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Closed form solutions of two nonlinear equation via the enhanced (G'/G)-expansion method

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Abstract: The enhanced (G'/G)-expansion method is highly effective and competent mathematical tool to examine exact traveling wave solutions of nonlinear evolution equations (NLEEs) arising in mathematical physics, applied mathematics, and engineering. Exact solutions of NLEEs play an important role to comprehend the obscurity of intricate physical phenomena. In this article, the enhanced (G'/G)-expansion method is suggested and executed to construct exact solutions of the first extended fifth order non-linear equation and the medium equal width equation. The solutions are presented in terms of the hyperbolic and the trigonometric functions involving free parameters. It is shown that the proposed method is effective and can be used for many other NLEEs in mathematical physics.

Subjects: Advanced Mathematics; Applied Mathematics; Mathematics Education

Keywords: the enhanced (G'/G)-expansion method; first extended fifth order non-linear equation; medium equal width (MEW) equation; nonlinear evolution equations (NLEEs); closed form wave solutions

1. Introduction

At the present time nonlinear evolution equations (NLEEs) appear in a broad range of scientific research in various fields. Since (NLEEs) and their exact solutions are frequently used to depict the

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PUBLIC INTEREST STATEMENT

Nonlinear evolution equations (NLEEs) frequently arise in formulating fundamental laws of nature and many problems naturally arising from solid-state physics, plasma physics, ocean and atmospheric waves, meteorology, etc. Closed form solutions to NLEEs play a significant role in nonlinear science, especially in nonlinear physical science, since it can provide much physical information and more insight into the physical aspects of the problem. As a result, various techniques have been developed by several groups of mathematicians and physicists to examine closed form solutions to NLEEs. In this article, we use the enhanced (G'/G)-expansion method to extract fresh and abundant exact traveling wave solutions to first extended fifth order non-linear equation and the medium equal width equation. Thus, we obtain closed form wave solutions of these two equations among them some are new solutions. We expect that the new exact traveling wave solutions will be helpful to clarify the associated phenomena.

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inner mechanism and obscurity of complex phenomena in various fields of science and engineering such as fluid dynamics, fluid mechanics, gas dynamics, elasticity, biochemistry, protein chemistry, chemically reactive materials, in ecology most population model, high energy physics, plasma physics, nuclear physics, optical fibers, meteorology, etc. Therefore, it is very crucial to search for further exact traveling solutions to NLEEs and gradually becomes one of the most important and significant tasks. As a result diverse groups of mathematicians, physicist, and engineers have been working in order to develop effective methods for obtaining exact solutions to NLEEs. For this reason, in the recent years several methods have been established to search exact solution, such as the homogeneous balance method (Wang, 1995; Zayed, Zedan, & Gepreel, 2004), the Jacobi-elliptic function expansion method (Chen & Wang, 2005; Liu, Fu, Liu, & Zhao, 2001), the functional variable method (Çevikel, Bekir, Akar, & San, 2012), the nonlinear transform method (Yang, Liu, & Yang, 2001), the Hirota's bilinear transformation method (Hirota, 1973; Hirota & Satsuma, 1981), the tanh-function method (Nassar, Abdel-Razek, & Seddeek, 2011), the extended tanh-method (Abdou, 2007; Fan, 2000), the complex hyperbolic function method (Chow, 1995; Wang & Zhou, 2003), the first integration method (Taghizadeh & Mirzazadeh, 2011), the Painleve expansion method (Weiss, Tabor, & Carnevale, 1982), the F-expansion method (Sirendaoreji, 2004), the Exp-function method (Akbar & Ali, 2011; Bekir & Boz, 2008; Naher, Abdullah, & Akbar, 2011, 2012), the modified Exp-function method (He, Li, & Long, 2012), the sine-cosine method (Wazwaz, 2004), the Adomian decomposition method (Adomian, 1994), the modified simple equation method (Akter & Akbar, 2015; Hossain & Akbar, 2017; Hossain, Akbar, & Wazwaz, 2017; Khan & Akbar, 2013a; Khan, Akbar, & Ali, 2013), the perturbation method (Biswas, Zony, & Zerrad, 2008), the $\exp(-\Phi(\eta))$-expansion method (Islam, Alam, Sazzad Hossain, Roshid, & Akbar, 2013; Khan & Akbar, 2013b), the variational method (Helal & Seadawy, 2009; Seadawy, 2011), the extended direct algebraic method (Seadawy, 2014, 2016), the (G'/G)-expansion method (Akbar, Ali, & Mohyud-Din, 2012; Akbar, Ali, & Zayed, 2012; Alam, Akbar, & Roshid, 2013; Naher & Abdullah, 2014a; Zayed & Shorog, 2013), the improve (G'/G)-expansion method (Naher & Abdullah, 2014b), etc. The recently developed enhanced (G'/G)-expansion method is getting popularity in use because of its straightforward calculation procedure and there is possible to obtain large number of solution.

