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STATISTICS | RESEARCH ARTICLE

Average run length of the long-memory autoregressive fractionally integrated moving average process of the exponential weighted moving average control chart

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Abstract: Measurement of control chart efficiency by comparison of average run length (ARL) is widely implemented in quality control. The aim of this study is to evaluate the ARL, which is a solution of the integral equation obtained from the Exponential Weighted Moving Average (EWMA) statistic with a long-memory Autoregressive Fractionally Integrated Moving Average (ARFIMA) process. The derivation of the analytical ARL of the EWMA control chart and proof of the existence and uniqueness of the analytical ARL by Fixed Point theory are shown. Moreover, the numerical ARL carried out by the Composite Midpoint Rule technique of the EWMA control chart is demonstrated. A comparison between the analytical and numerical ARL is also illustrated. The findings indicated that analytical ARL of the EWMA control chart is more quickly computational than the numerical ARL. Therefore, the analytical ARL is an alternative method for measuring the efficiency of the EWMA control chart with the long-memory ARFIMA process.

Subjects: Advanced Mathematics; Applied Mathematics; Statistics & Probability

Keywords: average run length; EWMA control chart; ARFIMA process; numerical integral equation

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PUBLIC INTEREST STATEMENT

The statistical quality control plays important roles as an effective technique for controlling production process. Several products from the process are controlled by the statistical quality control chart before delivering to the customers. Defective points on the products as the gathered data are collected from such products for analysis based on the statistics theory. Moreover, the gathered data are often formed by the time series, so quality control charts such as the EWMA control chart are developed for detecting the defective points on the time series data in order to be the great quality and reliable products. This is the advantages in application of the statistical quality control to solve the problems in real situations.

1. Introduction

Entrepreneurs need to modernize their business processes in order to achieve competitive advantages that are the lowest price and highest quality. As the consumers demand quality products, the organizations typically use Statistical Process Control (SPC) to control and monitor their production and service processes. SPC based on the statistical analysis is involved in quality control for monitoring and detecting unexpected deviations from the process average. Control charts such as the Shewhart chart (see Shewhart, 1931), Cumulative Sum (CUSUM) chart (see Page, 1954), and EWMA control chart (see Roberts, 1959) play important roles in SPC and statistical quality control analysis. The efficiency of controlling and monitoring relies on the SPC chart and the distribution of observations: simple observations and complex observations. A normal distribution random variable is an example of a simple observation because its mean and variance are constant and its error is an independent identically distributed random variable. Indeed, compared with simple observations, practical observations are complex observations that include trend, season, and autocorrelation.

Complex observations that form a discrete and continuous process are implemented in several disciplines, such as biological engineering, economics, and finance. The most complex observations include the time series process because it has a trend and autocorrelation. In other words, complex observations that fluctuate over a given time frame and time series involve real-value data, which depend on time. Two types of time series process are stationary and non-stationary processes. In particular, if the mean and variance of the observations are constant over time, then such a process is called a stationary process; in other cases, it is a non-stationary process. There are many well-known time series processes that are examples of complex observations, such as autoregressive (AR) process, autoregressive integrated moving average (ARIMA) process, and ARFIMA process.

