a more specific example on the latter case suppose that  $g_{t,\theta}(x) = \frac{\theta x}{1 + \eta_t x}$ , where  $\theta \in [0, 1]$  and  $(\eta_t)_{t \in \mathbb{Z}}$  is a strictly stationary and ergodic (double-) sequence of non-negative random variables. It is easy to see that if  $\mathbb{E}[\ln^+ z_0] < \infty$ , then a sufficient condition for a solution with the aforementioned properties uniformly over the possible values of  $\theta$  is that  $\sup \theta < 1$ , a condition that could perhaps be avoided.

Given the previous, the present note concerns the establishment of the existence of a unique stationary and ergodic solution over the set of integral numbers for systems of stochastic recurrence equations defined by stochastic self-maps on Polish metric spaces and its representation as a limit of relevant Picard iterates, by exploiting one of the numerous extensions of the Banach fixed point theorem. More specifically, our result is based on the fixed point theorem of Matkowski (see Matkowski, 1977) and thereby extends the previous result. As such it is possible that it could handle cases similar to the examples above, i.e. cases where the stochastic Lipschitz coefficients implied by the currently used method either do not exist, or lead to the imposition of unnecessarily strong conditions for the derivation of the solution.

In the following section we describe our framework, establish the main result, discuss the previous example in the light of that, and conclude by briefly mentioning potential generalizations.

## 2. Existence and uniqueness of stationary and ergodic solution to SRE's

In what follows  $(\Omega, \mathcal{F}, \mathbb{P})$  is a complete probability space,  $(E, d)$  is a Polish metric space,  $\mathcal{B}_E$  its Borel  $\sigma$ -algebra,  $\Theta$  is an arbitrary non empty set,  $\Phi_{t,\theta}: \Omega \times E \rightarrow E$ ,  $t \in \mathbb{Z}$ ,  $\theta \in \Theta$  are  $\mathcal{B}_E / \mathcal{F} \otimes \mathcal{B}_E$ -measurable self maps on  $E$ , and  $g_{t,\theta}: \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  are  $\mathcal{B}_E / \mathcal{F} \otimes \mathcal{B}_{\mathbb{R}_+}$ -measurable self maps on  $\mathbb{R}_+$ . The relevant supremum metric is denoted by  $d_{\Theta}^{\text{eas}}$  denotes exponentially almost sure convergence (see paragraph 2.5 in Straumann, 2004),  $\rightarrow_{\mathbb{P}}^{\text{a.s.}}$  denotes  $\mathbb{P}$  a.s. convergence, and  $\stackrel{\text{as}}{=}$  almost sure equality w.r.t.  $\mathbb{P}$ .  $\mathbb{E}$  denotes integration w.r.t.  $\mathbb{P}$ . If  $(X_t)_{t \in \mathbb{Z}}$  is an  $E$ -valued stochastic process defined on  $\Omega$ , then  $\sigma_{X_t} \equiv \sigma(X_t, X_{t-1}, \dots)$  denotes the  $\sigma$ -algebra generated by the  $(X_{t-k})_{k \in \mathbb{N}}$  collection of random elements. Finally for  $m \in \mathbb{N}$ ,

$$\Phi_{t,\theta}^{(m)} \equiv \begin{cases} \text{id}_E, & m = 0 \\ \Phi_{t,\theta} \circ \Phi_{t-1,\theta} \circ \dots \circ \Phi_{t-m+1,\theta}, & m > 0 \end{cases}.$$

The following theorem establishes the existence of a unique, up to indistinguishability, stationary and ergodic solution to the stochastic recurrence system defined by  $x_{t+1} = \Phi_{t,\theta}(x_t)$ , its continuity properties w.r.t.  $\theta$ , the form by which it approximates any other solution as well as the issue of its invertibility. In part, it is essentially based on the fixed point theorem of Matkowski (see Matkowski, 1977) in the particular probabilistic setting of a stochastic flow defined by stochastic recurrences. As mentioned above, it thereby generalizes the analogous result used in the time series literature that is based on the Banach fixed point theorem (see Theorem 20 of Bougerol, 1993 or equivalently Theorem 2.6.1 of Straumann, 2004).

