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*Corresponding author: Hager A. Ibrahim, Faculty of Commerce, Department of Statistics, AL-Azhar University (Girls' Branch), Cairo, Egypt
E-mail: dr.hager_ahmad@azhar.edu.eg

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Hiroshi Shiraishi, Keio university, Japan

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STATISTICS | RESEARCH ARTICLE

Direct L-moments for Type-I censored data with application to the Kumaraswamy distribution

Mahmoud Riad Mahmoud[1], Fatma A. Khalil[2], Ghada A. El-Kelany[2] and Hager A. Ibrahim[2]*

Abstract: This paper suggests a modification of L-moments method to make it suitable for censored data directly (namely: Direct L-moments). This study concentrates on Type-I censored data. The modification is applied to estimate the unknown parameters of Kumaraswamy (Kw) distribution. The suggested modification is compared with L-moments via partial probability-weighted moments (PPWM) method and maximum likelihood (ML) method. The results are achieved using a comparative numerical study in terms of estimate of the unknown parameters, relative bias and root of mean square error (RMSE) using Monte Carlo simulation.

Subjects: Science; Mathematics & Statistics; Applied Mathematics

Keywords: censored data; estimation; conventional moments; L-moments; Kumaraswamy distribution

1. Introduction

Finding robust and reliable estimates of unknown parameters has a great importance to the practitioner, whether he does this repeatedly as in industrial and medical applications, or occasionally as in business applications. For estimating the unknown parameters, conventional moments method considerable one of the most important methods of estimation. But it is often considerably less

ABOUT THE AUTHORS

Mahmoud Riad Mahmoud is an emeritus Professor of Statistics at Cairo University, Worked as Dean of the Institute of Statistical Studies and Research, interested in Theory of Statistics and Statistical Inference.

Fatma A. Khalil is an emeritus Associate Professor of Statistics in the Statistics department, Faculty of Commerce, AL-Azhar University (Girls' Branch), Cairo, Egypt, interested in Theory of Statistics and Statistical Inference.

Ghada A. El-Kelany, Assistant Professor in the department of statistics, Faculty of Commerce, AL-Azhar University (Girls' Branch), Cairo, Egypt. She received PhD in statistics in 2013 from AL-Azhar University (Girls' Branch), Cairo, Egypt, interested in Theory of Statistics and Statistical Inference.

Hager A. Ibrahim is PhD student at Faculty of Commerce, AL-Azhar University (Girls' Branch), Cairo, Egypt. She received MSc in Statistics 2014, Statistics department, Faculty of Commerce AL-Azhar University (Girls' Branch), Cairo, Egypt, interested in Theory of Statistics and Statistical Inference.

PUBLIC INTEREST STATEMENT

The Kumaraswamy distribution is a very good alternative to the beta distribution for modeling random processes which values are bounded from below and above. It has been utilized to model certain measurements in genetics, hydrology and ecology. Fitting this model to a given data-set requires the estimation of the parameters of the distribution. Among the methods used for estimating the parameters we find, the method of moments is used frequently. The use of the L-moments has been suggested as an alternative to the method of moments. In this article, the L-moments are utilized to estimate the parameters of the Kumaraswamy distribution.

accurate than those obtained using other methods, especially in the case of small samples (Bílková, 2014). Recently, L-moments method has been noticed as appealing alternative method to the conventional moments. Hosking (1990) pointed out, compared to the conventional moments, L-moments have lower sample variances and are more robust against outliers, the method of L-moments is analogous way to the method of conventional moments, L-moments estimators computed by equating the sample moments with the corresponding population moments, also, the L-moments ratios (L-skewness and L-kurtosis) are analogous to conventional moment ratios. L-moments have certain theoretical advantages over conventional moments (Zafirakou-Koulouris, Vogel, Craig, and Habermeier, 1998).

L-moments was first introduced by Sillitto (1951) and formally defined as expectations of certain linear combinations of order statistics by Hosking (1990). Hosking (1990) unified the theory of L-moments and provided guidelines for the practical use of L-moments. The method of L-moments was discussed from various earlier studies by Hosking (1990) who provided guidelines for the practical use of L-moments. Hosking (1990, 1992, 1994, 2006), Hosking, and Wallis (1995), Sillitto (1969), Elamir and Seheult (2001, 2004), Jones (2004), Asquith (2007), Abdul-Moniem (2007), Karvanen (2006), Bílková (2014) and Dutang (2016) considered various theoretical aspects and applications of L-moments for complete data.

L-moments are defined as linear compination of the expectation of the order statistics. Let X be a continuous random variable distributed with the distribution function $F(x)$ and quantile function $x(F)$. Consider $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$, the order statistics of a random sample. L-moments of the rth order of the random variable X is defined by Hosking (1990) as

$$\lambda_r = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k \binom{r-1}{k} E(X_{r-k:r}), \quad r = 1, 2, \ldots \tag{1.1}$$

Now, the expectation of the order statistics are given by:

$$E(X_{i:r}) = \frac{r!}{(i-1)!(r-i)!}\int_0^1 x(F)F^{i-1}(1-F)^{r-i}dF \tag{1.2}$$

The sample L-moments can be estimated unbiasedly from the sample order statistics by a formula suggested by Asquith (2007) as:

$$l_r = \frac{1}{r\binom{n}{r}}\sum_{i=1}^{n}\sum_{k=0}^{r-1}(-1)^k \binom{r-1}{k}\binom{i-1}{r-k-1}\binom{n-i}{k} X_{i:n} \tag{1.3}$$

For censored data, Hosking (1995) defined two variants of L-moment which are used with right censored observations. Similarly, Zafirakou-Koulouris et al. (1998) extended the applicability of L-moments ratio diagrams to left censored data via introducing a new partial probability weighted moment PPWM's based on L-moments definition. Wang (1990a, 1990b) extended the concept of PWM, suggested by Greenwood, Landwehr, Matalas, and Wallis (1979), to PPWM for the analysis of censored samples. The aim of this study is introduced a modification of L-moments method to make it suitable for Type-I censored data directly with application from Kumaraswamy distribution.

Kumaraswamy (1980) presented a distribution and called it the double bounded distribution to model hydrological random processes which are bounded at the lower and upper ends. The probability density function (pdf) of Kw distribution is

$$f(x;a, b) = abx^{a-1}(1-x^a)^{b-1}, \quad 0 < x < 1, \quad a, b > 0 \tag{1.4}$$

and cumulative distribution function (cdf) is

$$F(x;a,b) = 1 - (1 - x^a)^b \tag{1.5}$$

This article is organized as follows; Section 2 is concerned with L-moments for Type-I censored data via PPWM. A modification of L-moments, direct L-moments, is introduced in Section 3. direct L-moments for Kw distribution is presented in Section 4. L-moments via PPWM for Censored Data for Kw distribution are introduced in Section 5. ML method for censored data for Kw distribution is presented in Section 6. Simulation study and concluding remarks are presented in Section 7.