Generally, a time series is called a long-memory process with the fractional differencing parameter (d) if there exists a nonzero $d \in (0, 0.5)$. If $d = 0$, then the time series process is called a short-memory process. A long-memory process is a stationary process whose autocorrelation coefficients decay slowly, such as hyperbolic decay. The long-memory process is often founded in real situations, especially the econometric behavior which appears the long-memory property. The long-memory property is important and affects the validity and the confidence of the parameter estimation in the forecasting because this property connects between the present and the past data. Moreover, the long-memory property does not indicate the stationary process related to the phenomena of the unit root test which is interested in the case of at least of the roots equals unity, or the modulus of the root less than unity (see Phillips, 1987) and regime switching (see Diebold, & Inoue, 2001). The ARFIMA process was firstly presented by Granger and Joyeux (1980), and Hosking (1981). The ARFIMA process is an interesting mathematical notion based on sophisticated theory with applications in several areas, such as finance and banking, econometrics, environmental sciences, and quality control. The ARFIMA process reflects such a long-memory process. The ARIMA process requires an integer degree of d value, but the ARFIMA process is more generalized. Many researchers have applied the ARIMA and ARFIMA model to forecast given data and compare the accuracy of models based on the minimum of MAE, RMSE, MAPE, etc. The following studies are examples in which the ARFIMA model or a combined ARFIMA model was found to be better than single models, such as ARMA, ARIMA, or AR model. Richard and Sang (2002) developed trend-stationary ARFIMA and AR models for forecasting climatology data, i.e. annual temperatures and widths of tree rings. Later, Olbermann, Lopes and Reisen (2006) analyzed the estimation method for d value, to describe the property in variance of the first difference. Furthermore, the authors applied the ARFIMA model they developed to Brazilian exchange rates. Pan and Chen (2008) created and proposed ARFIMA and ARIMA models with an application in environmental science to air quality data in Taiwan. Lahiania and Scaillet (2009) came up with approaches related to the threshold problem in long-memory process data for forecasting U.S. unemployment rates. Amadeh, Amini and Effati (2013) presented ARIMA and ARFIMA models for forecasting with in-sample Persian Gulf oil F.O.B data downloaded from the OPEC website. Chi Xie and Gang (2015) presented a nonlinear model combining the ARFIMA model, the Support Vector Machine (SVM) model, and the Back-Propagation Neural Network (BPNN) model to forecast RMB exchange rates against the U.S. dollar (RMB/USD) and RMB exchange rates

against the euro (RMB/EUR) in China. The results indicated that the nonlinear combined method is better than the single models. On the other hand, Maqsood and Burney (2014) employed the ARMA and ARFIMA models to forecast the Karachi share index (KSE) in Pakistan. The results showed that ARMA performed better than ARFIMA with minimizing of RMSE and RMSPE. Meanwhile, a few studies on the theory of the ARFIMA model based on the stochastic process have been conducted. Kokoszka and Taqqu (1995) developed and analyzed an ARFIMA model under stable infinite variance, which was an innovative theory in the stochastic process field. As seen from the above literature review, the crucial point for modeling the ARFIMA model in applications with real data is estimating parameter d and the other parameters of AR and MA involved in the ARFIMA model. The d value was estimated by the GPH estimator method (see Geweke, & Porter-Hudak, 1983). Also, Barkoulas and Baum (1997) used spectral regression for estimating d value.

The ARL statistic is one of the indicators used for comparing the efficiency of quality control charts. ARL is applied for expectation of the number of observations in two states: in-control processes called ARL_0 and out-of-control processes called ARL_1 . That is, in an efficient quality control chart, ARL_0 should be the largest value, while ARL_1 should be the smallest value. The methods for calculating ARL in the EWMA control chart and related issues have been studied by several researchers. In general, the methods for evaluating the ARL of quality control charts can be divided into three main types: the Monte Carlo (MC) Method, the Markov Chain Approach (MCA), and the Numerical Integral Equation (NIE) method. Several researchers have used ARL to evaluate the efficiency of quality control charts. Crowder (1987) applied the numerical method of integration to evaluate the run length of a two-sided EWMA quality control chart with normal observations. Also, a computer program for evaluating the ARL in normal observations of the EWMA control chart was provided. Lucas and Saccucci (1990) used a method called MCA for evaluating ARL under the in-control process state. Suriyakat, Areepong, Sukparungsee and Mititelu (2012) derived and demonstrated explicit formulas for ARL in form integrations of the EWMA quality control charts with the AR(1) model with an exponential white noise. Recently, in order to evaluate AR in an EWMA chart for the Seasonal Autoregressive and Moving Average ($SARMA_L$) process with exponential white noise, where L is seasonal time, Phiyaphon, Areepong and Sukparungsee (2014) proved unique and existing analytical ARL initially from integral equations based on the $SARMA_L$ process. As seen from the above literature, designing the EWMA quality control chart of the long-memory ARFIMA process is a new topic and interesting as well as its ARL approximation is very important to represent the efficiency for detecting the out of control signals. Therefore, the goals of this research are to approximate the numerical ARL, derive the analytical ARL, and make a comparison between the numerical ARL and analytical ARL of the EWMA control chart using the long-memory ARFIMA process with the exponential white noise. Furthermore, the numerical method for the numerical ARL designed by the Composite Midpoint Rule technique as well as the existence and uniqueness of the analytical ARL based on the functional analysis concept are proposed. It was predicted that the numerical ARL converged to the analytical ARL. Also, the analytical ARL would affect the computational time of ARL evaluation of EWMA control chart using the long-memory ARFIMA process with the exponential white noise.