The objective of this article is to introduce and make use of the enhanced (G'/G)-expansion method to extract fresh and further general exact traveling wave solutions to the first extended fifth order nonlinear equation and medium equal (MEW) width equation. The rest of the article is arranged as follows: In Section 2, enhanced (G'/G)-expansion method is discussed. In Section 3, the enhanced (G'/G)-expansion method is applied to examine the NLEEs indicated above. In Section 4, we give the physical explanation and graphical illustrations of obtained results. In Section 5 conclusions are provided.

2. Interpretation of the enhanced(G'/G)-expansion method

In this section, we analyze the enhanced (G'/G)-expansion method for finding traveling wave solutions to NLEEs. Consider the nonlinear equation, say in two independent variables x and t in the form:

$$P\left(u, u_{t,} u_{x,} u_{tt,} u_{xx,} u_{xt,} \dots\right) = 0, \tag{2.1}$$

where P is a polynomial of $u(x, t)$ and its partial derivatives and $u = u(x, t)$ is an unknown function of x and t, which involves the highest degree nonlinear terms and the maximum number of derivatives. The important steps concerning this method are presented in the following:

Step 1: We introduce a compound variable ξ with respect to the real variables x and t,

$$u(x, t) = u(\xi), \quad \xi = x \pm \omega\, t, \tag{2.2}$$

where ω indicates the speed of the traveling wave.

The traveling wave transformation (2.2) allows us in reducing Equation (2.1) to an ordinary differential equation (ODE) for $u = u(\xi)$ in the form:

$$Q\left(u, u', u'', u''' \ldots\right) = 0, \tag{2.3}$$

where Q is a polynomial in $u(\xi)$ and its derivatives, and the primes specify the derivative with respect to ξ.

Step 2: Assume that the solution of Equation (2.3) can be expressed in the following form:

$$u(\xi) = \sum_{i=-n}^{n}\left(\frac{a_i(G'/G)^i}{\left(1+\lambda\left(\frac{G'}{G}\right)\right)^i} + b_i(G'/G)^{i-1}\sqrt{\sigma\left(1+\frac{(G'/G)^2}{\mu}\right)}\right), \tag{2.4}$$

in which $a_i, b_i(-n \le i \le n; n \in N)$ are constants to be determined later, $\sigma = \pm 1$, $\mu \neq 0$ and $G = G(\xi)$ satisfies the equation

$$G'' + \mu G = 0. \tag{2.5}$$

Step 3: The limiting value n can be evaluated by balancing the highest order derivative terms with the nonlinear terms of the highest degree present in Equation (2.3).

Step 4: Substituting (2.4) into (2.3) together with (2.5) and then collecting all terms of same powers of $(G'/G)^i$ and $(G'/G)^j\sqrt{\sigma\left(1+\frac{(G'/G)^2}{\mu}\right)}$ and setting each coefficient to zero yields a system of algebraic equations for $a_i, b_i\left(-n \le i \le n; n \in N\right)$, λ and ω. Solving this system of equations provide the values of the unknown parameters.

Step 5: From the general solution of equation (2.5), we obtain

when $\mu < 0$,

$$\frac{G'}{G} = \sqrt{-\mu}\tanh\left(\xi_0 + \sqrt{-\mu}\,\xi\right), \tag{2.6}$$

and

$$\frac{G'}{G} = \sqrt{-\mu}\coth\left(\xi_0 + \sqrt{-\mu}\xi\right). \tag{2.7}$$

Again when $\mu > 0$,

$$\frac{G'}{G} = \sqrt{\mu}\tan\left(\xi_0 - \sqrt{\mu}\,\xi\right), \tag{2.8}$$

and

$$\frac{G'}{G} = \sqrt{\mu}\cot\left(\xi_0 + \sqrt{\mu}\,\xi\right). \tag{2.9}$$

where ξ_0 is an arbitrary constant. Finally, substituting $a_i, b_i\left(-n \le i \le n; n \in N\right)$, λ and ω and solutions (2.6)–(2.9) into (2.4), we obtain further general and some fresh traveling wave solutions of (2.1).

3. Applications of the method

In this section, the enhanced (G'/G)-expansion method has been put to use to examine the closed form solutions leading to solitary wave solutions to the first extended fifth order non-linear equation and medium equal width equation.

