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On Hadamard-type inequalities for differentiable functions via Caputo k -fractional derivatives

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Abstract: In this paper, we prove a version of the Hadamard inequality for function f such that $f^{(n)}$ is convex via k -fractional Caputo derivatives. Using convexity of $|f^{(n)}|^q$, $q \geq 1$ we find the bounds of the difference of fractional differential inequality. Also we have found inequalities for Caputo fractional derivatives.

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1. Introduction

Fractional Calculus is the theory of integration and differentiation of arbitrary order. It neither means calculus of fraction nor fraction of any integration, differentiation or fraction of variation (see for details, Dalir & Bashour, 2010; Miller & Ross, 1993; Podlubni, 1998; Shantanu, 2011).

In recent years, notable contributions have been made on fractional calculus in numerical analysis and different areas of physics and engineering including fractal phenomena (see for details, Gorenó, 1997; Mainardi, 1997). Fourier, Euler and Laplace are among those mathematicians who worked a lot on fractional calculus and mathematical consequences. They gave its definition in their own style. Well-known definitions are the Grunwald–Letnikov and Riemann–Liouville definitions (see Loverro, 2004).

We give some preliminaries that we will use for our results. For this we will define convex functions, Hadamard inequality, Caputo fractional derivatives and finally Caputo k -fractional derivatives.

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PUBLIC INTEREST STATEMENT

Differentiation and integration are greatest discoveries of mathematical sciences. It is very difficult to find the branch of science which is not influenced by these notions. Fractional calculus is a collection of relatively little-known mathematical results concerning generalizations of differentiation and integration to noninteger orders. Nevertheless, the application of fractional calculus just emerged in the last three decades due to the progress in the area of chaos that revealed subtle relationships with the concepts of fractional calculus. There are now a growing number of research areas in mathematics and physics which employ fractional calculus.

Definition 1 A function $f:[a, b] \rightarrow \mathbb{R}$ is said to be convex if it satisfies the following inequality for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$.

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

If $-f$ is convex, then f is called concave function and vice versa.

THEOREM 1 (Farissi, 2010) If $f:[a, b] \rightarrow \mathbb{R}$ be a convex function, then the following inequality holds

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2},$$

where $a, b \in [a, b]$ and $a < b$.

This is classical Hadamard inequality published in Hadamard (1893). The history of this inequality can be found in Mitrovic' and Lackovicé (1985).

Hadamard inequality is in focus of researchers since it is found. In Sarikaya, Set, Yaldiz and Basak (2013) proved the Hadamard and the Hadamard-type inequalities for Riemann–Liouville fractional integrals which are given below:

THEOREM 2 Let $f:[a, b] \rightarrow \mathbb{R}$ be a positive function which is convex and integrable over $[a, b]$, with $a < b$. Then the following inequality for fractional integrals holds

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [I_{a^+}^\alpha f(b) + I_{b^-}^\alpha f(a)] \\ &\leq \frac{f(a)+f(b)}{2}, \end{aligned} \tag{1.1}$$

where $\alpha > 0$.

THEOREM 3 Let $f:[a, b] \rightarrow \mathbb{R}$ be a differentiable function in the interior of $[a, b]$, with $a < b$. Also if $|f'|$ is convex on $[a, b]$ then the following inequality for fractional integrals holds

$$\begin{aligned} &\left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [I_{a^+}^\alpha f(b) + I_{b^-}^\alpha f(a)] \right| \\ &\leq \frac{b-a}{2(\alpha+1)} \left(1 - \frac{1}{2^\alpha}\right) [|f'(a)| + |f'(b)|], \end{aligned} \tag{1.2}$$

where $\alpha > 0$.

In Farid (2016) gave the Hadamard inequalities for generalized fractional integrals which are as follows:

THEOREM 4 Let $f:[a, b] \rightarrow \mathbb{R}$ be a convex function and $g:[a, b] \rightarrow \mathbb{R}$ is nonnegative, integrable and symmetric to $\frac{a+b}{2}$. Then following inequalities for generalized fractional integral hold

$$\begin{aligned} &f\left(\frac{a+b}{2}\right) \left[(e_{\alpha, \beta, l, \omega, a^+}^{\gamma, \delta, k} g)(b) + (e_{\alpha, \beta, l, \omega, b^-}^{\gamma, \delta, k} g)(a) \right] \\ &\leq [(e_{\alpha, \beta, l, \omega, a^+}^{\gamma, \delta, k} fg)(b) + (e_{\alpha, \beta, l, \omega, b^-}^{\gamma, \delta, k} fg)(a)] \\ &\leq \frac{f(a)+f(b)}{2} \left[(e_{\alpha, \beta, l, \omega, a^+}^{\gamma, \delta, k} g)(b) + (e_{\alpha, \beta, l, \omega, b^-}^{\gamma, \delta, k} g)(a) \right]. \end{aligned} \tag{1.3}$$