The rest of this paper is organized as follows. In the next section, the generalization of ARFIMA model of the EWMA chart is described in Section 2. Analytical ARL method of the EWMA chart on ARFIMA process is derived in Section 3. Existence and uniqueness of analytical ARL are shown in Section 4. Numerical ARL method is illustrated in Section 5. Numerical results are contained in Section 6. Finally, Section 7 is the conclusions of this research.

2. Generalized ARFIMA model with the long-memory process of the EWMA chart

Let X_t be a sequence of random variables generated from the ARFIMA process on time frame: $t = 1, 2, 3, \dots, n$, with mean μ and variance σ^2 and the smoothing parameter $\lambda \in (0, 1]$.

The definition of the EWMA statistic at time t based on ARFIMA process is the following recurrence relation with the initial value, $Z_0 = \mu$:

3. Analytical average run length of the EWMA chart of the long-memory ARFIMA process

In this section, the analytical ARL of the EWMA chart in a long-memory ARFIMA observation is derived. Based on the method developed initially by Crowder (1987), the simple method of ARL approximation on a Gaussian distribution, the ARL approximation has been popularly adopted to measure and compare the efficiency of quality control charts. Let $L(u)$ denote ARL for long-memory ARFIMA observations with the initial value $Z_0 = u$. In this paper, the setting of control limits is focused on the specific upper control limit side. Namely, the Center Line (CL) is zero and the Upper Control Limit (UCL) is b .

According to the method of Champ and Rigdon (1991), substitute y for ε_t , where $\varepsilon_t \sim \text{Exp}(\beta)$ is exponential white noise error terms, then the formula for $L(u)$ can be rewritten as

$$L(u) = 1 + \int_0^b L((1 - \lambda)u + \lambda X_t) f(y) dy$$

Changing the integral variable, the integral equation is obtained as follows

$$L(u) = 1 + \frac{1}{\lambda} \int_0^b L(y) f\left(\frac{y - (1 - \lambda)u}{\lambda} - X_t\right) dy \tag{5}$$

$$= 1 + \frac{1}{\lambda\beta} \int_0^b L(y) \left(e^{-\frac{y}{\lambda\beta}} e^{\left\{ \frac{(1-\lambda)u}{\lambda\beta} + \frac{X_t}{\beta} \right\}} \right) dy.$$

$$\text{Let } C(u) = e^{\left\{ \frac{(1-\lambda)u}{\lambda\beta} + \frac{X_t}{\beta} \right\}}; \quad 0 \leq u \leq b$$

$$\text{Therefore,} \quad L(u) = 1 + \frac{C(u)}{\lambda\beta} \int_0^b L(y) e^{-\frac{y}{\lambda\beta}} dy, \quad 0 \leq u \leq b. \tag{6}$$

$$\text{Let } s = \int_0^b L(y) e^{-\frac{y}{\lambda\beta}} dy, \text{ so } L(u) = 1 + \frac{C(u)}{\lambda\beta} s.$$

$$= \int_0^b \left(1 + \frac{C(y)}{\lambda\beta} s \right) e^{-\frac{y}{\lambda\beta}} dy$$

$$= \lambda\beta \left(1 - e^{-\frac{b}{\lambda\beta}} \right) + \frac{s}{\lambda} e^{\frac{X_t}{\beta}} \left[1 - e^{-\frac{b}{\lambda\beta}} \right].$$

$$\text{Therefore, } s = \frac{\lambda\beta \left(1 - e^{-\frac{b}{\lambda\beta}} \right)}{1 - \frac{1}{\lambda} \left(1 - e^{-\frac{b}{\lambda\beta}} \right) e^{\frac{X_t}{\beta}}}.$$

Substitute constant s into Equation (6), then

$$L(u) = 1 + \frac{C(u)}{\lambda\beta} s = 1 + \frac{e^{\frac{(1-\lambda)u}{\lambda\beta}} e^{\frac{X_t}{\beta}} \lambda \left(1 - e^{-\frac{b}{\lambda\beta}} \right)}{\lambda - \left(1 - e^{-\frac{b}{\lambda\beta}} \right) e^{\frac{X_t}{\beta}}}.$$