3.1. Example 1

In this subsection, we will use the enhanced (G'/G)-expansion method to look for the exact solution and then the solitary wave solution to the following first extended fifth order non-linear equation of the form (Wazwaz, 2014)

$$u_{ttt} - u_{txxxx} - u_{txx} - 4\left(u_x u_t\right)_{xx} - 4\left(u_x u_{xt}\right)_x = 0 \tag{3.1}$$

The traveling wave transformation $u(x, t) = u(\xi), \xi = kx - \omega t$, converts (3.1) to the ODE in the form

$$-\omega^3 u''' + \omega k^4 u^{(v)} + \omega k^2 u''' + 4\omega k^3 \left(u' u'\right)'' + 4\omega k^3 \left(u' u''\right)' = 0 \tag{3.2}$$

Integrating (3.2) with respect to ξ twice and taking integration constant to zero, we obtain

$$k^4 u''' + 6k^3 u'^2 + (k^2 - \omega^2)u' = 0 \tag{3.3}$$

Taking homogeneous balance between the highest order derivative term u''' and the highest order nonlinear term u'^2 yields $n = 1$.

Therefore, the solution Equation (2.4) becomes

$$\begin{aligned} u(\xi) = {} & a_0 + \frac{a_1\left(\frac{G'}{G}\right)}{1 + \lambda\left(\frac{G'}{G}\right)} + \frac{a_{-1}\left(1 + \lambda\left(\frac{G'}{G}\right)\right)}{\left(\frac{G'}{G}\right)} + b_0\left(\frac{G'}{G}\right)^{-1}\sqrt{\sigma\left(1 + \frac{\left(\frac{G'}{G}\right)^2}{\mu}\right)} \\ & + b_1\sqrt{\sigma\left(1 + \frac{\left(\frac{G'}{G}\right)^2}{\mu}\right)} + b_{-1}\left(\frac{G'}{G}\right)^{-2}\sqrt{\sigma\left(1 + \frac{\left(\frac{G'}{G}\right)^2}{\mu}\right)} \end{aligned} \tag{3.4}$$

where $G = G(\xi)$ satisfies Equation (2.5).

Substituting (3.4) with the Equation (2.5) into Equation (3.3), we attain a polynomial of $(G'/G)^i$ and $(G'/G)^j\sqrt{\sigma\left(1 + \frac{(G'/G)^2}{\mu}\right)}$. From this polynomial we get the coefficients of $(G'/G)^i$ and $(G'/G)^j\sqrt{\sigma\left(1 + \frac{(G'/G)^2}{\mu}\right)}$. Equating them to zero, we achieve an over-determined system that contains thirty algebraic equations (for simplicity we skip to display them). Solving this system of algebraic equation, we get

Set 1: $\omega = k\sqrt{(1 - 4\mu k^2)}, \lambda = \lambda, a_{-1} = 0, a_0 = a_0, a_1 = k\left(1 + \mu\lambda^2\right), b_{-1} = b_0 = b_1 = 0.$

Set 2: $\omega = k\sqrt{(1 - \mu k^2)}, \lambda = 0, a_{-1} = 0, a_0 = a_0, a_1 = \frac{k}{2}, b_{-1} = 0, b_0 = 0, b_1 = \frac{k\sqrt{\mu}}{2\sqrt{\sigma}}.$

Set 3: $\omega = k\sqrt{(1 - 4\mu k^2)}, \lambda = \lambda, a_{-1} = -k\mu, a_0 = a_0, a_1 = 0, b_{-1} = 0, b_0 = 0, b_1 = 0.$

Set 4: $\omega = k\sqrt{(1 - 16\mu k^2)}, \lambda = 0, a_{-1} = -k\mu, a_0 = a_0, a_1 = k, b_{-1} = 0, b_0 = 0, b_1 = 0.$

Set 5: $\omega = k\sqrt{(1 - \mu k^2)}, \lambda = \lambda, a_{-1} = -\frac{k\mu}{2}, a_0 = a_0, a_1 = 0, b_{-1} = 0, b_0 = \frac{k\mu}{2\sqrt{\sigma}}, b_1 = 0.$

Now substituting solution set 1–5 with Equations (2.6)–(2.9) into Equation (3.4), we get sufficient traveling wave solution to Equation (3.1) as follows:

When $\mu < 0$, we get the hyperbolic solution,

Type-1:

$$u_1(\xi) = a_0 + k\left(1 + \mu\lambda^2\right)\frac{\sqrt{-\mu}\tanh\left(\xi_0 + \sqrt{-\mu}\xi\right)}{\left(1 + \lambda\sqrt{-\mu}\tanh\left(\xi_0 + \sqrt{-\mu}\xi\right)\right)} \tag{3.5}$$

$$u_2(\xi) = a_0 + k\left(1 + \mu\lambda^2\right)\frac{\sqrt{-\mu}\coth\left(\xi_0 + \sqrt{-\mu}\xi\right)}{\left(1 + \lambda\sqrt{-\mu}\coth\left(\xi_0 + \sqrt{-\mu}\xi\right)\right)} \tag{3.6}$$

where $\xi = x - k\sqrt{(1 - 4\mu k^2)}t$,

Type-2:

$$u_3(\xi) = a_0 + \frac{k}{2}\sqrt{-\mu}\tanh\left(\xi_0 + \sqrt{-\mu}\xi\right) + \frac{k}{2}\sqrt{\mu\left(1 - \left(\tanh\left(\xi_0 + \sqrt{-\mu}\xi\right)\right)^2\right)} \tag{3.7}$$

$$u_4(\xi) = a_0 + \frac{k}{2}\sqrt{-\mu}\coth\left(\xi_0 + \sqrt{-\mu}\xi\right) + \frac{k}{2}\sqrt{\mu\left(1 - \left(\coth\left(\xi_0 + \sqrt{-\mu}\xi\right)\right)^2\right)} \tag{3.8}$$

where $\xi = x - k\sqrt{(1 - \mu k^2)}t$

Type-3:

$$u_5(\xi) = a_0 - k\left(\lambda\mu + \sqrt{-\mu}\coth\left(\xi_0 + \sqrt{-\mu}\xi\right)\right) \tag{3.9}$$

$$u_6(\xi) = a_0 - k\left(\lambda\mu + \sqrt{-\mu}\tanh\left(\xi_0 + \sqrt{-\mu}\xi\right)\right) \tag{3.10}$$

where $\xi = x - k\sqrt{(1 - 4\mu k^2)}t$

Type-4:

$$u_7(\xi) = a_0 \pm k\sqrt{-\mu}\left(\tanh\left(\xi_0 + \sqrt{-\mu}\xi\right) - \coth\left(\xi_0 + \sqrt{-\mu}\xi\right)\right) \tag{3.11}$$

where $\xi = x - k\sqrt{(1 - 16\mu k^2)}t$

Type-5:

$$u_8(\xi) = a_0 - \frac{k\sqrt{-\mu}}{2}\coth\left(\xi_0 + \sqrt{-\mu}\xi\right)\left(\left(1 + \lambda\sqrt{-\mu}\tanh\left(\xi_0 + \sqrt{-\mu}\xi\right)\right) - \sqrt{-\mu\left(1 - \left(\tanh\left(\xi_0 + \sqrt{-\mu}\xi\right)\right)^2\right)}\right) \tag{3.12}$$

$$u_9(\xi) = a_0 - \frac{k\sqrt{-\mu}}{2}\tanh\left(\xi_0 + \sqrt{-\mu}\xi\right)\left(\left(1 + \lambda\sqrt{-\mu}\coth\left(\xi_0 + \sqrt{-\mu}\xi\right)\right) - \sqrt{-\mu\left(1 - \left(\coth\left(\xi_0 + \sqrt{-\mu}\xi\right)\right)^2\right)}\right) \tag{3.13}$$

where $\xi = x - k\sqrt{(1 - \mu k^2)}t$

Again, for $\mu > 0$, we get the following trigonometric solution:

Type-6:

$$u_{10}(\xi) = a_0 + k(1+\mu\lambda^2)\frac{\sqrt{\mu}\tan\left(\xi_0 - \sqrt{\mu}\xi\right)}{\left(1+\lambda\sqrt{\mu}\tan\left(\xi_0 - \sqrt{\mu}\xi\right)\right)} \tag{3.14}$$

$$u_{11}(\xi) = a_0 + k(1+\mu\lambda^2)\frac{\sqrt{\mu}\cot\left(\xi_0 + \sqrt{\mu}\xi\right)}{\left(1+\lambda\sqrt{\mu}\cot\left(\xi_0 + \sqrt{\mu}\xi\right)\right)} \tag{3.15}$$

where $\xi = x - k\sqrt{(1-\mu k^2)}t$

Type-7:

$$u_{12}(\xi) = a_0 + \frac{k\sqrt{\mu}}{2}\left(\tan(\xi_0 - \sqrt{\mu}\xi) + \sqrt{\left(1+\left(\tan(\xi_0 - \sqrt{\mu}\xi)\right)^2\right)}\right) \tag{3.16}$$

$$u_{13}(\xi) = a_0 + \frac{k\sqrt{\mu}}{2}\left(\cot(\xi_0 + \sqrt{\mu}\xi) + \sqrt{\left(1+\left(\cot(\xi_0 + \sqrt{\mu}\xi)\right)^2\right)}\right) \tag{3.17}$$