THEOREM 5 Let $f:[a, b] \rightarrow \mathbb{R}$ be a differentiable mapping in the interior of $[a, b]$ and $f' \in L[a, b]$ with $a < b$. If $|f'|$ is convex on $[a, b]$ and $g:[a, b] \rightarrow \mathbb{R}$ is continuous and symmetric to $\frac{a+b}{2}$, then following inequality for k -fractional integrals holds

$$\left| \left(\frac{f(a) + f(b)}{2} \right) [I_{a+}^{\alpha,k} g(b) + I_{b-}^{\alpha,k} g(a)] - [I_{a+}^{\alpha,k} (fg)(b) + I_{b-}^{\alpha,k} (fg)(a)] \right| \tag{1.4}$$

$$\leq \frac{(b-a)^{\frac{\alpha}{k}+1} \|g\|_{\infty}}{\left(\frac{\alpha}{k} + 1\right) \Gamma_k(\alpha + k)} \left(1 - \frac{1}{2^{\frac{\alpha}{k}}}\right) [|f'(a)| + |f'(b)|]$$

where $\alpha, k > 0$.

A lot of results related to fractional Hadamard inequality have been found in recent years (see, Farid, Javed, & Rehman, [in press](#); Farid, Naqvi, & Rehman, [2017](#)).

In Farid et al. ([2017](#)) give the version of Hadamard inequality for Caputo fractional derivatives. We are motivated to find fractional integral inequalities via Caputo k -fractional derivatives, a slight generalization of Caputo fractional derivatives.

Definition 2 Let $\alpha > 0$ and $\alpha \notin \{1, 2, 3, \dots\}$, $n = [\alpha] + 1$, $f \in AC^n[a, b]$, the space of functions having n th derivatives absolutely continuous. The right-sided and left-sided Caputo fractional derivatives of order α are defined as follows:

$$({}^C D_{a+}^{\alpha} f)(x) = \frac{1}{\Gamma(n - \alpha)} \int_a^x \frac{f^{(n)}(t)}{(x - t)^{\alpha - n + 1}} dt, \quad x > a \tag{1.5}$$

and

$$({}^C D_{b-}^{\alpha} f)(x) = \frac{(-1)^n}{\Gamma(n - \alpha)} \int_x^b \frac{f^{(n)}(t)}{(t - x)^{\alpha - n + 1}} dt, \quad x < b \tag{1.6}$$

If $\alpha = n \in \{1, 2, 3, \dots\}$ and usual derivative $f^{(n)}(x)$ of order n exists, then Caputo fractional derivative $({}^C D_{a+}^{\alpha} f)(x)$ coincides with $f^{(n)}(x)$, whereas $({}^C D_{b-}^{\alpha} f)(x)$ coincides with $f^{(n)}(x)$ with exactness to a constant multiplier $(-1)^n$. In particular we have

$$({}^C D_{a+}^0 f)(x) = ({}^C D_{b-}^0 f)(x) = f(x) \tag{1.7}$$

where $n = 1$ and $\alpha = 0$.

For further details see Kilbas, Srivastava and Trujillo ([2006](#)).

Definition 3 Let $\alpha > 0$, $k \geq 1$ and $\alpha \notin \{1, 2, 3, \dots\}$, $n = [\alpha] + 1$, $f \in AC^n[a, b]$. The right-sided and left-sided Caputo k -fractional derivatives of order α are defined as follows:

$$({}^C D_{a+}^{\alpha,k} f)(x) = \frac{1}{k \Gamma_k(n - \frac{\alpha}{k})} \int_a^x \frac{f^{(n)}(t)}{(x - t)^{\frac{\alpha}{k} - n + 1}} dt, \quad x > a \tag{1.8}$$

and

$$({}^C D_{b-}^{\alpha,k} f)(x) = \frac{(-1)^n}{k \Gamma_k(n - \frac{\alpha}{k})} \int_x^b \frac{f^{(n)}(t)}{(t - x)^{\frac{\alpha}{k} - n + 1}} dt, \quad x < b \tag{1.9}$$

where $\Gamma_k(\alpha)$ is the k -Gamma function defined as:

$$\Gamma_k(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-\frac{t}{k}} dt,$$

also

$$\Gamma_k(\alpha + k) = \alpha \Gamma_k(\alpha)$$

If $\alpha = n \in \{1, 2, 3, \dots\}$ and usual derivative $f^{(n)}(x)$ of order n exists, then Caputo k -fractional derivative $({}^C D_{a+}^{n,1} f)(x)$ coincides with $f^{(n)}(x)$, whereas $({}^C D_{b-}^{n,1} f)(x)$ coincides with $f^{(n)}(x)$ with exactness to a constant multiplier $(-1)^n$.