**THEOREM 1** Suppose that  $(\Phi_{t,\theta})_{t \in \mathbb{Z}}$  is stationary and ergodic for any  $\theta \in \Theta$ . Furthermore:

a. there exists a  $y \in E$  such that,

$$\mathbb{E}[\log^+ d_\Theta(\Phi_{0,\theta}(y), y)] < +\infty, \quad \mathbb{P} \text{ a.s.}, \quad (2)$$

b. for any  $t$  and  $\theta$ , for any  $x, y \in E$ ,

$$d(\Phi_{t,\theta}(x), \Phi_{t,\theta}(y)) \leq g_{t,\theta}(d(x, y)), \quad \mathbb{P} \text{ a.s.}, \quad (3)$$

and,

c. for any  $t \in \mathbb{Z}$  and  $\theta \in \Theta$ ,  $g_{t,\theta}$  is  $\mathbb{P}$  a.s. increasing, and for any  $z \in \mathbb{R}_+$ ,

$$|g_{t,\theta}^{(m)}(z)|_\Theta \xrightarrow{\text{eas}} 0 \text{ as } m \rightarrow \infty, \quad (4)$$

while for at least one  $t \in \mathbb{Z}$  the convergence is locally uniform in  $\mathbb{R}_+$ .

Then the SRE defined by

$$x_{t+1} = \Phi_{t,\theta}(x_t), \quad (5)$$

admits a stationary and ergodic solution  $(Y_{t,\theta})_{t \in \mathbb{Z}}$  for any  $\theta \in \Theta$  that has the representation

$$Y_{t+1,\theta} \stackrel{\text{as}}{=} \lim_{m \rightarrow \infty} \Phi_{t,\theta}^{(m)}(y), \quad (6)$$

and the convergence is uniform w.r.t.  $\theta$ . If  $(Y_{t,\theta}^*)_{t \in \mathbb{Z}}$  denotes any other stationary solution then

$$\mathbb{P}(d(Y_{t,\theta}, Y_{t,\theta}^*) = 0) = 1 \text{ for any } t \text{ and } \theta. \quad (7)$$

The random element  $Y_{t+1,\theta}$  is measurable w.r.t.  $\sigma(\Phi_{t,\theta}, \Phi_{t-1,\theta}, \dots)$ ,  $\theta \in \Theta$ . If  $\Phi_{t,\theta} = \Phi_\theta(X_t)$  for some stationary and ergodic  $(X_t)_{t \in \mathbb{Z}}$  where  $X_t$  assumes values in  $E$ , and  $\Phi_\theta: E \rightarrow E$  is  $\mathcal{B}_E/\mathcal{B}_E$ -measurable, then the random element  $Y_{t+1,\theta}$  is measurable w.r.t.  $\sigma_{X_t}$ . If  $\Theta$  is a compact topological space,  $\Phi_{t,\theta}(y)$  is  $\mathbb{P}$  a.s. continuous w.r.t.  $\theta$ , then the random element  $Y_{t+1,\theta}$  is  $\mathbb{P}$  a.s. continuous w.r.t.  $\theta$ . Finally, if  $\Theta$  is Polish, and  $(Y_{t,\theta}^*)_{t \in \mathbb{Z}}$  denotes any solution, for which  $\mathbb{E}[\log^+ d_\Theta(y^*, Y_{t,\theta}^*)] < +\infty$  and  $\mathbb{E}[\log^+ d_\Theta(y^*, Y_{t,\theta}^*)] < +\infty$  for some  $t \in \mathbb{Z}$  and  $y^* \in E$ , then

$$d_\Theta(Y_{t,\theta}, Y_{t,\theta}^*) \xrightarrow{\text{eas}} 0 \text{ as } t \rightarrow \infty. \quad (8)$$

*Proof* Fix  $y \in E$ . Suppose first that the  $\mathbb{P}$  a.s. limit in (6) exists. Then from the continuity of the metric for any  $\theta$

$$d\left(\lim_{m \rightarrow \infty} \Phi_{t,\theta}^{(m)}(y), \Phi_{t,\theta}(Y_t)\right) = \lim_{m \rightarrow \infty} d\left(\Phi_{t,\theta}^{(m)}(y), \Phi_{t,\theta}(Y_t)\right), \quad \mathbb{P} \text{ a.s.},$$

and that due to (3), (5) and (6)