where $\xi = x - k\sqrt{(1-\mu k^2)}t$

Type-8:

$$u_{14}(\xi) = a_0 - k\left(\lambda\mu + \sqrt{\mu}\cot(\xi_0 - \sqrt{\mu}\xi)\right) \tag{3.18}$$

$$u_{15}(\xi) = a_0 - k\left(\lambda\mu + \sqrt{\mu}\tan(\xi_0 + \sqrt{\mu}\xi)\right) \tag{3.19}$$

where $\xi = x - k\sqrt{(1-4\mu k^2)}t$

Type-9:

$$u_{16}(\xi) = a_0 + k\sqrt{\mu}\left(\tan\left(\xi_0 - \sqrt{\mu}\xi\right) - \cot\left(\xi_0 - \sqrt{\mu}\xi\right)\right) \tag{3.20}$$

$$u_{17}(\xi) = a_0 + k\sqrt{\mu}\left(\cot\left(\xi_0 + \sqrt{\mu}\xi\right) - \tan\left(\xi_0 + \sqrt{\mu}\xi\right)\right) \tag{3.21}$$

where $\xi = x - k\sqrt{(1-16\mu k^2)}t$

Type-10:

$$u_{18}(\xi) = a_0 - \frac{k\sqrt{\mu}}{2}\cot\left(\xi_0 - \sqrt{\mu}\xi\right)\left(\left(1+\lambda\sqrt{\mu}\tan(\xi_0 - \sqrt{\mu}\xi)\right) - \sqrt{\left(1+\left(\tan(\xi_0 - \sqrt{\mu}\xi)\right)^2\right)}\right) \tag{3.22}$$

$$u_{19}(\xi) = a_0 - \frac{k\sqrt{\mu}}{2}\tan\left(\xi_0 + \sqrt{-\mu}\xi\right)\left(\left(1+\lambda\sqrt{\mu}\cot(\xi_0 + \sqrt{\mu}\xi)\right) - \sqrt{\left(1+\left(\cot(\xi_0 + \sqrt{\mu}\xi)\right)^2\right)}\right) \tag{3.23}$$

where $\xi = x - k\sqrt{(1-\mu k^2)}t$

3.2. Example 2

In this subsection, we will use the enhanced (G'/G)-expansion method to look for the exact solution and then the solitary wave solution to the following medium equal width (MEW) equation of the form

$$u_t + 3u^2u_x - du_{xxt} = 0 \tag{3.24}$$

Which is related to the regularized long wave equation, has solitary waves with the same width of both positive and negative amplitudes. This is a nonlinear wave equation with cubic nonlinearity with pulselike solitary wave solution. This equation appears in many physical applications and is used as a model for nonlinear dispersive waves. The equation gives rise to equal width undular bore.

The traveling wave transformation $u(x,t)=u(\xi), \xi = x-\omega t$, converts (3.24) to the ODE in the form

$$d\omega u''' + 3u^2u' - \omega u' = 0. \tag{3.25}$$

Integrating (3.2) with respect to ξ, we obtain

$$d\omega u'' + u^3 - \omega u + C = 0 \tag{3.26}$$

where C is an integration constant.

Taking homogeneous balance between the highest order derivative term u'' and the highest order nonlinear term u^3 yields $n = 1$.

Therefore, the solution of Equation (3.26) becomes,

$$u(\xi) = a_0 + \frac{a_1\left(\frac{G'}{G}\right)}{1+\lambda\left(\frac{G'}{G}\right)} + \frac{a_{-1}\left(1+\lambda\left(\frac{G'}{G}\right)\right)}{\left(\frac{G'}{G}\right)} + b_0\left(\frac{G'}{G}\right)^{-1}\sqrt{\sigma\left(1+\frac{\left(\frac{G'}{G}\right)^2}{\mu}\right)} + b_1\sqrt{\sigma\left(1+\frac{\left(\frac{G'}{G}\right)^2}{\mu}\right)} + b_{-1}\left(\frac{G'}{G}\right)^{-2}\sqrt{\sigma\left(1+\frac{\left(\frac{G'}{G}\right)^2}{\mu}\right)} \tag{3.27}$$

where $G = G(\xi)$ satisfies Equation (2.5).