In particular we have

$$({}^C D_{a+}^{0,1} f)(x) = ({}^C D_{b-}^{0,1} f)(x) = f(x) \tag{1.10}$$

where $n, k = 1$ and $\alpha = 0$. For $k = 1$, Caputo k -fractional derivatives give the definition of Caputo fractional derivatives.

In this paper, in Section 2, we will give a version of the Hadamard inequality for function whose n th derivative is convex via Caputo k -fractional derivatives. In Section 3, we obtain some bounds of a difference of this inequality for function f for which $|f^{(n)}|^q, q \geq 1$ is convex. Also we deduce results for Caputo fractional derivatives.

In the whole paper, $C^n[a, b]$ denotes the space of n -times differentiable functions such that $f^{(n)}$ are continuous on $[a, b]$.

2. Hadamard-type inequalities for Caputo k -fractional derivatives

Here we give Caputo k -fractional Hadamard inequality.

THEOREM 6 Let $f: [a, b] \rightarrow \mathbb{R}$ be a positive function such that $f \in C^n[a, b], a < b$. If $f^{(n)}$ is a convex function on $[a, b]$, then the following inequalities for Caputo k -fractional derivatives hold

$$\begin{aligned} & f^{(n)}\left(\frac{a+b}{2}\right) \\ & \leq \frac{2^{n-\frac{\alpha}{k}-1} k \Gamma_k(n - \frac{\alpha}{k} + k)}{(b-a)^{n-\frac{\alpha}{k}}} \left[({}^C D_{(\frac{a+b}{2})+}^{\alpha,k} f)(b) + (-1)^n ({}^C D_{(\frac{a+b}{2})-}^{\alpha,k} f)(a) \right] \\ & \leq \frac{f^{(n)}(a) + f^{(n)}(b)}{2}. \end{aligned} \tag{2.1}$$

Proof. Since $f^{(n)}$ is convex, we have

$$f^{(n)}\left(\frac{x+y}{2}\right) \leq \frac{f^{(n)}(x) + f^{(n)}(y)}{2}. \tag{2.2}$$

Putting $x = \frac{t}{2}a + \frac{(2-t)}{2}b, y = \frac{(2-t)}{2}a + \frac{t}{2}b$ for $t \in [0, 1]$, we have $x, y \in [a, b]$, The above inequality becomes

$$\begin{aligned} & 2f^{(n)}\left(\frac{a+b}{2}\right) \\ & \leq f^{(n)}\left(\frac{t}{2}a + \frac{2-t}{2}b\right) + f^{(n)}\left(\frac{2-t}{2}a + \frac{t}{2}b\right). \end{aligned} \tag{2.3}$$

Multiplying both sides of above inequality with $t^{n-\frac{\alpha}{k}-1}$ and integrating over $[0, 1]$, we have

$$\begin{aligned}
 & 2f^{(n)}\left(\frac{a+b}{2}\right) \int_0^1 t^{n-\frac{\alpha}{k}-1} dt \\
 & \leq \int_0^1 \frac{f^{(n)}\left(\frac{t}{2}a + \frac{2-t}{2}b\right)}{t^{\frac{\alpha}{k}-n+1}} dt + \int_0^1 \frac{f^{(n)}\left(\frac{2-t}{2}a + \frac{t}{2}b\right)}{t^{\frac{\alpha}{k}-n+1}} dt \\
 & = \frac{2^{n-\frac{\alpha}{k}} k \Gamma_k\left(n - \frac{\alpha}{k}\right)}{(b-a)^{n-\frac{\alpha}{k}}} \left[\left({}^c D_{\left(\frac{a+b}{2}\right)^+}^{\alpha,k} f\right)(b) + (-1)^n \left({}^c D_{\left(\frac{a+b}{2}\right)^-}^{\alpha,k} f\right)(a) \right],
 \end{aligned}$$

which implies that

$$\begin{aligned}
 & f^{(n)}\left(\frac{a+b}{2}\right) \\
 & \leq \frac{2^{n-\frac{\alpha}{k}-1} k \Gamma_k\left(n - \frac{\alpha}{k} + k\right)}{(b-a)^{n-\frac{\alpha}{k}}} \left[\left({}^c D_{\left(\frac{a+b}{2}\right)^+}^{\alpha,k} f\right)(b) + (-1)^n \left({}^c D_{\left(\frac{a+b}{2}\right)^-}^{\alpha,k} f\right)(a) \right].
 \end{aligned} \tag{2.4}$$

Similarly, convexity of $f^{(n)}$ gives

$$\begin{aligned}
 & f^{(n)}\left(\frac{t}{2}a + \frac{2-t}{2}b\right) + f^{(n)}\left(\frac{2-t}{2}a + \frac{t}{2}b\right) \\
 & \leq \frac{t}{2}f^{(n)}(a) + \frac{2-t}{2}f^{(n)}(b) + \frac{2-t}{2}f^{(n)}(a) + \frac{t}{2}f^{(n)}(b).
 \end{aligned}$$

