On 2-absorbing and 2-absorbing primary hyperideals of a multiplicative hyperring

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Abstract: Let $R$ be a multiplicative hyperring. In this paper, we study 2-absorbing hyperideals which are a generalization of prime hyperideals and introduce the concept of 2-absorbing primary hyperideal which is a generalization of primary hyperideal.

Subjects: Algebra; Group Theory; Pure Mathematics

Keywords: prime hyperideal; 2-absorbing hyperideal; 2-absorbing primary hyperideal; hyperring

2010 Mathematics subject classification: 20N20

1. Introduction

The theory of algebraic hyperstructures (or hypersystems) is a well-established branch of classical algebraic theory. This theory was introduced in Marty (1934) at the 8th Congress of Scandinavian Mathematicians. Later on, many researchers have observed that the theory of hyperstructures also have many applications in both pure and applied sciences which a comprehensive review of this theory can be found in Corsini (1993), Davvaz and Leoreanu-Fotea (2007), Omidi and Davvaz (2016), Corsini (2003), and Vougiouklis (1994). For example, applications of hyperstructures in chemistry and physics can be studied in Chapter 8 in Davvaz and Leoreanu-Fotea (2007). The notion of multiplicative hyperring is introduced by Rota (1982).

A triple $(R, +, o)$ is called a multiplicative hyperring if

(1) $(R, +)$ is an abelian group;

(2) $(R, o)$ is semihypergroup;

(3) for all $a, b, c \in R$, we have $ao(b + c) \subseteq aob + aoc$ and $(b + c)oa \subseteq boa + coa$;

(4) for all $a, b \in R$, we have $ao(-b) = (-a)ob = -(aob)$.

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PUBLIC INTEREST STATEMENT

Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set.

The purpose of this paper is to introduce and to study the concepts of 2-absorbing and 2-absorbing primary hyperideals over a multiplicative hyperring. They are a generalization of prime hyperideals and a generalization of primary hyperideal, respectively.
For any 3 nonempty subsets A and B of R and x ∈ R, we define

\[ AoB = \bigcup_{a \in A, b \in B} aob, \quad Aox = Ao\{x\} \]

A nonempty subset I of a multiplicative hyperring R is a hyperideal if

1. If a, b ∈ I, then a − b ∈ I;
2. If x ∈ I and r ∈ R, then rax ⊆ I.

In an Algebra and its applications conference, a researcher introduced the concept of 2-absorbing hyperideal and obtains several results (Ghiasvand, 2014). Really, it is a generalization of prime hyperideal. Precisely, a nonzero proper hyperideal I of a multiplicative hyperring R is called to be 2-absorbing if \( xoyo \subseteq I \) whenever \( xoy \subseteq I \) or \( yoz \subseteq I \) or \( xoz \subseteq I \). In this paper, we aim to study the notion and introduce the concept of 2-absorbing primary hyperideal which is a generalization of primary hyperideal.

A nonzero proper hyperideal of R is called a prime hyperideal of R if \( aob \not\subseteq P \) for a, b ∈ R implies that a ∈ P or b ∈ P. The intersection of all prime hyperideals of R containing I is called the prime radical of I, being denoted by r(I). If the multiplicative hyperring R does not have any prime hyperideal containing I, we define \( r(I) = R \). Let C be the class of all finite products of elements of R i.e. \( C = \{ r_1, r_2, \ldots, r_n : r_i \in R, \ n \in \mathbb{N} \} \subseteq P(R) \). A hyperideal I of R is said to be a C-ideal of R if, for any \( A \in C \), \( A \cap I \neq \emptyset \Rightarrow A \subseteq I \). Let I be a hyperideal of R. Then, \( D = \{ r \in R : r^n \subseteq I \text{ for some } n \in \mathbb{N} \} \). The equality holds when I is a C-ideal of R (Dasgupta, 2007, proposition 3.2). In this paper, we assume that all 2-absorbing hyperideals are C-ideal.