Substituting (3.27) with the Equation (2.5) into Equation (3.26), we attain a polynomial of $(G'/G)^i$ and $(G'/G)^j\sqrt{\sigma\left(1+\frac{(G'/G)^2}{\mu}\right)}$. Equating the coefficient of these to zero, we achieve a system of algebraic equation which on solving, we get

$\omega = -\frac{a_{-1}^2}{2d\mu^2}, \lambda = \frac{\sqrt{-6d(4d\mu+1)}}{6d\mu}, a_{-1} = a_{-1}, a_0 = \frac{a_{-1}\sqrt{-6d(4d\mu+1)}}{6d\mu}, a_1 = \frac{a_{-1}(2d\mu-1)}{6d\mu^2}, b_{-1} = 0, b_0 = b_1 = 0$ and $C = \frac{a_{-1}^3(2d\mu-1)\sqrt{-6d(4d\mu+1)}}{18d^2\mu^3}$

Now substituting these values and Equations (2.6)–(2.9) into Equation (3.27), we deduce traveling wave solutions of Equation (3.24) as follows:

For another set $\omega = \omega, \lambda = \lambda, a_{-1} = 0, a_0 = a_0, a_1 = 0, b_{-1} = 0, b_0 = 0, b_1 = 0$ and $C = -a_0(a_0^2 - \omega)$ Equation (3.27) gives trivial solutions. So this case is rejected.

When $\mu < 0$, we get the hyperbolic solution,

Type-1:

$$u_{2_1}(\xi) = \frac{a_{-1}}{\sqrt{-\mu}}\left(\frac{(2d\mu-1)\tanh\left(\xi_0+\sqrt{-\mu}\xi\right)}{6d\mu \pm \sqrt{6d\mu(4d\mu+1)}\tanh\left(\xi_0+\sqrt{-\mu}\xi\right)} + \coth\left(\xi_0+\sqrt{-\mu}\xi\right)\right) \tag{3.28}$$

$$u_{2_2}(\xi)=\frac{a_{-1}}{\sqrt{-\mu}}\left(\frac{(2d\mu-1)+\coth\left(\xi_0+\sqrt{-\mu}\xi\right)}{6d\mu\pm\sqrt{6d\mu(4d\mu+1)}\coth\left(\xi_0+\sqrt{-\mu}\xi\right)}+\tanh\left(\xi_0+\sqrt{-\mu}\xi\right)\right) \tag{3.29}$$

where $\xi=x-\frac{a_{-1}^2}{2d\mu^2}t$,

Again, for $\mu > 0$, we get the following trigonometric solution:

Type-2:

$$u_{2_3}(\xi)=\frac{a_{-1}}{\sqrt{\mu}}\left(\frac{(2d\mu-1)\tan\left(\xi_0-\sqrt{\mu}\xi\right)}{6d\mu\pm\sqrt{-6d\mu(4d\mu+1)}\tan\left(\xi_0-\sqrt{\mu}\xi\right)}+\cot\left(\xi_0-\sqrt{-\mu}\xi\right)\right) \tag{3.30}$$

$$u_{2_4}(\xi)=\frac{a_{-1}}{\sqrt{\mu}}\left(\frac{(2d\mu-1)\cot\left(\xi_0+\sqrt{\mu}\xi\right)}{6d\mu\pm\sqrt{-6d\mu(4d\mu+1)}\cot\left(\xi_0+\sqrt{\mu}\,\xi\right)}+\tan\left(\xi_0+\sqrt{-\mu}\xi\right)\right) \tag{3.31}$$