Multiplying both sides of above inequality with $t^{n-\frac{\alpha}{k}-1}$ and integrating over $[0, 1]$, we have

$$\begin{aligned}
 & \int_0^1 \frac{f^{(n)}\left(\frac{t}{2}a + \frac{2-t}{2}b\right)}{t^{\frac{\alpha}{k}-n+1}} dt + \int_0^1 \frac{f^{(n)}\left(\frac{2-t}{2}a + \frac{t}{2}b\right)}{t^{\frac{\alpha}{k}-n+1}} dt \\
 & \leq [f^{(n)}(a) + f^{(n)}(b)] \int_0^1 t^{n-\frac{\alpha}{k}-1} dt,
 \end{aligned}$$

from which we have

$$\begin{aligned}
 & \frac{2^{n-\frac{\alpha}{k}-1} k \Gamma_k\left(n - \frac{\alpha}{k} + k\right)}{(b-a)^{n-\frac{\alpha}{k}}} \left[\left({}^c D_{\left(\frac{a+b}{2}\right)^+}^{\alpha,k} f\right)(b) + (-1)^n \left({}^c D_{\left(\frac{a+b}{2}\right)^-}^{\alpha,k} f\right)(a) \right] \\
 & \leq \frac{f^{(n)}(a) + f^{(n)}(b)}{2}.
 \end{aligned} \tag{2.5}$$

Combining inequality (2.4) and inequality (2.5), we get inequality (2.1). □

COROLLARY 1 If we take $k = 1$ in the above theorem, then we get the following inequality for Caputo fractional derivatives (Farid et al., 2017)

$$\begin{aligned}
 & f^{(n)}\left(\frac{a+b}{2}\right) \\
 & \leq \frac{2^{n-\alpha-1} \Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}} \left[\left({}^c D_{\left(\frac{a+b}{2}\right)^+}^{\alpha} f\right)(b) + (-1)^n \left({}^c D_{\left(\frac{a+b}{2}\right)^-}^{\alpha} f\right)(a) \right] \\
 & \leq \frac{f^{(n)}(a) + f^{(n)}(b)}{2}.
 \end{aligned}$$

3. Caputo k -fractional inequalities related to Hadamard inequality

For next results, we need the following lemma.

LEMMA 1 Let $f:[a, b] \rightarrow \mathbb{R}$ be a function such that $f \in C^n[a, b]$, $a < b$. Then the following equality for Caputo k -fractional derivatives holds

$$\begin{aligned} & \frac{2^{n-\frac{\alpha}{k}-1} k \Gamma_k \left(n - \frac{\alpha}{k} + k \right)}{(b-a)^{n-\frac{\alpha}{k}}} \left[\left({}^c D_{\left(\frac{a+b}{2}\right)^+}^{\alpha,k} f \right) (b) + (-1)^n \left({}^c D_{\left(\frac{a+b}{2}\right)^-}^{\alpha,k} f \right) (a) \right] \\ & - f^{(n)} \left(\frac{a+b}{2} \right) \\ & = \frac{b-a}{4} \left[\int_0^1 t^{n-\frac{\alpha}{k}} f^{(n+1)} \left(\frac{t}{2} a + \frac{2-t}{2} b \right) dt \right. \\ & \left. - \int_0^1 t^{n-\frac{\alpha}{k}} f^{(n+1)} \left(\frac{2-t}{2} a + \frac{t}{2} b \right) dt \right]. \end{aligned} \tag{3.1}$$

Proof. We note that

$$\begin{aligned} & \frac{b-a}{4} \left[\int_0^1 t^{n-\frac{\alpha}{k}} f^{(n+1)} \left(\frac{t}{2} a + \frac{2-t}{2} b \right) dt \right] \\ & = \frac{b-a}{4} \left[t^{n-\frac{\alpha}{k}} \frac{2}{a-b} f^{(n)} \left(\frac{t}{2} a + \frac{2-t}{2} b \right) \Big|_0^1 \right. \\ & \left. - \int_0^1 \left(n - \frac{\alpha}{k} \right) t^{n-\frac{\alpha}{k}-1} \frac{2}{a-b} f^{(n)} \left(\frac{t}{2} a + \frac{2-t}{2} b \right) dt \right] \\ & = \frac{b-a}{4} \left[-\frac{2}{b-a} f^{(n)} \left(\frac{a+b}{2} \right) \right. \\ & \left. - \frac{2 \left(n - \frac{\alpha}{k} \right)}{(b-a)} \int_b^{\frac{a+b}{2}} \left(\frac{b-a}{2(b-x)} \right)^{\frac{\alpha}{k}-n+1} \frac{2}{b-a} f^{(n)}(x) dx \right] \\ & = \frac{b-a}{4} \left[-\frac{2}{b-a} f^{(n)} \left(\frac{a+b}{2} \right) \right. \\ & \left. + \frac{2^{n-\frac{\alpha}{k}+1} k \Gamma_k \left(n - \frac{\alpha}{k} + k \right)}{(b-a)^{n-\frac{\alpha}{k}+1}} \left({}^c D_{\left(\frac{a+b}{2}\right)^+}^{\alpha,k} f \right) (b) \right]. \end{aligned} \tag{3.2}$$