Consequently, the analytical ARL by solving the integral equation as

$$L(u) = 1 - \frac{\lambda \left(1 - e^{-\frac{b}{\lambda\beta}} \right) e^{\frac{(1-\lambda)u}{\lambda\beta}}}{\left(1 - e^{-\frac{b}{\lambda\beta}} \right) - \lambda e^{-\frac{X_t}{\beta}}} \tag{7}$$

Since the process is an in-control state with exponential parameter $\beta = \beta_0$, the explicit formula for the ARL_0 form is as follows

$$L_0(u) = 1 - \frac{\lambda(1 - e^{-\frac{b}{\lambda\beta_0}})e^{-\frac{(1-\lambda)u}{\lambda\beta_0}}}{(1 - e^{-\frac{b}{\beta_0}}) - \lambda e^{-\frac{X_1}{\beta_0}}} \tag{8}$$

On the other hand, if the process is an out-of-control state with exponential parameter $\beta = \beta_1$, where $\beta = \beta_0(1 + \delta)$, where δ is shift size, the explicit formula for ARL_1 can be written as follows

$$L_1(u) = 1 - \frac{\lambda(1 - e^{-\frac{b}{\lambda\beta_1}})e^{-\frac{(1-\lambda)u}{\lambda\beta_1}}}{(1 - e^{-\frac{b}{\beta_1}}) - \lambda e^{-\frac{X_1}{\beta_1}}} \tag{9}$$

4. Existence and uniqueness of analytical average run length of the EWMA chart of the long-memory ARFIMA process

In this section, the definitions related to the proof of the existence and uniqueness of the analytical ARL are as follows:

Definition 1 Let M be a nonempty set and $\vartheta: M \times M \rightarrow \mathbb{R}$ be called a metric M . For all $x, y, z \in M$, denote (M, ϑ) a metric space, which satisfies the following conditions:

- (1) $\vartheta(x, y) = 0$ if and only if $x = y$,
- (2) $\vartheta(x, y) = \vartheta(y, x)$,
- (3) $\vartheta(x, z) = \vartheta(y, x) + \vartheta(x, z)$.

Definition 2 A sequence (x_n) of points of (M, ϑ) is a Cauchy's sequence if $\vartheta(x_m, x_n) \rightarrow 0$ for $m, n \rightarrow \infty$.

Definition 3 A metric space is complete if every Cauchy sequence converges to $x \in M$. That is to say, if $\vartheta(x_m, x_n) \rightarrow 0$ as $m, n \rightarrow \infty$, then there exists $x \in M$ such that $\vartheta(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$.

Definition 4 An operator $T: M \rightarrow M$ is a contraction mapping, or contraction, if there exists $\rho \in [0, 1)$ such that $\vartheta(Tx, Ty) \leq \rho\vartheta(x, y)$, for all $x, y \in M$.

Definition 5 An element $x \in M$ is a fixed point of an operator T , if $Tx = x$.

Definition 6 A uniform norm or maximum norm is defined by $\| \bullet \|_{\infty} = \sup_{0 \leq u \leq b} |L(u)|$

THEOREM 1 *Banach's Fixed Point Theorem (see Sofonea, Han, & Shillor, 2006)*

If $T: M \rightarrow M$ is a contraction mapping on a complete metric space (M, ϑ) , then there exists a unique solution $x \in M$ of $Tx = x$.

An integral equation of the second kind corresponding to the analytical ARL of the EWMA chart with a long-memory ARFIMA process is defined as

$$L(u) - \frac{1}{\lambda\beta} \int_0^b L(y) \left(e^{-\frac{y}{\lambda\beta} \left\{ \frac{(1-\lambda)u}{\lambda\beta} + \frac{X_1}{\beta} \right\}} \right) dy = 1 \tag{10}$$

with $L: [0, b] \rightarrow \mathbb{R}$ as an unknown function and $k(u, y) = \frac{1}{\lambda\beta} e^{-\frac{y}{\lambda\beta} \left\{ \frac{(1-\lambda)u}{\lambda\beta} + \frac{X_1}{\beta} \right\}}$, which is a kernel function, and the mapping T is defined as

$$TL(u) = 1 + \frac{1}{\lambda\beta} \int_0^b L(y) \left(e^{-\frac{y}{\lambda\beta} \left\{ \frac{(1-\lambda)u}{\lambda\beta} + \frac{X_1}{\beta} \right\}} \right) dy \tag{11}$$