2. L-moments for censored data via PPWM

Wang (1990a, b) introduced the concept of PPWM. Let $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$, the order statistics of a random sample of size n that comes from the distribution of the random variable X. Wang (1990a) defined a lower censoring PPWM as follows:

$$\beta_r^{'} = \int_c^1 x(F)F^r dF, \quad r = 0, 1, 2, \ldots \tag{2.1}$$

where $c = F(T)$ is the random fraction of observations that are uncensored, T being the lower bound censoring threshold. He showed that the following statistic is an unbiased estimator of $\beta_r^{'}$:

$$b_r^{'} = \frac{1}{n}\sum_{i=1}^{n} \frac{\binom{i-1}{r}}{\binom{n-1}{r}} x_{(i)}^{*}, \quad r = 0, 1, 2, \ldots \tag{2.2}$$

where,

$$x_{(i)}^{*} = \begin{cases} 0, & x_{(i)} \le T \\ x_{(i)}, & x_{(i)} > T \end{cases}$$

Similar quantities for upper bound censoring are defined by Wang (1990b) as follows:

$$\beta_r^{''} = \int_0^c x(F)F^r dF, \quad r = 0, 1, 2, \ldots \tag{2.3}$$

where $c = F(T)$, T now being the upper bound censoring threshold. Given a complete sample $x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)}$, the following statistic is an unbiased estimator of $\beta_r^{''}$:

$$b_r^{''} = \frac{1}{n}\sum_{i=1}^{n} \frac{\binom{i-1}{r}}{\binom{n-1}{r}} x_{(i)}^{**}, \quad r = 0, 1, 2, \ldots \tag{2.4}$$

where,

$$x_{(i)}^{**} = \begin{cases} x_{(i)}, & x_{(i)} \le T \\ 0, & x_{(i)} > T \end{cases}$$

Hosking (1995) introduced two different PPWM for using in L-moments with right censored data, Type-A PPWM and Type-B PPWM respectively.

- **Type-A PPWM**

$$\beta_r^{A} = \frac{1}{c^{r+1}} \int_0^c x(F)F^r dF, \quad r = 0, 1, 2, \ldots \tag{2.5}$$

where, $c = F(T)$. He derived Type-A PPWM estimator as follows: where m is observed (uncensored) data.

- **Type-B PPWM**

$$\beta_r^B = \int_0^c x(F)F^r dF + \frac{1-c^{r+1}}{r+1}x(c), \quad r = 0, 1, 2, \ldots \tag{2.6}$$

He derived Type-B PPWM estimator as follows:

$$b_r^B = \frac{1}{n}\left[\sum_{i=1}^{m} \frac{\binom{i-1}{r}}{\binom{m-1}{r}} X_{i:n} + \sum_{i=m+1}^{n} \frac{\binom{i-1}{r}}{\binom{n-1}{r}} T\right], \quad r = 0, 1, 2, \ldots \tag{2.7}$$

Zafirakou-Koulouris et al. (1998) derived PPWM for left censoring, following the same approach introduced by Hosking (1995) for right censoring. They derived Type-A' and Type-B' PPWM for left censoring respectively.

- **Type-A' PPWM**

$$\beta_r^{A'} = \frac{1}{(1-c)^{r+1}} \int_c^1 x(F)(F-c)^r dF, \quad r = 0, 1, 2, \ldots \tag{2.8}$$

They presented unbiased estimators of Type-A' PPWM for left censoring as follows:

$$b_r^{A'} = \frac{1}{s}\sum_{i=1}^{s} \frac{\binom{i-1}{r}}{\binom{s-1}{r}} X_{n-s+i:n}, \quad r = 0, 1, 2, \ldots, \quad s = n-m+1 \tag{2.9}$$

where s is censored data.

- **Type-B' PPWM**

$$\beta_r^{B'} = x(c)\frac{c^{r+1}}{r+1} + \int_c^1 F^r x(F) dF, \quad r = 0, 1, 2, \ldots \tag{2.10}$$

They presented unbiased estimators of Type-B' PPWM for left censoring as follows:

$$b_r^{B'} = \frac{1}{n}\left[\sum_{i=1}^{n-k} \frac{\binom{i-1}{r}}{\binom{n-1}{r}} T + \sum_{i=n-k+1}^{n} \frac{\binom{i-1}{r}}{\binom{n-1}{r}} X_{i:n}\right], \quad r = 0, 1, 2, \ldots \tag{2.11}$$

For any distribution the rth L-moments is related to the rth PPWM, see Hosking (1990), via

$$\lambda_{r+1} = \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k}\binom{r+k}{k} \beta_k, \quad r = 0, 1, 2, \ldots \tag{2.12}$$

from which the first four L-moments in terms of PPWM are

$$\begin{aligned} \lambda_1 &= \beta_0, \\ \lambda_2 &= -\beta_0 + 2\beta_1, \\ \lambda_3 &= \beta_0 - 6\beta_1 + 6\beta_2, \\ \lambda_4 &= -\beta_0 + 12\beta_1 - 30\beta_2 + 20\beta_3. \end{aligned} \tag{2.13}$$

3. Direct L-moments for censored data

All the research sighted above discussed L-moments method with censored data to estimate the parameters relied on using PPWM. The aim of this section is introducing a modification of L-moments (namely: Direct L-moments) method to make it suitable for censored data directly.

3.1. Direct L-moments for right censored data

Let $x_1, x_2, \ldots, x_n$ be a Type-I censored random sample of size n from a distribution with distribution function $F(x)$ and quantile function $x(u)$. Let the threshold T satisfy $F(T) = c$ and c is the fraction of observed data.

$$\underbrace{x_{1:n} \le x_{2:n} \le \cdots \le x_{m:n}}_{\text{m(observed)}} \le T \le \underbrace{x_{m+1:n} \le \cdots \le x_{n-1:n} \le x_{n:n}}_{\text{n-m(censored)}}$$

3.1.1. Direct L-moments for right censored data (Type-AD)

The quantile function of Type-AD L-moments is

$$y^A(u) = x(uc) \quad 0 < u < 1$$

substitution into Equation (1.1) leads to the Type-AD L-moments where:

$$\begin{aligned} \mu_r^A &= \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k \binom{r-1}{k} \frac{r!}{(r-k-1)!k!} \int_0^1 y^A(u) u^{r-k-1}(1-u)^k du \\ &= \frac{1}{rc^r}\sum_{k=0}^{r-1}(-1)^k \binom{r-1}{k} \frac{r!}{(r-k-1)!k!} \int_0^c x(u) u^{r-k-1}(c-u)^k du \end{aligned} \tag{3.1}$$

The first four L-moments for Type-AD right censoring are calculated as follows:

$$\begin{aligned} \mu_1^A &= \frac{1}{c}\int_0^c x(u)du, \\ \mu_2^A &= \frac{2}{c^2}\int_0^c ux(u)du - \mu_1^A, \\ \mu_3^A &= \frac{6}{c^3}\int_0^c u^2x(u)du - \frac{6}{c^2}\int_0^c ux(u)du + \mu_1^A, \\ \mu_4^A &= \frac{20}{c^4}\int_0^c u^3x(u)du - \frac{30}{c^3}\int_0^c u^2x(u)du + \frac{12}{c^2}\int_0^c ux(u)du - \mu_1^A. \end{aligned} \tag{3.2}$$

The standard method to compute L-moments estimator is equating the sample L-moments (m_r^A) with the corresponding population L-moments (μ_r^A). Type-AD L-moments estimators of the uncensored data of m observations are given by

$$m_r^A = \frac{1}{r\binom{m}{r}} \sum_{i=1}^{m}\sum_{k=0}^{r-1}(-1)^k \binom{r-1}{k}\binom{i-1}{r-k-1}\binom{m-i}{k} X_{i:n} \tag{3.3}$$