Among many results in this paper, it is shown (Theorem 2.8) that a nonzero proper hyperideal I of R is a 2-absorbing hyperideal if and only if whenever \( J o K o L \subseteq I \) for some hyperideals J, K, L of R, then \( J o K \subseteq I \) or \( K o L \subseteq I \) or \( J o L \subseteq I \). It is shown (Theorem 2.2) that if \( I_1, I_2, \ldots, I_n \) are 2-absorbing hyperideals of R and I is a hyperideal of R contained in \( \bigcup_{i=1}^{n} I_i \) and \( r(I_j) \not\subseteq E_{x_i} \) for all \( x_i \in r(I_j) \setminus I_i \), where \( i \neq k \) (recall that \( E_{x_i} = \{ r \in R \mid rax \subseteq I_j \} \)), then \( I \subseteq I_j \) for some \( 1 \leq i \leq n \). It is shown (Theorem 4.13) that a nonzero hyperideal I of R is a 2-absorbing primary hyperideal if and only if whenever \( I_o I_1 I_2 oI_3 \subseteq I \) for some hyperideals \( I_1, I_2, I_3 \) of R, then \( I_1 oI_2 \subseteq I \) or \( I_2 oI_3 \subseteq I \) or \( I_3 \subseteq r(I) \) or \( I_2 oI_3 \subseteq r(I) \).

2. 2-absorbing hyperideals in multiplicative hyperrings

Definition 2.1 A nonzero proper hyperideal I of a multiplicative hyperring R is called to be 2-absorbing if \( xoyo \subseteq I \) whenever \( xoy \subseteq I \) or \( yoz \subseteq I \) or \( xoz \subseteq I \).

Example 2.2 In the multiplicative hyperring of integers \( \mathbb{Z}_n \) with \( A = \{5, 7\} \), the principal hyperideals \( < 2 > \) and \( < 3 > \) are prime hyperideals (by Proposition 4.3 in Dasgupta, 2007). Hence, hyperideal \( < 2 > \cap < 3 > \) is a 2-absorbing hyperideal.

Theorem 2.3 Let I be a 2-absorbing hyperideal of R such that \( r(I) = P \) is a prime hyperideal of R and suppose that I ≠ P. For each \( x \in P \setminus I \) let \( E_x = \{ y \in R \mid xoy \subseteq I \} \). Then \( E_x \) is a prime hyperideal of R containing P. Furthermore, either \( E_y \subseteq E_x \) or \( E_y \subseteq E_x \) for every \( x, y \in P \setminus I \).

Proof. Assume that \( x \in P \setminus I \). Since \( P^2 \subseteq I \) (by Theorem 4, in Ghiasvand, 2014), we have \( P \subseteq E_x \). Let \( P \neq E_y \) and \( yoz \subseteq E_y \) for some \( y, z \in R \). Since \( P \subseteq E_y \), we may assume that \( y \notin P \) and \( z \notin P \), and thus \( yoz \not\subseteq I \). By \( yoz \not\subseteq I \), we have \( yoz \not\subseteq I \). Since \( I \) is a 2-absorbing hyperideal of \( P \), \( yoz \not\subseteq I \). Therefore, \( yoz \not\subseteq I \). Hence \( E_y \) is a prime hyperideal of \( P \) containing \( P \). Let \( x, y \in P \setminus I \) and suppose that \( z \in E_y \setminus E_x \). Since \( P \subseteq E_y \), \( z \in E_y \setminus P \). We show that \( E_y \subseteq E_x \). Let \( w \in E_y \setminus P \). Since \( P \subseteq E_x \), we may assume that \( w \in E_x \setminus P \). Since \( z \notin P \) and \( w \notin P \), we conclude that \( zow \not\subseteq I \). Since \( zox(y + y)w \not\subseteq I \) and \( zow, zoy \not\subseteq I \), we conclude that \( (x + y)ow \not\subseteq I \). Hence \( wox \not\subseteq I \) since \( (x + y)ow \not\subseteq I \) and \( way \not\subseteq I \). Thus \( w \in E_x \subseteq E_y \).
Theorem 2.4 Let I be a 2-absorbing hyperideal of R such that I \neq r(I) = P \cap Q where P and Q are the only nonzero distinct prime hyperideals of R that are minimal over I. Then for each x \in r(I) \setminus I, E_y = \{ y \in R \mid xoy \subseteq I \} is a prime hyperideal of R containing P and Q. Furthermore, either E_y \subseteq E_x or E_y \subseteq E_x for every x, y \in r(I) \setminus I.

Proof. Assume that x \in r(I) \setminus I. Since PoQ \subseteq I by Theorem 4 in Ghasvand (2014), we have xoP \subseteq I are xoQ \subseteq I. Thus P \subseteq E_x are Q \subseteq E_y. Let yoz \subseteq E_y for some y, z \in R. Since P \subseteq E_x or Q \subseteq E_y, we may assume that y, z \in P and y, z \notin Q, and thus yoz \notin I. Since yoz \subseteq E_y, we have yozx \subseteq I. Since I is a 2-absorbing hyperideal of R and yoz \notin I, we have either yoxz \subseteq I or zoxz \subseteq I, and thus either y \in E_x or z \in E_y. Hence E_y is a prime hyperideal of R. By a similar argument to that in the proof of Theorem 2.3, we can show easily that either E_y \subseteq E_x or E_x \subseteq E_y for every x, y \in r(I) \setminus I.