THEOREM 2 *Analytical ARL, or $L(u)$, of an integral equation of the second kind corresponding to the EWMA chart with a long-memory ARFIMA process has existence and uniqueness.*

Proof. (Existence)

Let $C([0, b])$ be a set of all continuous functions on $[0, b]$, $L_0 \in C([0, b])$, and $(L_n)_{n \geq 0}$ be a Cauchy's sequence of analytical ARL that satisfies $L_{n+1} = TL_n$. Consider

$$\vartheta(L_{n+1}, L_n) = \vartheta(TL_n, TL_{n-1}) \leq \rho \vartheta(TL_n, TL_{n-1}); \rho \in [0, 1).$$

Iteratively, $\vartheta(L_{n+1}, L_n) \leq \rho^n \vartheta(L_1, L_0)$, for $n \geq 0$.

Thus, For $n \geq m$

$$\vartheta(L_n, L_m) \leq \vartheta(L_n, L_{n-1}) + \dots + \vartheta(L_{m+1}, L_m) \leq (\rho^{n-1} + \dots + \rho^m) \vartheta(L_1, L_0) \leq \frac{\rho^m}{1 - \rho} \vartheta(L_1, L_0)$$

$(L_n)_{n \geq 0}$ is a Cauchy's sequence and $\lim_{n \rightarrow \infty} T^n L_0 \rightarrow L$. That is, $TL = L$.

Therefore, the existing continuous function $L: [0, b] \rightarrow \mathbb{R}$ satisfied integral Equation (10).

(Uniqueness)

Let $k(u, y) = \frac{1}{\lambda \rho} e^{-\frac{y}{\rho} \left\{ \frac{(1-\lambda)u}{c} + \frac{X_t}{\rho} \right\}}$ be a continuous function, where $k: [0, b] \times [0, b] \rightarrow \mathbb{R}$.

To prove that T is a contraction mapping on the complete metric space $(C([0, b]), \|\bullet\|_\infty)$, it is necessary to show that the inequality $\sup_{0 \leq x \leq b} \left\{ \int_0^b |k(u, y)| dy \right\} < 1$ is satisfied.

For any $L_1, L_2 \in C([0, b])$,

$$\|TL_1 - TL_2\|_\infty = \sup_{0 \leq u \leq b} \left| \int_0^b k(u, y)(L_1(y) - L_2(y)) dy \right| \leq \rho \|L_1 - L_2\|_\infty$$

$$\text{where } \rho = \sup_{0 \leq x \leq b} \left\{ \int_0^b |k(u, y)| dy \right\} < 1.$$

After applying Banach's Fixed Point Theorem, so T is a contraction mapping. Therefore, there is a unique continuous function $L: [0, b] \rightarrow \mathbb{R}$ that satisfied integral Equation (10).

5. Numerical average run length of the EWMA chart of the long-memory ARFIMA process

An integral equation of the second kind for the ARL of the EWMA chart of the long-memory ARFIMA process is formed Equation (5) where $f\left(\frac{y-(1-\lambda)u}{\lambda} - X_t\right)$ is a kernel function and $L(u)$ is the unknown function or the ARL function.

The Composite Midpoint Rule is applied to divide the domain interval $[0, b]$ into m sub-grids with equal length. Namely,

$$\int_0^b L(y) f\left(\frac{y-(1-\lambda)u}{\lambda} - X_t\right) dy \approx \sum_{j=1}^m w_j f(a_j) \tag{12}$$

$$\text{where } a_j = \frac{b}{m} \left(j - \frac{1}{2}\right) \text{ and } w_j = \frac{b}{m}; j = 1, 2, \dots, m.$$

Substitute Equation (12) into Equation (5) to acquire the system of linear equations

$$\tilde{L}(a_i) \approx 1 + \frac{1}{\lambda} \sum_{j=1}^m w_j \tilde{L}(a_j) f\left(\frac{a_j - (1-\lambda)a_i}{\lambda} - X_t\right); i = 1, 2, \dots, m \tag{13}$$