$$\begin{aligned} d(\Phi_{t,\theta}^{(m)}(y), \Phi_{t,\theta}(Y_t)) &\leq g_{t,\theta}(d(\Phi_{t-1,\theta}^{(m-1)}(y), Y_t)), \mathbb{P} \text{ a.s.}, \\ &\leq g_{t,\theta}(d(\Phi_{t-1,\theta}^{(m-1)}(y), \lim_{n \rightarrow \infty} \Phi_{t-1,\theta}^{(n)}(y))), \mathbb{P} \text{ a.s.}, \\ &= \lim_{n \rightarrow \infty} g_{t,\theta}(d(\Phi_{t-1,\theta}^{(m-1)}(y), \Phi_{t-1,\theta}^{(n)}(y))), \mathbb{P} \text{ a.s.}, \end{aligned}$$

and analogously

$$\begin{aligned} g_{t,\theta}(d(\Phi_{t-1,\theta}^{(m-1)}(y), \lim_{n \rightarrow \infty} \Phi_{t-1,\theta}^{(n)}(y))) &\leq g_{t,\theta}^{(2)}(d(\Phi_{t-2,\theta}^{(m-2)}(y), \Phi_{t-2,\theta}^{(m-1)}(y))), \mathbb{P} \text{ a.s.}, \\ &\leq g_{t,\theta}^{(m)}(d(\Phi_{t-m,\theta}^{(0)}(y), \Phi_{t-m,\theta}(y))), \mathbb{P} \text{ a.s.}, \\ &= g_{t,\theta}^{(m)}(d(y, \Phi_{t-m,\theta}(y))), \mathbb{P} \text{ a.s.} \end{aligned}$$

This along with (4) implies that  $d(Y_{t+1,\theta}, \Phi_{t,\theta}(Y_t)) = 0$ ,  $\mathbb{P}$  a.s., which implies that the process  $(Y_{t,\theta})_{t \in \mathbb{Z}}$  is a solution to (5). Furthermore if the limit exists then the stationarity, ergodicity and the measurability w.r.t.  $\sigma(\Phi_{t,\theta}, \Phi_{t-1,\theta}, \dots)$  of  $Y_{t,\theta}$  follows from Corollary 2.1.3. of Straumann (2004) while measurability w.r.t.  $\sigma_{X_t}$  follows trivially. Due to completeness the proof of the existence of the limit, reduces to the proof that  $(\Phi_{t,\theta}^{(m)}(y))_{m \in \mathbb{N}}$  is a Cauchy sequence for any  $t, \theta$ . Using the same reasoning as before we have that

$$\begin{aligned} \lim_{m \rightarrow \infty} d(\Phi_{t,\theta}^{(m+1)}(y), \Phi_{t,\theta}^{(m)}(y)) &\leq \lim_{m \rightarrow \infty} g_{t,\theta}(d(\Phi_{t-1,\theta}^{(m)}(y), \Phi_{t,\theta}^{(m-1)}(y))), \mathbb{P} \text{ a.s.}, \\ &\leq \lim_{m \rightarrow \infty} g_{t,\theta}^{(m)}(d(\Phi_{t-m,\theta}(y), y)), \mathbb{P} \text{ a.s.}, \end{aligned}$$

and the latter is due to monotonicity  $\mathbb{P}$  a.s. less than or equal

$$\lim_{m \rightarrow \infty} g_{t,\theta}^{(m)}(d_{\Theta}(\Phi_{t-m,\theta}(y), y)).$$

Stationarity and (2) imply that  $d_{\Theta}(\Phi_{t-m,\theta}(y), y) < +\infty$   $\mathbb{P}$  a.s. and then (4) implies that the last limit is zero  $\mathbb{P}$  a.s. Hence the limit exists. For the uniqueness up to indistinguishability result in (7) suppose again without loss of generality that the locally uniform version of (4) holds for  $t = 0$ . Then

$$d(Y_{1,\theta}, Y_{1,\theta}^*) \leq \lim_{m \rightarrow \infty} g_{0,\theta}^{(m)}(d(\Phi_{-m,\theta}(y), Y_1^*)), \mathbb{P} \text{ a.s.}$$

The existence of the limit in (6) along with the locally uniform nature of (4) imply that the left hand-side is  $\mathbb{P}$  almost surely zero which then implies that  $\mathbb{P}(d(Y_{1,\theta}, Y_{1,\theta}^*) = 0) = 1$ . Stationarity and measurability of  $d$  imply (7). The uniformity over  $\Theta$  in (6) and the compactness of  $\Theta$  imply the continuity result. Now, let  $\Theta$  be Polish and  $(Y_{t,\theta}^*)_{t \in \mathbb{Z}}$  denote any other solution of (5). Suppose without loss of generality that the log-moment conditions described in the additional prerequisites of (8) are valid for  $t = 0$ . In a completely analogous manner to the previous we obtain that

$$\lim_{t \rightarrow \infty} d_{\Theta}(Y_{t+1,\theta}, Y_{t+1,\theta}^*) \leq \lim_{t \rightarrow \infty} |g_{t,\theta}^{(t)}|_{\Theta}(d_{\Theta}(Y_{0,\theta}, Y_{0,\theta}^*)), \mathbb{P} \text{ a.s.}$$