Similarly,

$$\begin{aligned} & -\frac{b-a}{4} \left[\int_0^1 t^{n-\frac{\alpha}{k}} f^{(n+1)} \left(\frac{2-t}{2} a + \frac{t}{2} b \right) dt \right] \\ & = -\frac{b-a}{4} \left[\frac{2}{b-a} f^{(n)} \left(\frac{a+b}{2} \right) \right. \\ & \left. - \frac{2^{n-\frac{\alpha}{k}+1} k \Gamma_k \left(n - \frac{\alpha}{k} + k \right)}{(b-a)^{n-\frac{\alpha}{k}+1}} (-1)^n \left({}^c D_{\left(\frac{a+b}{2}\right)^-}^{\alpha,k} f \right) (a) \right]. \end{aligned} \tag{3.3}$$

Subtracting (3.3) from (3.2), we get (3.1). □

Using the above lemma, we give the following Hadamard-type inequality via Caputo k -fractional derivatives.

THEOREM 7 Let $f: [a, b] \rightarrow \mathbb{R}$ be a function such that $f \in C^n[a, b]$, $a < b$. If $|f^{(n+1)}|^q$ is convex on $[a, b]$ for $q \geq 1$, then the following inequality for Caputo k -fractional derivatives holds

$$\begin{aligned}
 & \left| \frac{2^{n-\frac{\alpha}{k}-1} k \Gamma_k \left(n - \frac{\alpha}{k} + k \right)}{(b-a)^{n-\frac{\alpha}{k}}} \left[\left({}^C D_{\left(\frac{a+b}{2}\right)_+}^{\alpha,k} f \right) (b) + (-1)^n \left({}^C D_{\left(\frac{a+b}{2}\right)_-}^{\alpha,k} f \right) (a) \right] \right. \\
 & \quad \left. - f^{(n)} \left(\frac{a+b}{2} \right) \right| \\
 & \leq \frac{b-a}{4 \left(n - \frac{\alpha}{k} + 1 \right)} \left(\frac{1}{2 \left(n - \frac{\alpha}{k} + 2 \right)} \right)^{\frac{1}{q}} \left[\left[\left(n - \frac{\alpha}{k} + 1 \right) |f^{(n+1)}(a)|^q \right. \right. \\
 & \quad \left. \left. + \left(n - \frac{\alpha}{k} + 3 \right) |f^{(n+1)}(b)|^q \right]^{\frac{1}{q}} + \left[\left(n - \frac{\alpha}{k} + 3 \right) |f^{(n+1)}(a)|^q \right. \right. \\
 & \quad \left. \left. + \left(n - \frac{\alpha}{k} + 1 \right) |f^{(n+1)}(b)|^q \right]^{\frac{1}{q}} \right]. \tag{3.4}
 \end{aligned}$$

Proof. Using Lemma 1 and convexity of $|f^{(n+1)}|$ and for $q = 1$, we have

$$\begin{aligned}
 & \left| \frac{2^{n-\frac{\alpha}{k}-1} k \Gamma_k \left(n - \frac{\alpha}{k} + k \right)}{(b-a)^{n-\frac{\alpha}{k}}} \left[\left({}^C D_{\left(\frac{a+b}{2}\right)_+}^{\alpha,k} f \right) (b) + (-1)^n \left({}^C D_{\left(\frac{a+b}{2}\right)_-}^{\alpha,k} f \right) (a) \right] \right. \\
 & \quad \left. - f^{(n)} \left(\frac{a+b}{2} \right) \right| \\
 & \leq \frac{b-a}{4} \int_0^1 t^{n-\frac{\alpha}{k}} \left| f^{(n+1)} \left(\frac{t}{2} a + \frac{2-t}{2} b \right) \right| dt \\
 & \quad + \left| f^{(n+1)} \left(\frac{2-t}{2} a + \frac{t}{2} b \right) \right| dt. \\
 & = \frac{b-a}{4 \left(n - \frac{\alpha}{k} + 1 \right)} [|f^{(n+1)}(a)| + |f^{(n+1)}(b)|].
 \end{aligned}$$