3.1.2. Direct L-moments for right censored data (Type-BD)

The quantile function of Type-BD L-moments is

$$y^B(u) = \begin{cases} x(u), & 0 < u < c \\ x(c), & c \le u < 1 \end{cases}$$

substitution into Equation (1.1) leads to the Type-BD L-moments where:

$$\mu_r^B = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k \binom{r-1}{k}\frac{r!}{(r-k-1)!k!}\int_0^1 y^B(u)u^{r-k-1}(1-u)^k du$$

$$\begin{aligned}
\mu_r^B &= \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k \binom{r-1}{k}\frac{r!}{(r-k-1)!k!} \\
&\quad \times\left[\int_0^c x(u)u^{r-k-1}(1-u)^k du + \int_c^1 x(c)u^{r-k-1}(1-u)^k du\right] \\
&= \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k \binom{r-1}{k}\frac{r!}{(r-k-1)!k!} \\
&\quad \times\left[\int_0^c x(u)u^{r-k-1}(1-u)^k du + x(c)\left(\int_0^1 u^{r-k-1}(1-u)^k du - \int_0^c u^{r-k-1}(1-u)^k du\right)\right] \\
&= \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k \binom{r-1}{k}\frac{r!}{(r-k-1)!k!} \\
&\quad \times\left[\int_0^c x(u)u^{r-k-1}(1-u)^k du + x(c)[\beta(r-k, k+1) - \beta(c,\ r-k,\ k+1)]\right]
\end{aligned} \tag{3.4}$$

The first four L-moments for Type-BD right censoring are calculated as follows:

$$\begin{aligned}
\mu_1^B &= \int_0^c x(u)du + x(c)[1 - \beta(c,\ 1,\ 1)], \\
\mu_2^B &= 2\int_0^c ux(u)du - \int_0^c x(u)du, \\
\mu_3^B &= 6\int_0^c u^2x(u)du - 6\int_0^c ux(u)du + \int_0^c x(u)du + 2x(c)[2\beta(c,2,2) - \beta(c,\ 3,\ 1)], \\
\mu_4^B &= 20\int_0^c u^3x(u)du - 30\int_0^c u^2x(u)du + 12\int_0^c ux(u)du - \int_0^c x(u)du.
\end{aligned} \tag{3.5}$$

The standard method to compute L-moments estimator is equating the rth sample L-moments (m_r^B) with the corresponding population L-moments (μ_r^B). Type-BD L-moments estimators are computed from the complete sample, where $n - m$ censored data are replaced by the censoring threshold T given by:

$$m_r^B = \frac{1}{r\binom{n}{r}}\left[\sum_{i=1}^{m}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\binom{i-1}{r-k-1}\binom{n-i}{k}X_{i:n} + \left(\sum_{i=m+1}^{n}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\binom{i-1}{r-k-1}\binom{n-i}{k}\right)T\right] \quad (3.6)$$

3.2. Direct L-moments for left censored data

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n. Type-I left censoring occurs when the observations below the fixed threshold T are censored:

$$\underbrace{x_{1:n} \le x_{2:n} \le \cdots \le x_{s:n}}_{\text{s(censored)}} \le T \le \underbrace{x_{s+1:n} \le \cdots \le x_{n-1:n} \le x_{n:n}}_{\text{n-s(observed)}}$$

Let the threshold T satisfy $F(T) = h$ and h is the fraction of censored data.

3.2.1. Direct L-moments for left censored data (Type-A′D)

The quantile function of Type-A'D L-moments is

$$y^{A'}(u) = x((1-h)u + h) \quad 0 < u < 1$$

substitution into (1.1) leads to the Type-A'D L-moments where:

$$\mu_r^{A'} = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\frac{r!}{(r-k-1)!k!}\int_0^1 y^{A'}(u)u^{r-k-1}(1-u)^k du$$
$$= \frac{1}{r(1-h)^r}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\frac{r!}{(r-k-1)!k!}\int_h^1 x(u)(u-h)^{r-k-1}(1-u)^k du \quad (3.7)$$

The first four L-moments for Type-A'D left censoring are calculated as follows:

$$\mu_1^{A'} = \frac{1}{1-h}\int_h^1 x(u)du,$$
$$\mu_2^{A'} = \frac{1}{(1-h)^2}\left[2\int_h^1 ux(u)du - (h+1)\int_h^1 x(u)du\right],$$
$$\mu_3^{A'} = \frac{1}{(1-h)^3}\left[6\int_h^1 u^2x(u)du - 6(h+1)\int_h^1 ux(u)du + (h^2+4h+1)\int_h^1 x(u)du\right], \quad (3.8)$$
$$\mu_4^{A'} = \frac{1}{(1-h)^4}\left[20\int_h^1 u^3x(u)du - 30(h+1)\int_h^1 u^2x(u)du + 12(h^2+3h+1)\int_h^1 ux(u)du - (h^3+9h^2+9h+1)\int_h^1 x(u)du\right].$$

The standard method to compute L-moments estimator is equating the sample L-moments ($m_r^{A'}$) with the corresponding population L-moments ($\mu_r^{A'}$). Type-A'D L-moments estimators for left censored data given by:

$$m_r^{A'} = \frac{1}{r\binom{n-s}{r}} \sum_{i=1}^{n-s} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \binom{i-1}{r-k-1} \binom{n-s-i}{k} X_{s+i:n} \tag{3.9}$$

3.2.2. Direct L-moments for left censored data (Type-B'D)

The quantile function of Type-B'D L-moments is

$$y^{B'}(u) = \begin{cases} x(h), & 0 < u \le h \\ x(u), & h < u < 1 \end{cases}$$

substitution into Equation (1.1) leads to the Type-B'D L-moments where:

$$\begin{aligned} \mu_r^{B'} &= \frac{1}{r} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \frac{r!}{(r-k-1)!k!} \int_0^1 y^{B'}(u) u^{r-k-1} (1-u)^k du \\ &= \frac{1}{r} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \frac{r!}{(r-k-1)!k!} \left[\int_0^h x(h) u^{r-k-1} (1-u)^k du + \int_h^1 x(u) u^{r-k-1} (1-u)^k du \right] \\ &= \frac{1}{r} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \frac{r!}{(r-k-1)!k!} \left[x(h)\beta(h, r-k, k+1) + \int_h^1 x(u) u^{r-k-1} (1-u)^k du \right] \end{aligned} \tag{3.10}$$

The first four L-moments for Type-B'D left censoring are calculated as follow:

$$\begin{aligned} \mu_1^{B'} &= x(h)\beta(h, 1, 1) + \int_h^1 x(u) du, \\ \mu_2^{B'} &= 2\int_h^1 u x(u) du - \int_h^1 x(u) du, \\ \mu_3^{B'} &= 6\int_h^1 u^2 x(u) du - 6\int_h^1 u x(u) du + \int_h^1 x(u) du - 4x(h)\beta(h, 2, 2), \\ \mu_4^{B'} &= 20\int_h^1 u^3 x(u) du - 30\int_h^1 u^2 x(u) du + 12\int_h^1 u x(u) du - \int_h^1 x(u) du. \end{aligned} \tag{3.11}$$

The standard method to compute L-moments estimator is equating the sample L-moments ($m_r^{B'}$) with the corresponding population L-moments ($\mu_r^{B'}$). Type-B'D L-moments estimators for left censored data given by

$$m_r^{B'} = \frac{1}{r\binom{n}{r}}\left[\sum_{i=1}^{s}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\binom{i-1}{r-k-1}\binom{n-i}{k}T + \sum_{i=s+1}^{n}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\binom{i-1}{r-k-1}\binom{n-i}{k}X_{i:n}\right] \tag{3.12}$$

4. Direct L-moments for censored data for Kw distribution

In this section, the population L-moments of order r for Kw distribution is introduced.