Due to the monotonicity of  $g_{t,\theta}$  this implies that

$$\lim_{t \rightarrow \infty} d_{\Theta}(Y_{t+1,\theta}, Y_{t+1,\theta}^*) \leq \lim_{t \rightarrow \infty} |g_{t,\theta}^{(t)}|_{\Theta}(d_{\Theta}(y^*, Y_{0,\theta}^*) + d_{\Theta}(Y_{0,\theta}, y^*)), \mathbb{P} \text{ a.s.},$$

and the log-moment conditions along with (4) imply (8). □

The  $g_{t,\theta}$  is essentially a random  $\mathbb{P}$  a.s. comparison (or Matkowski) function and  $\Phi_{t,\theta}$  is analogously a random  $\mathbb{P}$  a.s. Matkowski (generalized) contraction (see Radu, 2011). This generalized contractive property can in some cases be established by appropriate modifications of Blackwell's Lemma (see for example Chapter 4 in Corbae, Stinchcombe, & Zeman, 2009). An instance of a suchlike modification is the following.

**LEMMA 1** In addition to the framework of Theorem 1, suppose also that  $E$  is a Euclidean vector lattice w.r.t. the partial order  $\geq$ , that the topology of  $E$  is induced by a norm  $\|\cdot\|$  compatible with the order (see Chapter 2 in Le Cam, 2012), and that for any “constant” vector  $(a)$  at  $a \geq 0$ ,  $\|(a)\| = a$ , while for any  $x \in E$ ,  $(\|x\|) \geq x$ . Suppose that the following conditions hold for any  $t, \theta$ ,

- a. (Monotonicity) for any  $x, y \in E$  such that  $x \geq y$  then  $\Phi_{t,\theta}(x) \geq \Phi_{t,\theta}(y)$ ,  $\mathbb{P}$  a.s.,
- b. (Discounting) for any  $y \geq 0$  and any  $x \in E$ ,  $\Phi_{t,\theta}(x) + (g_{t,\theta}(|y|)) \geq \Phi_{t,\theta}(y)$ .

Then (3) holds.

*Proof* Let  $x, y \in E$ . Due to the properties of  $(E, \geq)$  we have that  $(\|x - y\|) + x \geq y$  and  $(\|x - y\|) + y \geq x$ . This, discounting and monotonicity imply that  $\Phi_{t,\theta}(x) + (g_{t,\theta}(\|x - y\|)) \geq \Phi_{t,\theta}(y)$ ,  $\mathbb{P}$  a.s. and  $\Phi_{t,\theta}(y) + (g_{t,\theta}(\|x - y\|)) \geq \Phi_{t,\theta}(x)$ ,  $\mathbb{P}$  a.s. Hence  $(g_{t,\theta}(\|x - y\|)) \geq (\Phi_{t,\theta}(x) - \Phi_{t,\theta}(y))^+ + (\Phi_{t,\theta}(x) - \Phi_{t,\theta}(y))^-$ ,  $\mathbb{P}$  a.s. and thereby due to the norm-order compatibility  $\|(g_{t,\theta}(\|x - y\|))\| \geq \|\Phi_{t,\theta}(x) - \Phi_{t,\theta}(y)\|$ ,  $\mathbb{P}$  a.s. The assumed behavior of the norm on constant vectors yields the result.  $\square$

An obvious instance of applicability of the above lemma is when  $E = \mathbb{R}^n$ , equipped with the max-norm and where  $\geq$  is defined pointwisely. The example of the previous section also adheres to this structure.