The matrix represented by the above system of m linear equations with m unknown variables can be formed

$$L_{m \times 1} = (I_m - R_{m \times m})^{-1} \mathbf{1}_{m \times 1} \tag{14}$$

where $L_{m \times 1} = [\tilde{L}(a_1), \tilde{L}(a_2), \dots, \tilde{L}(a_m)]^T$, $\mathbf{1}_{m \times 1} = [1, 1, \dots, 1]^T$, a unit vector, $R_{m \times m}$ is a matrix with dimension $m \times m$ with element $r_{ij} \approx \frac{1}{\lambda} w_j f\left(\frac{a_j - (1-\lambda)a_i}{\lambda} - X_t\right)$, $I_{m \times 1} = [diag(1, 1, \dots, 1)]$ is a unit diagonal matrix.

Finally, u is substituted for a_i in $\tilde{L}(a_i)$, the numerical integration for function ARL(u) called approximated ARL as follows

$$\tilde{L}(u) \approx 1 + \frac{1}{\lambda} \sum_{j=1}^m w_j \tilde{L}(a_j) f\left(\frac{a_j - (1-\lambda)u}{\lambda} - X_t\right) \tag{15}$$

6. Numerical results

In this section, the Absolute Percentage Relative Error (APRE) to measure the accuracy of ARL is defined as

$$APRE = \frac{|L(u) - \tilde{L}(u)|}{L(u)} \times 100\% \tag{16}$$

where $L(u)$ is an analytical ARL, $\tilde{L}(u)$ is a numerical ARL.

Using Equations (8) and (15) to evaluate the ARL of the EWMA control chart of the long-memory ARFIMA process with exponential white noise and the shift parameter $\beta = \beta_0(1 + \delta)$, where δ is shift size, the numerical value of the analytical ARL, numerical ARL, and APRE are shown as Tables 1 and 2. The figures in the parentheses account for the computational CPU time (PC System: Windows 7 Ultimate, 4.00 GB RAM, 64-bit Operating System) in minutes.

Table 1. Comparison of the numerical values of the ARL on ARFIMA (2, 0.1, 1) for $\phi_1 = 0.1, \phi_2 = 0.2, \theta_1 = 0.1, u = 1$ between explicit formula and the NIE method

Shift size (δ) with $\beta_0 = 1$	$\lambda = 0.1, b = 0.00116835$			$\lambda = 0.2, b = 0.04815825$		
	Analytical ARL	Numerical ARL	APRE (%)	Analytical ARL	Numerical ARL	APRE (%)
0.00	370.000278327767 (0.013)	370.000278325667 (6.895)	5.676×10^{-10}	370.000719625898 (0.014)	370.000718469701 (6.833)	3.125×10^{-07}
0.01	330.531062710523 (0.013)	330.531062708684 (13.665)	5.564×10^{-10}	298.249344990459 (0.015)	298.24934411144 (13.697)	2.947×10^{-07}
0.03	265.483124226259 (0.013)	265.483124224842 (20.42)	5.337×10^{-10}	208.503539782817 (0.016)	208.503539221868 (20.452)	2.690×10^{-07}
0.05	214.996836316501 (0.014)	214.996836315396 (27.082)	5.140×10^{-10}	155.324129650318 (0.016)	155.324129261563 (27.316)	2.503×10^{-07}
0.10	131.174032164052 (0.014)	131.17403216344 (33.759)	4.666×10^{-10}	87.1431882494976 (0.016)	87.1431880604958 (34.149)	2.169×10^{-07}
0.30	26.746104745948 (0.016)	26.7461047458612 (40.529)	3.245×10^{-10}	21.3731121727153 (0.016)	21.3731121424153 (40.95)	1.418×10^{-07}
0.40	14.5165662689905 (0.015)	14.5165662689513 (47.331)	2.700×10^{-10}	13.5036772575531 (0.016)	13.5036772416677 (47.798)	1.176×10^{-07}
0.50	8.69830365910755 (0.015)	8.69830365908805 (54.117)	2.242×10^{-10}	9.32436340150487 (0.016)	9.32436339234663 (54.599)	9.822×10^{-08}