- **Type-AD**

From 0 (3.2), the first four L-moments for Type-I right censoring with Type-AD for Kw distribution are calculated as follows:

$$\mu_1^A = \frac{1}{c}\int_0^c [1-(1-u)^{\frac{1}{b}}]^{\frac{1}{a}}du.$$

Putting $z=(1-u)^{\frac{1}{b}}$, this led to $du=-bz^{b-1}\,dz$; and, $0<u<c$ gives $1<z<(1-c)^{\frac{1}{b}}$, then

$$\mu_1^A = \frac{b}{c}\left[\int_0^1 z^{b-1}(1-z)^{\frac{1}{a}}dz - \int_0^{(1-c)^{\frac{1}{b}}} z^{b-1}(1-z)^{\frac{1}{a}}dz\right]$$
$$= \frac{b}{c}\left[\beta\left(b, \frac{1}{a}+1\right) - \beta\left((1-c)^{\frac{1}{b}}, b, \frac{1}{a}+1\right)\right],$$
$$\mu_2^A = \frac{(2-c)b}{c^2}\left[\beta\left(b, \frac{1}{a}+1\right) - \beta\left((1-c)^{\frac{1}{b}}, b, \frac{1}{a}+1\right)\right] - \frac{2b}{c^2}\left[\beta\left(2b, \frac{1}{a}+1\right) - \beta\left((1-c)^{\frac{1}{b}}, 2b, \frac{1}{a}+1\right)\right],$$
$$\mu_3^A = \frac{6b}{c^3}\left[\beta\left(3b, \frac{1}{a}+1\right) - \beta\left((1-c)^{\frac{1}{b}}, 3b, \frac{1}{a}+1\right) - (2-c)\left[\beta\left(2b, \frac{1}{a}+1\right) - \beta\left((1-c)^{\frac{1}{b}}, 2b, \frac{1}{a}+1\right)\right] + \left(1-c+\frac{c^2}{6}\right)\left[\beta\left(b, \frac{1}{a}+1\right) - \beta\left((1-c)^{\frac{1}{b}}, b, \frac{1}{a}+1\right)\right]\right],$$
$$\mu_4^A = \frac{(20-30c+12c^2-c^3)b}{c^4}\left[\beta\left(b, \frac{1}{a}+1\right) - \beta\left((1-c)^{\frac{1}{b}}, b, \frac{1}{a}+1\right)\right] - \frac{(60-60c+12c^2)b}{c^4}\left[\beta\left(2b, \frac{1}{a}+1\right) - \beta\left((1-c)^{\frac{1}{b}}, 2b, \frac{1}{a}+1\right)\right] + \frac{(60-30c)b}{c^4}\left[\beta\left(3b, \frac{1}{a}+1\right) - \beta\left((1-c)^{\frac{1}{b}}, 3b, \frac{1}{a}+1\right)\right] - \frac{20b}{c^4}\left[\beta\left(4b, \frac{1}{a}+1\right) - \beta\left((1-c)^{\frac{1}{b}}, 4b, \frac{1}{a}+1\right)\right]. \tag{4.1}$$

- **Type-BD**

From Equation (3.2), the first four L-moments for Type-I right censoring with Type-BD for Kw distribution are calculated as follows:

$$\begin{aligned}
\mu_1^B &= b\left[\beta\left(b, \frac{1}{a}+1\right) - \beta\left((1-c)^{\frac{1}{b}}, b, \frac{1}{a}+1\right)\right] + \left(1-(1-c)^{\frac{1}{b}}\right)^{\frac{1}{a}}(1-\beta(c,1,1)),\\
\mu_2^B &= b\left[\beta\left(b, \frac{1}{a}+1\right) - \beta\left((1-c)^{\frac{1}{b}}, b, \frac{1}{a}+1\right)\right] - 2b\left[\beta\left(2b, \frac{1}{a}+1\right) - \beta\left((1-c)^{\frac{1}{b}}, 2b, \frac{1}{a}+1\right)\right],\\
\mu_3^B &= 6b\left[\beta\left(3b, \frac{1}{a}+1\right) - \beta\left((1-c)^{\frac{1}{b}}, 3b, \frac{1}{a}+1\right)\right] - 6b\left[\beta\left(2b, \frac{1}{a}+1\right) - \beta\left((1-c)^{\frac{1}{b}}, 2b, \frac{1}{a}+1\right)\right]\\
&\quad + b\left[\beta\left(b, \frac{1}{a}+1\right) - \beta\left((1-c)^{\frac{1}{b}}, b, \frac{1}{a}+1\right)\right] - 2\left(1-(1-c)^{\frac{1}{b}}\right)^{\frac{1}{a}}[\beta(c,3,1) - 2\beta(c,2,2)],\\
\mu_4^B &= b\left[\beta\left(b, \frac{1}{a}+1\right) - \beta\left((1-c)^{\frac{1}{b}}, b, \frac{1}{a}+1\right)\right] - 12b\left[\beta\left(2b, \frac{1}{a}+1\right) - \beta\left((1-c)^{\frac{1}{b}}, 2b, \frac{1}{a}+1\right)\right]\\
&\quad + 30b\left[\beta\left(3b, \frac{1}{a}+1\right) - \beta\left((1-c)^{\frac{1}{b}}, 3b, \frac{1}{a}+1\right)\right] - 20b\left[\beta\left(4b, \frac{1}{a}+1\right) - \beta\left((1-c)^{\frac{1}{b}}, 4b, \frac{1}{a}+1\right)\right].
\end{aligned} \tag{4.2}$$

- **Type-A'D**

From Equation (3.8), the first four L-moments for Type-I left censoring with Type-A'D for Kw distribution are calculated as follows:

$$\begin{aligned}
\mu_1^{A'} &= \frac{b}{1-h}\beta\left((1-h)^{\frac{1}{b}}, b, \frac{1}{a}+1\right),\\
\mu_2^{A'} &= \frac{b}{(1-h)}\beta\left((1-h)^{\frac{1}{b}}, b, \frac{1}{a}+1\right) - \frac{2b}{(1-h)^2}\beta\left((1-h)^{\frac{1}{b}}, 2b, \frac{1}{a}+1\right),\\
\mu_3^{A'} &= \frac{b}{(1-h)^2}\beta\left((1-h)^{\frac{1}{b}}, b, \frac{1}{a}+1\right) - \frac{6b}{(1-h)^2}\beta\left((1-h)^{\frac{1}{b}}, 2b, \frac{1}{a}+1\right)\\
&\quad + \frac{6b}{(1-h)^3}\beta\left((1-h)^{\frac{1}{b}}, 3b, \frac{1}{a}+1\right),\\
\mu_4^{A'} &= \frac{b}{1-h}\beta\left((1-h)^{\frac{1}{b}}, b, \frac{1}{a}+1\right) - \frac{12b}{(1-h)^2}\beta\left((1-h)^{\frac{1}{b}}, 2b, \frac{1}{a}+1\right)\\
&\quad + \frac{30b}{(1-h)^3}\beta\left((1-c)^{\frac{1}{b}}, 3b, \frac{1}{a}+1\right) - \frac{20b}{(1-h)^4}\beta\left((1-c)^{\frac{1}{b}}, 4b, \frac{1}{a}+1\right).
\end{aligned} \tag{4.3}$$