Returning to the results of Theorem (1), notice that, as in Banach case, the unique in the sense of indistinguishability, stationary and ergodic solution is characterized as a  $\mathbb{P}$  a.s. limit of Picard iterations. This solution is adapted to the  $(\sigma(\Phi_{t,\theta}, \Phi_{t-1,\theta}, \dots))_{t \in \mathbb{Z}}$  filtration, and subsequently adapted to the “richer”  $(\sigma(\Phi_{t,\theta}, \Phi_{t-1,\theta}, \dots, \theta \in \Theta))_{t \in \mathbb{Z}}$  filtration. When the recursion is constructed by the  $(X_t)_{t \in \mathbb{Z}}$  process then the aforementioned solution is invertible, i.e. adapted to the  $(\sigma_{X_t})_{t \in \mathbb{Z}}$  filtration. The continuity property w.r.t.  $\theta$  of the solution characterization result in (6) can be also described as follows.

**COROLLARY 1** Suppose that  $\Theta$  is a compact topological space. Then for any  $\theta, \theta_n \in \Theta$ , such that  $\theta_n \rightarrow \theta$  as  $n \rightarrow \infty$ , the unique stationary and ergodic solution characterized in (6) of the recursion defined by  $x_{t+1} = \Phi_{t,\theta_n}(x_t)$ ,  $\mathbb{P}$  a.s. converges to the analogous solution of the recursion defined by  $x_{t+1} = \Phi_{t,\theta}(x_t)$ .

The corollary above, or more precisely the continuity property established in Theorem (1) essentially provides with a partial answer to Research Problem (d.) in Bharucha-Reid (1976) concerning the convergence of random fixed points.

Finally, when  $\Theta$  is a Polish space, then we obtain an even stronger version of the uniqueness property, in the sense that any other solution of the recursion, converges exponentially fast to the aforementioned one.

As mentioned above, the previous theorem admits as a particular case the standard Banach type argument in which  $g_{t,\theta}(z) = \Lambda_{t,\theta}z$  and  $\mathbb{E}[\sup_{\Theta} \ln \Lambda_{t,\theta}] < 0$ . As such it can be used to obtain weaker sufficient conditions in cases where Lipschitz coefficients are not well defined, or are inadequate, or have properties that imply strong restrictions. Let us in this respect examine the aforementioned example in the light of the previous results.

*Example* (-continued) In the light of the framework established above we have that here  $\Phi_{t,\theta}(x) \equiv (z_t + g_{t,\theta}(x))^+$ . Suppose furthermore that all the requirements of Theorem (1) are satisfied except for (3),  $g_{t,\theta}$  satisfies the “subadditivity” property,  $\forall x \in \mathbb{R}_+, \delta > 0, g_{t,\theta}(x + \delta) \leq g_{t,\theta}(x) + g_t(\delta)$ ,  $\mathbb{P}$  a.s. The structure of  $E$  required for Lemma (1) obviously holds in this case. Furthermore, monotonicity is obvious, while this the Lipschitzian property of  $(\cdot)^+$  and the subadditivity above imply that for any  $x \in \mathbb{R}$  and  $\delta > 0$ ,

$$\Phi_{t,\theta}(\mathbf{x} + \delta) \leq \Phi_{t,\theta}(\mathbf{x}) + g_{t,\theta}(\delta), \mathbb{P} \text{ a.s.}$$

Hence Lemma (1) is applicable and thereby all the assertions of Theorem (1) hold. As described in the original formulation of the example, consider now the more specific case where  $\Phi_{t,\theta} = \left(z_t + \frac{\theta x}{1+\eta_t x}\right)^+$ ,  $t \in \mathbb{Z}$ , with  $\theta$  and  $\eta_t$  as already defined. It is easy to see that if  $\mathbb{E}[\ln^+ z_0] < \infty$ , all the previous hold and thereby Theorem (1) is applicable without any further restrictions on  $\Theta$ . Hence the results of the theorem above can be used in order to avoid unnecessary restrictions. Notice also that similar cases of this formulation could be further constructed via the use of the results in Section 2 of Jaśkiewicz, Matkowski, and Nowak (2014).

Finally, let us mention that the results of this note could perhaps be considered as simply indicative of a possible more general situation. Given that in the Metric Fixed Point Theory a plethora of extensions and generalizations of the Banach Fixed Point Theorem is being established (for indicative examples see Ćirić, 1974 and the references therein), it is possible that subsequent stochastic formulations of suchlike extensions could be relevant to the establishment of relatively weak conditions for results such as the above concerning richer classes of stochastic recurrence equations. Furthermore, appropriate reformulations of the results above could also be useful in stochastic extensions of the dynamic programming considerations in Jaśkiewicz et al. (2014) that could be of importance in the field of Dynamic Macroeconomics under general schemata of discounting. We leave such considerations for further research.

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