For $q > 1$, we proceed as follows. Using Lemma 1 we have

$$\begin{aligned}
 & \left| \frac{2^{n-\frac{\alpha}{k}-1} k \Gamma_k \left(n - \frac{\alpha}{k} + k \right)}{(b-a)^{n-\frac{\alpha}{k}}} \left[\left({}^C D_{\left(\frac{a+b}{2}\right)_+}^{\alpha,k} f \right) (b) + (-1)^n \left({}^C D_{\left(\frac{a+b}{2}\right)_-}^{\alpha,k} f \right) (a) \right] \right. \\
 & \quad \left. - f^{(n)} \left(\frac{a+b}{2} \right) \right| \\
 & \leq \frac{b-a}{4} \left[\int_0^1 t^{n-\frac{\alpha}{k}} \left| f^{(n+1)} \left(\frac{t}{2} a + \frac{2-t}{2} b \right) \right| dt \right. \\
 & \quad \left. + \int_0^1 t^{n-\frac{\alpha}{k}} \left| f^{(n+1)} \left(\frac{2-t}{2} a + \frac{t}{2} b \right) \right| dt \right].
 \end{aligned}$$

Using power mean inequality, we get

$$\begin{aligned} & \left| \frac{2^{n-\frac{\alpha}{k}-1} k \Gamma_k \left(n - \frac{\alpha}{k} + k \right)}{(b-a)^{n-\frac{\alpha}{k}}} \left[\left({}^c D_{\left(\frac{a+b}{2}\right)^+}^{\alpha, k} f \right) (b) + (-1)^n \left({}^c D_{\left(\frac{a+b}{2}\right)^-}^{\alpha, k} f \right) (a) \right] \right. \\ & \quad \left. - f^{(n)} \left(\frac{a+b}{2} \right) \right| \\ & \leq \frac{b-a}{4} \left(\frac{1}{n - \frac{\alpha}{k} + 1} \right)^{\frac{1}{p}} \left[\left[\int_0^1 t^{n-\frac{\alpha}{k}} \left| f^{(n+1)} \left(\frac{t}{2} a + \frac{2-t}{2} b \right) \right|^q dt \right]^{\frac{1}{q}} \right. \\ & \quad \left. + \left[\int_0^1 t^{n-\frac{\alpha}{k}} \left| f^{(n+1)} \left(\frac{2-t}{2} a + \frac{t}{2} b \right) \right|^q dt \right]^{\frac{1}{q}} \right]. \end{aligned}$$

Convexity of $|f^{(n+1)}|^q$ gives

$$\begin{aligned} & \left| \frac{2^{n-\frac{\alpha}{k}-1} k \Gamma_k \left(n - \frac{\alpha}{k} + k \right)}{(b-a)^{n-\frac{\alpha}{k}}} \left[\left({}^c D_{\left(\frac{a+b}{2}\right)^+}^{\alpha, k} f \right) (b) + (-1)^n \left({}^c D_{\left(\frac{a+b}{2}\right)^-}^{\alpha, k} f \right) (a) \right] \right. \\ & \quad \left. - f^{(n)} \left(\frac{a+b}{2} \right) \right| \\ & \leq \frac{b-a}{4} \left(\frac{1}{n - \frac{\alpha}{k} + 1} \right)^{\frac{1}{p}} \left[\left[\int_0^1 t^{n-\frac{\alpha}{k}} \left(\frac{t}{2} |f^{(n+1)}(a)|^q + \frac{2-t}{2} |f^{(n+1)}(b)|^q \right) dt \right]^{\frac{1}{q}} \right. \\ & \quad \left. + \left[\int_0^1 t^{n-\frac{\alpha}{k}} \left(\frac{2-t}{2} |f^{(n+1)}(a)|^q + \frac{t}{2} |f^{(n+1)}(b)|^q \right) dt \right]^{\frac{1}{q}} \right] \\ & = \frac{b-a}{4} \left(\frac{1}{n - \frac{\alpha}{k} + 1} \right)^{\frac{1}{p}} \left[\left[\frac{|f^{(n+1)}(a)|^q}{2(n - \frac{\alpha}{k} + 2)} + \frac{|f^{(n+1)}(b)|^q}{n - \frac{\alpha}{k} + 1} - \frac{|f^{(n+1)}(b)|^q}{2(n - \frac{\alpha}{k} + 2)} \right]^{\frac{1}{q}} \right. \\ & \quad \left. + \left[\frac{|f^{(n+1)}(a)|^q}{n - \frac{\alpha}{k} + 1} - \frac{|f^{(n+1)}(a)|^q}{2(n - \frac{\alpha}{k} + 2)} + \frac{|f^{(n+1)}(b)|^q}{2(n - \frac{\alpha}{k} + 2)} \right]^{\frac{1}{q}} \right], \end{aligned}$$

which after a little computation gives the required result. □

COROLLARY 2 If we take $k = 1$ in the above theorem, then we get the following result for Caputo fractional derivatives (Farid et al., 2017)