where $\xi=x-\frac{a_{-1}^2}{2d\mu^2}t$,

4. Physical explanation and graphical illustrations

In this section, we have discussed about the obtained solution of first extended fifth order non-linear equation and medium equal width (MEW) equation. From the above solution, it has been detected that $\sigma=\pm 1$ and $\mu \neq 0$. For negative values of μ, the hyperbolic solutions $u_1(\xi)-u_9(\xi)$ of the new fifth order non-linear equation are obtained through type 1 to 5 and when $\mu > 0$, trigonometric solutions $u_{10}(\xi)-u_{19}(\xi)$ through type 6 to 10 are obtained. The solutions $u_2(\xi), u_6(\xi)$, and $u_8(\xi)$ demonstrate the nature of kink wave. Solutions $u_1(\xi), u_5(\xi)$ and $u_7(\xi)$ demonstrate the nature of singular kink wave. Moreover, solutions $u_{10}(\xi)-u_{19}(\xi)$ demonstrate the nature of periodic traveling wave. The solution $u_4(\xi)$ express the nature of soliton solution where $u_3(\xi)$ and $u_9(\xi)$ represent the singular solution. The graphical illustrations of some obtained solutions are given below in the figures. Figure 1 represents the kink shape solution of $u_2(\xi)$ in (3.6) for $\mu=-2, \sigma=1, \xi_0=4, a_0=2, \lambda=1$ and $k=1$ within the interval $-10 \le x \le 10$ and $-10 \le t \le 10$. Soliton solution $u_4(\xi)$ in (3.9) for $\mu=-1, \sigma=-1, \xi_0=2, a_0=3, \lambda=0, k=1$ and Singular kink wave solution $u_5(\xi)$ in (3.8) for $\mu=-2, \sigma=1, \xi_0=1, a_0=3, \lambda=1, k=1$ within the interval $-10 \le x \le 10$ and $-10 \le t \le 10$ has been shown in Figures 2 and 3, respectively. Figure 4 represents the Periodic solution $u_{19}(\xi)$ in (3.23) for $\mu=1/8, \sigma=1, \xi_0=2, a_0=2, \lambda=1, k=1$ within the interval $-10 \le x \le 10$ and $-10 \le t \le 10$. From the solutions of the medium equal width (MEW) equation, it is observed that the negative values of μ offer the hyperbolic solutions $u_{2_1}(\xi)$- $u_{2_2}(\xi)$ and the positive values of μ, recommend the trigonometric solutions $u_{2_3}(\xi)-u_{2_4}(\xi)$. The solution (3.28) is represented in Figure 5 which shows the shape of singular kink type traveling wave solution with $\mu=-2, \sigma=1, \xi_0=2, a_{-1}=2, d=1$ within the interval $-10 \le x \le 10$ and $-5 \le t \le 5$. The solution in (3.29) also represents singular kink type traveling wave solution which is similar to Figure 5. The Periodic traveling wave solution in (3.30) is represented by Figure 6 for $\mu=\frac{1}{2}, \xi_0=3, a_{-1}=2, d=\frac{1}{4}$ within the interval $-10 \le x \le 10$ and $-5 \le t \le 5$. The solution in (3.31) represents Periodic traveling wave solution which is also similar to Figure 6. So for simplicity we ignored these figures.

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Figure 1. Kink solution $u_2(\xi)$ in (3.6) for $\mu = -2, \sigma = 1, \xi_0 = 4, a_0 = 2$, $\lambda = 1$ and $k = 1$.

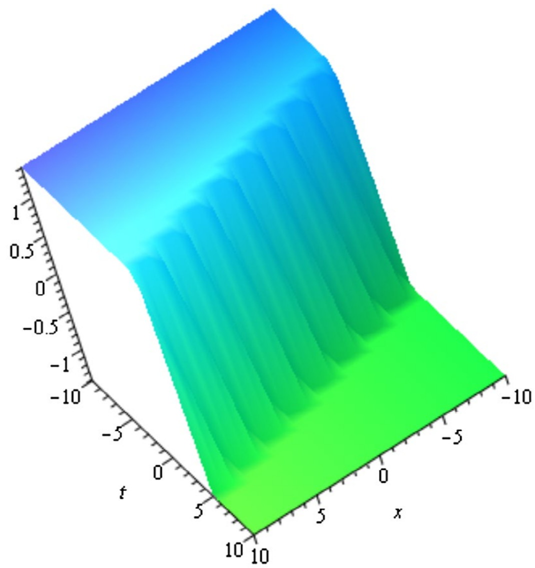


Figure 2. Soliton solution $u_4(\xi)$ in (3.9) for $\mu = -1, \sigma = -1, \xi_0 = 2, a_0 = 3$, $\lambda = 0$ and $k = 1$.

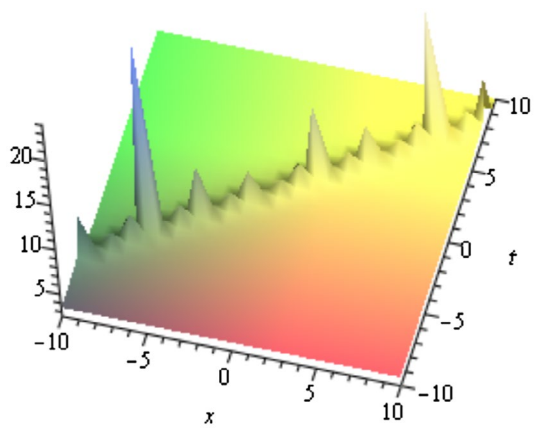


Figure 3. Singular Kink solution $u_5(\xi)$ in (3.8) for $\mu = -2, \sigma = 1, \xi_0 = 3, a_0 = 1$, $\lambda = 1$ and $k = 1$