- **Type-B'D**

From Equation (3.11), the first four L-moments for Type-I left censoring with Type-B'D for Kw distribution are calculated as follows:

$$\begin{aligned}
\mu_1^{B'} &= \left(1-(1-h)^{\frac{1}{b}}\right)^{\frac{1}{a}}\beta(h,1,1) + b\beta\left((1-h)^{\frac{1}{b}}, b, \frac{1}{a}+1\right),\\
\mu_2^{B'} &= b\beta\left((1-h)^{\frac{1}{b}}, b, \frac{1}{a}+1\right) - 2b\beta\left((1-h)^{\frac{1}{b}}, 2b, \frac{1}{a}+1\right),\\
\mu_3^{B'} &= b\beta\left((1-h)^{\frac{1}{b}}, b, \frac{1}{a}+1\right) - 6b\beta\left((1-h)^{\frac{1}{b}}, 2b, \frac{1}{a}+1\right) + 6b\beta\left((1-h)^{\frac{1}{b}}, 3b, \frac{1}{a}+1\right)\\
&\quad - 4\left(1-(1-h)^{\frac{1}{b}}\right)^{\frac{1}{a}}\beta(h,2,2),\\
\mu_4^{B'} &= b\beta\left((1-h)^{\frac{1}{b}}, b, \frac{1}{a}+1\right) - 12b\beta\left((1-h)^{\frac{1}{b}}, 2b, \frac{1}{a}+1\right) + 30b\beta\left((1-h)^{\frac{1}{b}}, 3b, \frac{1}{a}+1\right)\\
&\quad - 20b\beta\left((1-h)^{\frac{1}{b}}, 4b, \frac{1}{a}+1\right).
\end{aligned} \tag{4.4}$$

5. L-moments via PPWM for censored data for Kw distribution

In this section, different types PPWM of Kw distribution are introduced which we use in determining the L-moments of a Type-I censored data.

- **Type-A**

From Equation (2.5), the first four PPWM for Type-I right censoring with Type-A for Kw distribution are calculated as follows:

$$\begin{aligned}
\beta_0^A &= \frac{b}{c}\left[\beta(b, \frac{1}{a}+1) - \beta((1-c)^{\frac{1}{b}}, b, \frac{1}{a}+1)\right],\\
\beta_1^A &= \frac{b}{c^2}\left[\left(\beta(b, \frac{1}{a}+1) - \beta((1-c)^{\frac{1}{b}}, b, \frac{1}{a}+1)\right) - \left(\beta(2b, \frac{1}{a}+1) - \beta((1-c)^{\frac{1}{b}}, 2b, \frac{1}{a}+1)\right)\right],\\
\beta_2^A &= \frac{b}{c^3}\left[\left(\beta(b, \frac{1}{a}+1) - \beta((1-c)^{\frac{1}{b}}, b, \frac{1}{a}+1)\right) - 2\left(\beta(2b, \frac{1}{a}+1) - \beta((1-c)^{\frac{1}{b}}, 2b, \frac{1}{a}+1)\right)\right.\\
&\quad \left. + \left(\beta(3b, \frac{1}{a}+1) - \beta((1-c)^{\frac{1}{b}}, 3b, \frac{1}{a}+1)\right)\right],\\
\beta_3^A &= \frac{b}{c^4}\left[\left(\beta(b, \frac{1}{a}+1) - \beta((1-c)^{\frac{1}{b}}, b, \frac{1}{a}+1)\right) - 3\left(\beta(2b, \frac{1}{a}+1) - \beta((1-c)^{\frac{1}{b}}, 2b, \frac{1}{a}+1)\right)\right.\\
&\quad \left. + 3\left(\beta(3b, \frac{1}{a}+1) - \beta((1-c)^{\frac{1}{b}}, 3b, \frac{1}{a}+1)\right) - \left(\beta(4b, \frac{1}{a}+1) - \beta((1-c)^{\frac{1}{b}}, 4b, \frac{1}{a}+1)\right)\right].
\end{aligned} \tag{5.1}$$

- **Type-B**

From Equation (2.6), the first four PPWM for Type-I right censoring with Type-B for Kw distribution are calculated as follows:

$$\begin{aligned}
\beta_0^B &= b\left[\beta(b, \frac{1}{a}+1) - \beta((1-c)^{\frac{1}{b}}, b, \frac{1}{a}+1)\right] + (1-c)\left[1-(1-c)^{\frac{1}{b}}\right]^{\frac{1}{a}},\\
\beta_1^B &= b\left(\beta(b, \frac{1}{a}+1) - \beta((1-c)^{\frac{1}{b}}, b, \frac{1}{a}+1)\right) - b\left(\beta(2b, \frac{1}{a}+1) - \beta((1-c)^{\frac{1}{b}}, 2b, \frac{1}{a}+1)\right)\\
&\quad + \frac{1-c^2}{2}\left[1-(1-c)^{\frac{1}{b}}\right]^{\frac{1}{a}},\\
\beta_2^B &= b\left(\beta(b, \frac{1}{a}+1) - \beta((1-c)^{\frac{1}{b}}, b, \frac{1}{a}+1)\right) - 2b\left(\beta(2b, \frac{1}{a}+1) - \beta((1-c)^{\frac{1}{b}}, 2b, \frac{1}{a}+1)\right)\\
&\quad + b\left(\beta(3b, \frac{1}{a}+1) - \beta((1-c)^{\frac{1}{b}}, 3b, \frac{1}{a}+1)\right) + \frac{1-c^3}{3}\left[1-(1-c)^{\frac{1}{b}}\right]^{\frac{1}{a}},\\
\beta_3^B &= b\left(\beta(b, \frac{1}{a}+1) - \beta((1-c)^{\frac{1}{b}}, b, \frac{1}{a}+1)\right) - 3b\left(\beta(2b, \frac{1}{a}+1) - \beta((1-c)^{\frac{1}{b}}, 2b, \frac{1}{a}+1)\right)\\
&\quad + 3b\left(\beta(3b, \frac{1}{a}+1) - \beta((1-c)^{\frac{1}{b}}, 3b, \frac{1}{a}+1)\right) - b\left(\beta(4b, \frac{1}{a}+1) - \beta((1-c)^{\frac{1}{b}}, 4b, \frac{1}{a}+1)\right)\\
&\quad + \frac{1-c^4}{4}\left[1-(1-c)^{\frac{1}{b}}\right]^{\frac{1}{a}}.
\end{aligned} \tag{5.2}$$

- **Type-A'**

From Equation (2.8), the first four PPWM for Type-I right censoring with Type-A' for Kw distribution are calculated as follows:

$$\beta_0^{A'} = \frac{b}{1-h}\beta((1-h)^{\frac{1}{b}}, b, \frac{1}{a}+1),$$
$$\beta_1^{A'} = \frac{b}{1-h}\beta(1-h)^{\frac{1}{b}}, b, \frac{1}{a}+1) - \frac{b}{(1-h)^2}\beta((1-h)^{\frac{1}{b}}, 2b, \frac{1}{a}+1),$$
$$\beta_2^{A'} = \frac{b}{1-h}\beta(1-h)^{\frac{1}{b}}, b, \frac{1}{a}+1) - \frac{2b}{(1-h)^2}\beta((1-h)^{\frac{1}{b}}, 2b, \frac{1}{a}+1) + \frac{b}{(1-h)^3}\beta((1-h)^{\frac{1}{b}}, 3b, \frac{1}{a}+1),$$
$$\beta_3^{A'} = \frac{b}{1-h}\beta(1-h)^{\frac{1}{b}}, b, \frac{1}{a}+1) - \frac{3b}{(1-h)^2}\beta((1-h)^{\frac{1}{b}}, 2b, \frac{1}{a}+1) + \frac{3b}{(1-h)^3}\beta(1-h)^{\frac{1}{b}}, 3b, \frac{1}{a}+1) - \frac{b}{(1-h)^4}\beta((1-h)^{\frac{1}{b}}, 4b, \frac{1}{a}+1). \quad (5.3)$$