$$\begin{aligned} & \left| \frac{2^{n-\alpha-1} \Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}} \left[\left({}^c D_{\left(\frac{a+b}{2}\right)^+}^{\alpha} f \right) (b) + (-1)^n \left({}^c D_{\left(\frac{a+b}{2}\right)^-}^{\alpha} f \right) (a) \right] \right. \\ & \quad \left. - f^{(n)} \left(\frac{a+b}{2} \right) \right| \\ & \leq \frac{b-a}{4(n-\alpha+1)} \left(\frac{1}{2(n-\alpha+2)} \right)^{\frac{1}{q}} \left[[(n-\alpha+1)|f^{(n+1)}(a)|^q \right. \\ & \quad \left. + (n-\alpha+3)|f^{(n+1)}(b)|^q \right]^{\frac{1}{q}} + [(n-\alpha+3)|f^{(n+1)}(a)|^q \\ & \quad \left. + (n-\alpha+1)|f^{(n+1)}(b)|^q \right]^{\frac{1}{q}}. \end{aligned}$$

THEOREM 8 Let $f:[a, b] \rightarrow \mathbb{R}$ be a function such that $f \in C^n[a, b]$, $a < b$. If $|f^{(n+1)}|^q$ is convex on $[a, b]$ for $q > 1$, then the following inequality for Caputo k -fractional derivatives holds

$$\begin{aligned} & \left| \frac{2^{n-\frac{\alpha}{k}-1} k \Gamma_k \left(n - \frac{\alpha}{k} + k \right)}{(b-a)^{n-\frac{\alpha}{k}}} \left[\left({}^C D_{\left(\frac{a+b}{2}\right)^+}^{\alpha, k} f \right)(b) + (-1)^n \left({}^C D_{\left(\frac{a+b}{2}\right)^-}^{\alpha, k} f \right)(a) \right] \right. \\ & \quad \left. - f^{(n)} \left(\frac{a+b}{2} \right) \right| \\ & \leq \frac{b-a}{4} \left(\frac{1}{np - \frac{\alpha p}{k} + 1} \right)^{\frac{1}{p}} \left[\left(\frac{|f^{(n+1)}(a)|^q + 3|f^{(n+1)}(b)|^q}{4} \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\frac{3|f^{(n+1)}(a)|^q + |f^{(n+1)}(b)|^q}{4} \right)^{\frac{1}{q}} \right] \\ & \leq \frac{b-a}{4} \left(\frac{4}{3(np - \frac{\alpha p}{k} + 1)} \right)^{\frac{1}{p}} [|f^{(n+1)}(a)| + |f^{(n+1)}(b)|], \end{aligned} \tag{3.5}$$

with $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 1, we have

$$\begin{aligned} & \left| \frac{2^{n-\frac{\alpha}{k}-1} k \Gamma_k \left(n - \frac{\alpha}{k} + k \right)}{(b-a)^{n-\frac{\alpha}{k}}} \left[\left({}^C D_{\left(\frac{a+b}{2}\right)^+}^{\alpha, k} f \right)(b) + (-1)^n \left({}^C D_{\left(\frac{a+b}{2}\right)^-}^{\alpha, k} f \right)(a) \right] \right. \\ & \quad \left. - f^{(n)} \left(\frac{a+b}{2} \right) \right| \\ & \leq \frac{b-a}{4} \left[\int_0^1 t^{n-\frac{\alpha}{k}} \left| f^{(n+1)} \left(\frac{t}{2} a + \frac{2-t}{2} b \right) \right| dt \right. \\ & \quad \left. + \int_0^1 t^{n-\frac{\alpha}{k}} \left| f^{(n+1)} \left(\frac{2-t}{2} a + \frac{t}{2} b \right) \right| dt \right]. \end{aligned}$$

From Hölder's inequality, we get

$$\begin{aligned} & \left| \frac{2^{n-\frac{\alpha}{k}-1} k \Gamma_k \left(n - \frac{\alpha}{k} + k \right)}{(b-a)^{n-\frac{\alpha}{k}}} \left[\left({}^C D_{\left(\frac{a+b}{2}\right)^+}^{\alpha, k} f \right)(b) + (-1)^n \left({}^C D_{\left(\frac{a+b}{2}\right)^-}^{\alpha, k} f \right)(a) \right] \right. \\ & \quad \left. - f^{(n)} \left(\frac{a+b}{2} \right) \right| \\ & \leq \frac{b-a}{4} \left[\left[\int_0^1 t^{np-\frac{\alpha p}{k}} dt \right]^{\frac{1}{p}} \left[\int_0^1 \left| f^{(n+1)} \left(\frac{t}{2} a + \frac{2-t}{2} b \right) \right|^q dt \right]^{\frac{1}{q}} \right. \\ & \quad \left. + \left[\int_0^1 t^{np-\frac{\alpha p}{k}} dt \right]^{\frac{1}{p}} \left[\int_0^1 \left| f^{(n+1)} \left(\frac{2-t}{2} a + \frac{t}{2} b \right) \right|^q dt \right]^{\frac{1}{q}} \right]. \end{aligned}$$