Table 2. Comparison of the numerical values of the ARL on ARFIMA (2, 0.3, 1) for $\phi_1 = 0.1, \phi_2 = 0.2, \theta_1 = 0.1, u = 1$ between explicit formula and the NIE method

Shift size (δ) with $\beta_0 = 1$	$\lambda = 0.1, b = 0.000952043$			$\lambda = 0.2, b = 0.03902366$		
	Analytical ARL	Numerical ARL	APRE (%)	Analytical ARL	Numerical ARL	APRE (%)
0.00	370.000102121068 (0.013)	370.000102119674 (13.806)	3.767×10^{-10}	370.000489748709 (0.014)	370.000488992539 (13.713)	2.044×10^{-07}
0.01	329.834194954745 (0.013)	329.834194953526 (20.593)	3.696×10^{-10}	294.623262926142 (0.014)	294.62326235931 (20.531)	1.924×10^{-07}
0.03	263.844709287366 (0.013)	263.844709286432 (27.41)	3.540×10^{-10}	202.681554867049 (0.014)	202.68155451156 (27.270)	1.754×10^{-07}
0.05	212.837836137424 (0.013)	212.837836136697 (41.045)	3.416×10^{-10}	149.42252748169 (0.015)	149.422527237917 (34.088)	1.631×10^{-07}
0.10	128.688468651728 (0.014)	128.688468651329 (47.737)	3.100×10^{-10}	82.5156392321039 (0.015)	82.5156391153942 (40.906)	1.414×10^{-07}
0.30	25.529523652078 (0.014)	25.5295236520231 (54.539)	2.150×10^{-10}	19.7172740590727 (0.015)	19.7172740408606 (47.707)	9.237×10^{-08}
0.40	13.732940036868 (0.015)	13.7329400368437 (61.2)	1.784×10^{-10}	12.3841472074836 (0.016)	12.3841471980154 (54.54)	7.645×10^{-08}
0.50	8.18150577624434 (0.014)	8.18150577623225 (67.986)	1.478×10^{-10}	8.52239783752942 (0.016)	8.52239783210926 (61.373)	6.360×10^{-08}

Tables 1 and 2 show the figures of the analytical ARL and numerical ARL given $ARL_0 = 370, m = 1, 000$, varied with different shift sizes: 0.00, 0.01, 0.03, 0.05, 0.10, 0.30, 0.40, and 0.50. The smoothing parameters, λ , of the EWMA control chart were set as 0.1 and 0.2. The order of d value was also set as 0.1 and 0.3, corresponding to the long-memory ARFIMA process. Overall, the figures of numerical ARL approach the figures of the analytical ARL, which is the exact solution of ARL integral Equation (10). For all tables, the shift sizes increase, while the figures of both ARLs decrease significantly, starting approximately at 370 of ARL_0 . Likewise, the figures of ARPE show a clear decrease. On the contrary, the computational CPU time for numerical ARL increases continuously in all tables, starting at around seven minutes for $d = 0.1$ and around 14 min for $d = 0.3$, and the CPU time of the analytical ARL is steady between 0.013 and 0.016 min, while the shift sizes increase. Therefore, the analytical method of evaluation ARL is better than numerical method because both methods return almost the same ARL but analytical method takes the time less than numerical method.

7. Conclusions

This paper compared analytical ARL to numerical ARL. Furthermore, the existence and uniqueness of the analytical ARL was proved. The research indicated that the NIE method shows good agreement with the exact solution of the analytical ARL of the integral equation because the numerical ARL converges to the analytical ARL when the shift size increases. In evaluating both ARLs, the computational time of the ARL evaluation was found to be quite small. To sum up, the explicit formula for the ARL is an alternative for evaluating the ARL of EWMA quality control charts of the long-memory ARFIMA process because the CPU computational time for analyzing ARLs is very small compared to the CPU computational time for numerical ARLs. As discussed previously, the explicit formula is a very fast method with good potential; nevertheless, the limitation is that some analytical ARLs cannot be derived directly as explicit formulas due to the unfeasibility of computing the terms of the integral in integral equations. Future research should be focused on computing numerical ARL in case of integral equations with complicated integrands and the real-life situation data with long-memory behavior is an alternative and should be applied to evaluate the ARL of EWMA control chart.

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