- **Type-B'**

From Equation (2.10), the first four PPWM for Type-I right censoring with Type-B' for Kw distribution are calculated as follows:

$$\beta_0^{B'} = b\beta((1-h)^{\frac{1}{b}}, b, \frac{1}{a}+1) + h(1-(1-h)^{\frac{1}{b}})^{\frac{1}{a}},$$
$$\beta_1^{B'} = b\beta(1-h)^{\frac{1}{b}}, b, \frac{1}{a}+1) - b\beta((1-h)^{\frac{1}{b}}, 2b, \frac{1}{a}+1) + \frac{h^2}{2}(1-(1-h)^{\frac{1}{b}})^{\frac{1}{a}},$$
$$\beta_2^{B'} = b\beta(1-h)^{\frac{1}{b}}, b, \frac{1}{a}+1) - 2b\beta((1-h)^{\frac{1}{b}}, 2b, \frac{1}{a}+1) + b\beta((1-h)^{\frac{1}{b}}, 3b, \frac{1}{a}+1) + \frac{h^3}{3}(1-(1-h)^{\frac{1}{b}})^{\frac{1}{a}},$$
$$\beta_3^{B'} = b\beta(1-h)^{\frac{1}{b}}, b, \frac{1}{a}+1) - 3b\beta((1-h)^{\frac{1}{b}}, 2b, \frac{1}{a}+1) + 3b\beta((1-h)^{\frac{1}{b}}, 3b, \frac{1}{a}+1) - b\beta((1-h)^{\frac{1}{b}}, 4b, \frac{1}{a}+1) + \frac{h^4}{4}(1-(1-h)^{\frac{1}{b}})^{\frac{1}{a}}. \quad (5.4)$$

6. Maximum likelihood method for censored data for Kw distribution

6.1. ML method for right censored data for Kw distribution

Let $x_1, x_2, \ldots, x_n$ is a sample size of Kw distribution. The likelihood function of the parameters (a and b) of Kw distribution, based on Type-I right censoring, is given by:

$$L(a, b; \underline{x}) \propto \prod_{i=1}^{m} f(x_i) \times [1 - F(T)]^{n-m} \quad (6.1)$$

Therefore, the log-likelihood function is

$$\ell = \log L(a, b; \underline{x}) = \sum_{i=1}^{m} \log f(x_i) + (n-m)\log(1 - F(T)) \quad (6.2)$$

To estimate the unknown parameters, a and b, the first partial derivations of the log likelihood function, ℓ, with respect to a and b respectively is needed. Setting $\frac{\partial \ell}{\partial a} = 0$ and $\frac{\partial \ell}{\partial b} = 0$, we get the likelihood equations:

$$\frac{m}{b} + \sum_{i=1}^{m} \log(1 - x_i^a) + (n-m)\log(1 - T^a) = 0 \quad (6.3)$$

and,

$$\frac{m}{\hat{a}} + \sum_{i=1}^{m} [1 - (\hat{b} - 1) x_i^{\hat{a}} (1 - x_i^{\hat{a}})^{-1}] \log x_i - \hat{b}(n - m) T^{\hat{a}} (1 - T^{\hat{a}})^{-1} \log T = 0 \tag{6.4}$$

These equations constitute a system of two nonlinear equations must be solved in a and b to get the maximum likelihood estimator (MLEs) for right censored data of these parameters. It is obvious that the system of nonlinear equation has no closed form solution. So, a numerical technique is required to get the estimates of the unknown parameters.

6.2. ML method for left censored data for Kw distribution

Let $x_1, x_2, \ldots, x_n$ is a sample size of Kw distribution. The likelihood function of the parameters (a and b) of Kw distribution, based on Type-I left censoring, is given by:

$$L(a, b; \underline{x}) \propto [F(T)]^s \times \prod_{i=s+1}^{n} f(x_i) \tag{6.5}$$

Therefore, the log-likelihood function is

$$\ell = \log L(a, b; \underline{x}) = s \log(F(T)) + \sum_{i=s+1}^{n} \log f(x_i) \tag{6.6}$$

To estimate the unknown parameters, a and b, the first partial derivations of the log likelihood function, ℓ, with respect to a and b respectively is needed. Setting $\frac{\partial \ell}{\partial a} = 0$ and $\frac{\partial \ell}{\partial b} = 0$, we get the likelihood equations:

$$\frac{sbT^a (1 - T^a)^{b-1} \log T}{1 - (1 - T^a)^b} + \sum_{i=s+1}^{n} \left[1 - (b - 1)(1 - x_i^a)^{-1} x_i^a) \log x_i + \frac{1}{a} \right] = 0 \tag{6.7}$$

and,

$$\frac{-s(1 - T^a)^b \log(1 - T^a)}{1 - (1 - T^a)^b} + \sum_{i=s+1}^{n} \left[\log(1 - x_i^a) + \frac{1}{b} \right] = 0 \tag{6.8}$$

These equations constitute a system of two nonlinear equations must be solved in a and b to get the MLEs for left censored data of these parameters. It is obvious that the system of nonlinear equation has no closed form solution. So, a numerical technique is required to get the estimates of the unknown parameters.

7. Simulation study and concluding remarks

This section is devoted to present the modification of L-moments method (namely: Direct L-moments) in estimation process using a comparative numerical study. The two unknown parameters of Kumaraswamy distribution are estimated using direct L-moments method, L-moments via PPWM method and ML method for Type-I censored data(right and left censoring). A comparative study based on relative bias (RB) and root of mean square errors. All computations are performed using Mathematica-10 programs. The simulation study is conducted according to the following steps:

Table 1. The estimates, relative bias (RB) and RMSE for two parameters of Kw distribution using Direct L-moments, L-moments and ML method based on right censoring ($a = 2$ and $b = 5$)

Type (A)

T	*n*	Par.	Direct L-moments (Type-AD)			L-moments (Type-A)			ML		
			Estimate	RB	RMSE	Estimate	RB	RMSE	Estimate	RB	RMSE
0.7	50	*a*	2.0314	0.0157	2.0543	2.0315	0.0158	2.0544	2.0628	0.0314	2.0840
		b	5.3562	0.0712	5.6187	5.3563	0.0713	5.6188	5.4893	0.0978	5.7344
	75	*a*	2.0163	0.0081	2.0313	2.0162	0.0081	2.0312	2.0364	0.0182	2.0500
		b	5.2262	0.0452	5.3889	5.2263	0.0452	5.3887	5.3045	0.0609	5.4507
	100	*a*	2.0140	0.0070	2.0252	2.0141	0.0071	2.0253	2.0309	0.0154	2.0413
		b	5.1694	0.0338	5.2868	5.1694	0.0339	5.2869	5.2413	0.0482	5.3504
0.9	50	*a*	2.0182	0.0091	2.0398	2.0183	0.0092	2.0399	2.0640	0.0320	2.0844
		b	5.2738	0.0036	5.5092	5.2739	0.0037	5.5093	5.5080	0.1016	5.7551
	75	*a*	2.0086	0.0042	2.0222	2.0087	0.0043	2.0223	2.0417	0.0320	2.0546
		b	5.1813	0.0017	5.3273	5.1814	0.0907	5.3274	5.3502	0.1016	5.4921
	100	*a*	2.0083	0.0041	2.0187	2.0084	0.0042	2.0188	2.0317	0.0158	2.0413
		b	5.1265	0.0016	5.2334	5.1266	0.0017	5.2335	5.2293	0.0458	5.3279