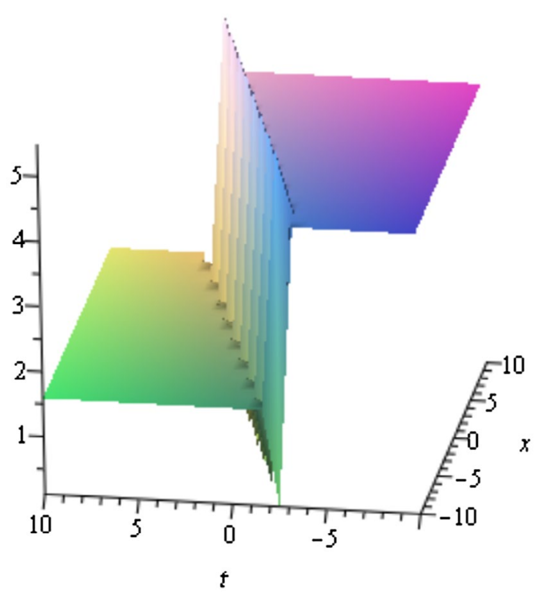


Figure 4. Periodic solution $u_{19}(\xi)$ in (3.23) for $\mu = 1/8, \sigma = 1, \xi_0 = 2, a_0 = 2$, $\lambda = 1$ and $k = 1$.

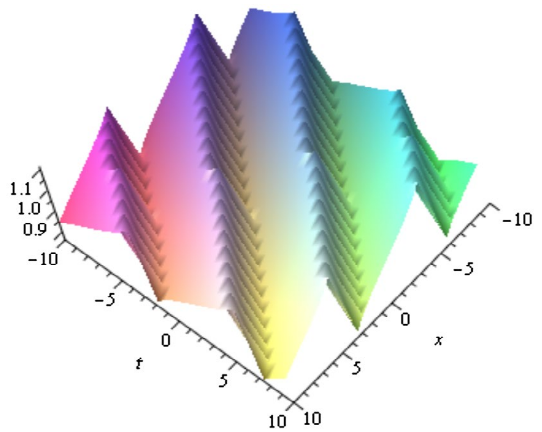


Figure 5. Singular Kink solution $u_{2_1}(\xi)$ in (3.28) for $\mu = 2, d = 2, a_{-1} = 2$ and $\xi_0 = 2$.

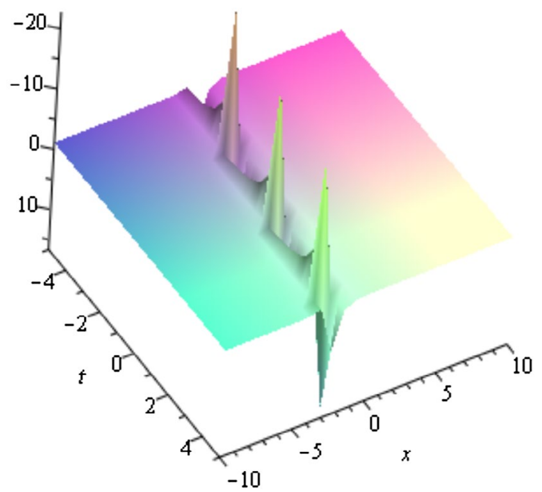
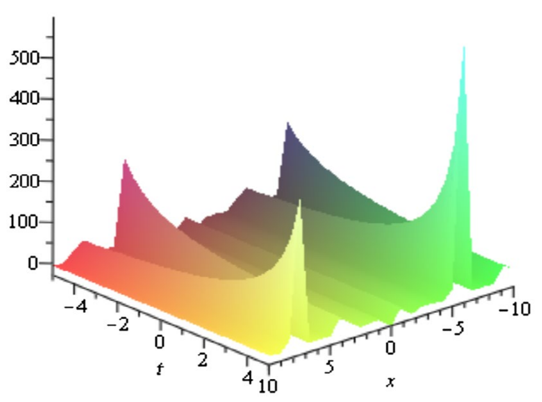


Figure 6. Periodic solution $u_{2_3}(\xi)$ in (3.30) for $\mu = \frac{1}{2}, d = \frac{1}{4}, a_{-1} = 2$ and $\xi_0 = 3$.



5. Conclusion

In this article, enhanced (G'/G)-expansion method has been successfully used to find the exact traveling wave solutions of first extended fifth order non-linear equation and medium equal width equation. The solutions are verified to check the correctness of the solutions by putting them back into the original equation and found correct. The key advantage of the enhanced (G'/G)-expansion method against other methods is that the method provides more general and huge amount of new exact traveling wave solutions with several free parameters in a uniform way. The exact solutions have its great importance to rendering the inner mechanism of the physical problems Therefore this method is very easy and straightforward to handling. Also it is quite capable and can be applied for finding exact solutions of other NLEEs in mathematical physics.

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