Convexity of $|f^{(n+1)}|^q$ gives

$$\begin{aligned} & \left| \frac{2^{n-\frac{\alpha}{k}-1} k \Gamma_k \left(n - \frac{\alpha}{k} + k \right)}{(b-a)^{n-\frac{\alpha}{k}}} \left[\left({}^c D_{\left(\frac{a+b}{2}\right)^+}^{\alpha,k} f \right) (b) + (-1)^n \left({}^c D_{\left(\frac{a+b}{2}\right)^-}^{\alpha,k} f \right) (a) \right] \right. \\ & \quad \left. - f^{(n)} \left(\frac{a+b}{2} \right) \right| \\ & \leq \frac{b-a}{4} \left(\frac{1}{np - \frac{\alpha p}{k} + 1} \right)^{\frac{1}{p}} \left[\int_0^1 \left(\frac{t}{2} |f^{(n+1)}(a)|^q + \frac{2-t}{2} |f^{(n+1)}(b)|^q \right) dt \right]^{\frac{1}{q}} \\ & \quad + \left[\int_0^1 \left(\frac{2-t}{2} |f^{(n+1)}(a)|^q + \frac{t}{2} |f^{(n+1)}(b)|^q \right) dt \right]^{\frac{1}{q}} \\ & = \frac{b-a}{4} \left(\frac{1}{np - \frac{\alpha p}{k} + 1} \right)^{\frac{1}{p}} \left[\left[\frac{|f^{(n+1)}(a)|^q + 3|f^{(n+1)}(b)|^q}{4} \right]^{\frac{1}{q}} \right. \\ & \quad \left. + \left[\frac{3|f^{(n+1)}(a)|^q + |f^{(n+1)}(b)|^q}{4} \right]^{\frac{1}{q}} \right]. \end{aligned}$$

For second inequality of (3.5), we use Minkowski's inequality as follows

$$\begin{aligned} & \left| \frac{2^{n-\frac{\alpha}{k}-1} k \Gamma_k \left(n - \frac{\alpha}{k} + k \right)}{(b-a)^{n-\frac{\alpha}{k}}} \left[\left({}^c D_{\left(\frac{a+b}{2}\right)^+}^{\alpha,k} f \right) (b) + (-1)^n \left({}^c D_{\left(\frac{a+b}{2}\right)^-}^{\alpha,k} f \right) (a) \right] \right. \\ & \quad \left. - f^{(n)} \left(\frac{a+b}{2} \right) \right| \\ & \leq \frac{b-a}{16} \left(\frac{4}{np - \frac{\alpha p}{k} + 1} \right)^{\frac{1}{p}} \left[\left[|f^{(n+1)}(a)|^q + 3|f^{(n+1)}(b)|^q \right]^{\frac{1}{q}} \right. \\ & \quad \left. + \left[3|f^{(n+1)}(a)|^q + |f^{(n+1)}(b)|^q \right]^{\frac{1}{q}} \right] \\ & \leq \frac{b-a}{16} \left(\frac{4}{np - \frac{\alpha p}{k} + 1} \right)^{\frac{1}{p}} \left(3^{\frac{1}{q}} + 1 \right) \left(|f^{(n+1)}(a)| + |f^{(n+1)}(b)| \right) \\ & \leq \frac{b-a}{4} \left(\frac{4}{3(np - \frac{\alpha p}{k} + 1)} \right)^{\frac{1}{p}} \left(|f^{(n+1)}(a)| + |f^{(n+1)}(b)| \right). \quad \square \end{aligned}$$

COROLLARY 3 For $k = 1$ in above theorem, we get the following inequality for Caputo fractional derivatives (Farid et al., 2017)

$$\begin{aligned} & \left| \frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}} \left[\left({}^c D_{\left(\frac{a+b}{2}\right)^+}^\alpha f \right)(b) + (-1)^n \left({}^c D_{\left(\frac{a+b}{2}\right)^-}^\alpha f \right)(a) \right] \right. \\ & \left. - f^{(n)}\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{b-a}{4} \left(\frac{1}{np-\alpha p+1} \right)^{\frac{1}{p}} \left[\left(\frac{|f^{(n+1)}(a)|^q + 3|f^{(n+1)}(b)|^q}{4} \right)^{\frac{1}{q}} \right. \\ & \left. + \left(\frac{3|f^{(n+1)}(a)|^q + |f^{(n+1)}(b)|^q}{4} \right)^{\frac{1}{q}} \right] \\ & \leq \frac{b-a}{4} \left(\frac{4}{3(np-\alpha p+1)} \right)^{\frac{1}{p}} [|f^{(n+1)}(a)| + |f^{(n+1)}(b)|], \end{aligned}$$

with $\frac{1}{p} + \frac{1}{q} = 1$.

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