Type (B)

T	*n*	Par.	Direct L-moments (Type-BD)			L-moments (Type-B)			ML		
			Estimate	RB	RMSE	Estimate	RB	RMSE	Estimate	RB	RMSE
0.7	50	*a*	1.3786	−0.3106	1.4793	2.0116	0.0058	2.0335	2.0628	0.0314	2.0840
		b	3.0822	−0.3835	3.6782	5.2201	0.4403	5.4657	5.4893	0.0978	5.7344
	75	*a*	1.3571	−0.3214	1.4241	2.0046	0.0023	2.0189	2.0364	0.0182	2.0500
		b	2.8561	−0.4287	3.2269	5.1419	0.0283	5.2896	5.3045	0.0609	5.4507
	100	*a*	2.0044	0.0022	2.0149	1.3571	−0.3214	1.4098	2.0309	0.0154	2.0413
		b	5.1199	0.0239	5.2288	2.7921	−0.4415	3.0766	5.2413	0.0482	5.3504
0.9	50	*a*	2.0084	0.0042	2.0310	2.0144	0.0072	2.0359	2.0640	0.0320	2.0844
		b	5.2291	0.0458	5.4864	5.2505	0.0501	5.5008	5.5080	0.1016	5.7551
	75	*a*	2.0027	0.0042	2.0170	2.0078	0.0072	2.0215	2.0417	0.0320	2.0546
		b	5.1566	0.0458	5.3074	5.1768	0.0501	5.3228	5.3502	0.1016	5.4921
	100	*a*	1.9998	−0.0001	2.0107	2.0062	0.0030	2.0164	2.0317	0.0158	2.0413
		b	5.0743	0.0148	5.1813	5.1008	0.0201	5.2028	5.2293	0.0458	5.3279

Table 2. The estimates, relative bias (RB) and RMSE for two parameters of Kw distribution using Direct L-moments, L-moments and ML method based on right censoring ($a = 5$ and $b = 2$)

Type (A)

T	**n**	**Par.**	**Direct L-moments (Type-AD)**			**L-moments (Type-A)**			**ML**		
			Estimate	**RB**	**RMSE**	**Estimate**	**RB**	**RMSE**	**Estimate**	**RB**	**RMSE**
0.97	50	*a*	5.0712	0.0142	5.1476	5.0713	0.0143	5.1477	5.1297	0.0259	5.2160
		b	2.0880	0.0440	2.1564	2.0881	0.0441	2.1565	2.1188	0.0594	2.1792
	75	*a*	5.0571	0.0114	5.1071	5.0538	0.0107	5.1039	5.1105	0.0221	5.1574
		b	2.0560	0.0280	2.1002	2.0607	0.0303	2.1055	2.0838	0.0419	2.1208
	100	*a*	5.0537	0.0107	5.0895	5.0538	0.0108	5.0896	5.0997	0.0199	5.1320
		b	2.0457	0.0228	2.0761	2.0458	0.0229	2.0762	2.0712	0.0356	2.0968
0.99	50	*a*	5.0905	0.0181	5.1644	5.0906	0.0182	5.1645	5.1503	0.0300	5.2419
		b	2.0778	0.0389	2.1447	2.0779	0.0389	2.1447	2.1132	0.0566	2.1775
	75	*a*	5.0905	0.0181	5.1644	5.0906	0.0182	5.1645	5.1503	0.0300	5.2419
		b	2.0778	0.0388	2.1447	2.0779	0.0389	2.1448	2.1132	0.1132	2.1775
	100	*a*	5.0233	0.0046	5.0580	5.0234	0.0047	5.0581	5.0793	0.0158	5.1109
		b	2.0282	0.0141	2.0580	2.0283	0.0142	2.0581	2.0612	0.0306	2.0864

Type (B)

T	*n*	Par.	Direct L-moments (Type-BD)			L-moments (Type-B)			ML		
			Estimate	RB	RMSE	Estimate	RB	RMSE	Estimate	RB	RMSE
0.97	50	*a*	3.6948	−0.2610	4.0025	5.0909	0.0181	5.1659	5.1297	0.0259	5.2160
		b	1.4093	−0.2953	1.6287	2.0792	0.0396	2.1467	2.1188	0.0594	2.1792
	75	*a*	3.5680	−0.2863	3.7787	5.0380	0.0076	5.0868	5.1105	0.0221	5.1574
		b	1.3030	−0.3484	1.4549	2.0463	0.0231	2.0892	2.0838	0.0419	2.1208
	100	*a*	3.5166	−0.2966	3.6817	5.0280	0.0056	5.0639	5.0997	0.0199	5.1320
		b	1.2488	−0.3755	1.3623	2.0307	0.0153	2.0613	2.0712	0.0356	2.0968
0.99	50	*a*	4.8858	−0.0228	4.9998	5.0850	0.0170	5.1590	5.1503	0.0300	5.2419
		b	1.9721	−0.0139	2.0681	2.0732	0.0366	2.1399	2.1132	0.0566	2.1775
	75	*a*	4.8858	−0.0228	4.9998	5.0850	0.0170	5.1590	5.1503	0.0300	5.2419
		b	1.9721	−0.0139	2.0681	2.0732	0.0366	2.1399	2.1132	0.1132	2.1775
	100	*a*	4.8005	−0.0398	4.8564	5.0200	0.0040	5.0546	5.0793	0.0158	5.1109
		b	1.9007	−0.0496	1.9506	2.0253	0.0126	2.0550	2.0612	0.0306	2.0864

Table 3. The estimates, relative bias (RB) and RMSE for two parameters of Kw distribution using Direct L-moments, L-moments and ML method based on left censoring ($a = 0.75$ and $b = 5$)

Type (A)

T	*n*	Par.	Direct L-moments (Type-*A*′D)			L-moments (Type-*A*′)			ML		
			Estimate	RB	RMSE	Estimate	RB	RMSE	Estimate	RB	RMSE
0.009	50	*a*	0.7575	0.0101	0.7683	0.7576	0.0102	0.7684	0.9154	0.2205	1.0295
		b	5.3733	0.0746	5.6788	5.3734	0.0747	5.6789	5.5116	0.1023	5.6601
	75	*a*	0.7555	0.0074	0.7624	0.7556	0.0075	0.7625	0.8823	0.1764	0.8829
		b	5.2386	0.0477	5.4167	5.2387	0.0478	5.4168	5.1804	0.3639	5.9432
	100	*a*	0.7543	0.0057	0.7597	0.7544	0.0058	0.7598	0.8492	0.1323	0.9402
		b	5.1843	0.0368	5.3185	5.1844	0.0369	5.3186	5.3512	0.0702	5.4322
0.001	50	*a*	0.7531	0.0041	0.7626	0.7532	0.0042	0.7627	0.7759	0.0345	0.7836
		b	5.3061	0.0612	5.5695	5.3062	0.0613	5.5696	5.5438	0.1087	5.7935
	75	*a*	0.7528	0.0037	0.7592	0.7529	0.0038	0.7593	0.7740	0.0316	0.7813
		b	5.1981	0.0396	5.3627	5.1982	0.0397	5.3628	5.5360	0.1022	5.8375
	100	*a*	0.7524	0.0032	0.7571	0.7523	0.0033	0.7570	0.7624	0.0165	0.7660
		b	5.1566	0.0313	5.2741	5.1565	0.0314	5.2740	5.5513	0.1102	5.9415

Type (B)

T	*n*	Par.	Direct L-moments (Type-*B*′D)			L-moments (Type-*B*′)			ML		
			Estimate	RB	RMSE	Estimate	RB	RMSE	Estimate	RB	RMSE
0.009	50	*a*	0.7836	0.0448	0.7932	0.7802	0.0402	0.7897	0.9154	0.2205	1.0295
		b	5.7021	0.1404	6.0311	5.6536	0.1307	5.9709	5.5116	0.1023	5.6601
	75	*a*	0.7834	0.0446	0.7896	0.7800	0.0400	0.7800	0.8823	0.1764	0.8829
		b	5.5759	0.1151	5.7686	5.5693	0.1138	5.5694	5.1804	0.3639	5.9432
	100	*a*	0.7832	0.0443	0.7880	0.7799	0.0399	0.7847	0.8492	0.1323	0.9402
		b	5.5264	0.1052	5.6709	5.4850	0.0970	5.6253	5.3512	0.0702	5.4322
0.001	50	*a*	0.7507	0.0009	0.7604	0.7505	0.0006	0.7603	0.7759	0.0345	0.7836
		b	5.2779	0.0555	5.5474	5.2760	0.0552	5.5448	5.5438	0.1087	5.7935
	75	*a*	0.7527	0.0037	0.7591	0.7526	0.0035	0.7590	0.7740	0.0316	0.7813
		b	5.1986	0.0397	5.3628	5.1971	0.0394	5.3610	5.5360	0.1022	5.8375
	100	*a*	0.7539	0.0052	0.7638	0.7538	0.0050	0.7636	0.7624	0.0165	0.7660
		b	5.3209	0.0641	5.5959	5.3189	0.0637	5.5934	5.5513	0.1102	5.9415

Table 4. The estimates, relative bias (RB) and RMSE for two parameters of Kw distribution using Direct L-moments, L-moments and ML method based on left censoring ($a = 5$ and $b = 0.75$)

Type (A)

T	*n*	Par.	Direct L-moments (Type-A'D)			L-moments (Type-A')			ML		
			Estimate	RB	RMSE	Estimate	RB	RMSE	Estimate	RB	RMSE
0.4	50	*a*	5.1873	0.0374	5.3201	5.1874	0.0375	5.3202	5.2863	0.0572	5.3738
		b	0.7738	0.0317	0.7938	0.7739	0.0318	0.7939	1.9498	2.5998	5.9238
	75	*a*	5.1055	0.0211	5.1917	5.1056	0.0212	5.1918	5.1784	0.0356	5.2530
		b	0.7608	0.0144	0.7732	0.7609	0.0145	0.7733	0.7739	0.0318	0.7827
	100	*a*	5.0958	0.0191	5.1567	5.1125	0.0225	5.1103	5.1421	0.0284	5.1939
		b	0.7590	0.0121	0.7683	0.7648	0.0198	0.7683	0.7677	0.0237	0.7741
0.2	50	*a*	5.1193	0.0238	5.2424	5.1194	0.0239	5.2425	4.8815	0.0236	5.0052
		b	0.7687	0.0248	0.7880	0.7688	0.0249	0.7881	0.7462	0.0049	0.7599
	75	*a*	5.0784	0.0156	5.1593	5.0785	0.0157	5.1594	5.1065	0.0213	5.1697
		b	0.7618	0.0158	0.7744	0.7619	0.0159	0.7745	0.7872	0.0497	0.7968
	100	*a*	5.0638	0.0127	5.1268	5.0639	0.0128	5.1269	5.1254	0.0250	5.1910
		b	0.7568	0.0092	0.7658	0.7569	0.0093	0.7659	0.7662	0.0216	0.7721

Type (B)

T	*n*	Par.	Direct L-moments (Type-B'D)			L-moments (Type-B')			ML		
			Estimate	RB	RMSE	Estimate	RB	RMSE	Estimate	RB	RMSE
0.4	50	*a*	5.8078	0.1615	5.9809	5.2321	0.0464	5.3460	5.2863	0.0572	5.3738
		b	0.9283	0.2378	0.9908	0.7867	0.0489	0.8039	1.9498	2.5998	5.9238
	75	*a*	5.7013	0.1402	5.8025	5.1598	0.0319	5.2317	5.1784	0.0356	5.2530
		b	0.8969	0.1959	0.9291	0.7742	0.0323	0.7846	0.7739	0.0318	0.7827
	100	*a*	5.6874	0.1374	5.7620	5.1516	0.0303	5.2015	5.1421	0.0284	5.1939
		b	0.8900	0.1867	0.9138	0.7724	0.0298	0.7801	0.7677	0.0237	0.7741
0.2	50	*a*	5.1285	0.0257	5.2510	5.1168	0.0233	5.2399	4.8815	0.0236	5.0052
		b	0.7711	0.0282	0.7907	0.7685	0.0247	0.7878	0.7462	0.0049	0.7599
	75	*a*	5.0883	0.0176	5.1687	5.0756	0.0151	5.1566	5.1065	0.0213	5.1697
		b	0.7643	0.0191	0.7770	0.7617	0.0156	0.7742	0.7872	0.0497	0.7968
	100	*a*	5.0734	0.0146	5.1362	5.0606	0.0121	5.1236	5.1254	0.0250	5.1910
		b	0.7593	0.0124	0.7684	0.7566	0.0088	0.7655	0.7662	0.0216	0.7721

(1) Generate n random sample size (25, 50, 75 and 100) drawn randomly from Kw distribution with some different values of parameters. However, the results were not impressive for small sample sizes (less than 50) and therefore it was not reported.
(2) The generated data is ordered.
(3) The point of censored data is fixed, namely threshold (T).
(4) Applying formulas mentioned in (3.3), (3.6), (3.9) and (3.12).
(5) Equate step (4) with the corresponding population moments to get the estimates $\hat{a}$ and $\hat{b}$.
(6) The simulation process repeated 5,000 times.
(7) The simulation results are reported from Tables 1 to 4.

7.1. Concluding remarks

Tables 1–4 reporting the estimates of the two unknown parameters of Kumaraswamy distribution and their characteristics, it is observed that:

- As expected, the relative bias and RMSE decreases as sample sizes increases with reasonable results obtained starting from $n = 75$.
- The results suggest that the direct L-moments method is better, in terms of accuracy and precision, than L-moments via PPWM and ML methods.
- In all results, relative bias and root mean square errors decrease as censoring levels increase.
- In the case of right and left censoring, the estimates of direct L-moments method with Type-AD estimates are very close to L-moments with Type-A estimates.
- In the case of right and left censoring, the estimates of direct L-moments method with Type-BD estimates are much more accurate than L-moments with Type-B estimates.
- In general the variance of estimates is small and this helps to obtain short confidence interval.
- It is recommended to use the direct L-moments method with Type-AD estimates in the case of right and left censoring.

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Author details
Mahmoud Riad Mahmoud[1]
E-mail: mrmahmoud@cu.edu.eg
Fatma A. Khalil[2]
E-mail: dr_fkhalil@yahoo.com
Ghada A. El-Kelany[2]
E-mail: gh_elkelany@azhar.edu.eg
Hager A. Ibrahim[2]
E-mail: dr.hager_ahmad@azhar.edu.eg
[1] Department of Mathematical Statistics, Institute of Statistics Studies and Research, Cairo University, Cairo, Egypt.
[2] Faculty of Commerce, Department of Statistics, AL-Azhar University (Girls' Branch), Cairo, Egypt.

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