Theorem 2.5 Let I be a 2-absorbing hyperideal and p, q be two prime hyperideals of R. Then

(i) If r(I) = P, then E_y = \{ y \in R \mid xoy \subseteq I \} is a 2-absorbing hyperideal of R, for all x \in R \setminus P with r(E_y) = P and T = \{ E_y \mid x \in R \} is a totally ordered set.

(ii) If r(I) = P \cap Q, then E_y is a 2-absorbing hyperideal of R, for all x \in R \setminus P \cup Q with r(E_y) = P \cap Q and T = \{ E_y \mid x \in R \setminus P \cup Q \} is a totally ordered set.

(iii) If r(I) = P \cap Q, then E_y = Q, for all x \in P \setminus Q and E_y = P, for all x \in Q \setminus P.

Proof.

(i) Let x \in R \setminus P, a, b, c \in R and aoboc \subseteq E_y. Hence, aobocax \subseteq I and I is a 2-absorbing hyperideal of R. Thus aox \subseteq I or bocax \subseteq I or aob \subseteq I. If aox \subseteq I or bocax \subseteq I we are done. If aob \subseteq I, then aob \subseteq I or aoc \subseteq I or boc \subseteq I which imply that aobocax \subseteq I or aocox \subseteq I or bocx \subseteq I that is the claim. Thus E_y is a 2-absorbing hyperideal of R and it is easy to see that E_y \subseteq P. Suppose that x, y \in R \setminus P. It is clear that xoy \subseteq R \setminus P, E_y \subseteq E_{xoy}, E_y \subseteq E_{xox} and E_y \subseteq E_{xox}. To establish the reverse inclusion, let z \in E_{xox} which implies that either xoax \subseteq I or zox \subseteq I since xox \subseteq I. Thus E_{xox} \subseteq E_y \cup E_y and therefore E_{xox} = E_y \cup E_y. Hence, either E_{xox} = E_y or E_{xox} = E_y. Now, either E_y \subseteq E_y or E_y \subseteq E_y which shows that T = \{ E_y \mid x \in R \setminus P \} is a totally ordered set. On the other hand, Theorem 2.3, shows that E_y is a prime hyperideal of R containing P, for all x \in P \setminus I and T = \{ E_y \mid x \in P \setminus I \} is totally ordered. Therefore, T = \{ E_y \mid x \in R \} is totally ordered.

(ii) By using an argument similar to that of (i) we can show that E_y is a 2-absorbing hyperideal of R, E_y \subseteq P \cap Q and r(E_y) = P \cap Q. Moreover, it is easy to see that T = \{ E_y \mid x \in R \setminus P \cup Q \} is a totally ordered set.

(iii) Suppose that r(I) = P \cap Q and x \in P \setminus Q. If y \in P \cap Q, then xoy \subseteq PoQ \subseteq I. Hence, y \in E_y and so P \cap Q \subseteq E_y. Also, E_y \subseteq Q. Now, assume that y \in Q. Then xoy \subseteq PoQ \subseteq I and it follows that y \in E_y. Thus Q = E_y.

Theorem 2.6 Suppose that I is an hyperideal of R such that I \neq r(I) and r(I) is a prime hyperideal of R. Then the following statements are equivalent:

(i) I is a 2-absorbing hyperideal of R;
(ii) E_y = \{ y \in R \mid xoy \subseteq I \} is a prime hyperideal of R for each x \in r(I) \setminus I.

Proof.

(i) \Rightarrow (ii). This can be proved by using Theorem 2.3.

(ii) \Rightarrow (i). Assume that xoyz \subseteq I for some x, y, z \in R. Since r(I) is a prime hyperideal of R, we may assume that x \in r(I). If x \in I, then xoy \subseteq I and we are done. Hence suppose that x \in r(I) \setminus I. Thus yoz \subseteq E_y. Since E_y is a prime hyperideal of R by Theorem 2.3, we have either xoy \subseteq I or zox \subseteq I. Thus I is a 2-absorbing hyperideal of R.
Theorem 2.7 Let $I$ be an hyperideal of $R$ such that $I \neq r(I) = P \cap Q$ where $P$ and $Q$ are nonzero distinct prime hyperideals of $R$ that are minimal over $I$. Then the following statements are equivalent:

(i) $I$ is a 2-absorbing hyperideal of $R$;
(ii) $PoQ \subseteq I$ and $E_y = \{y \in R \mid yax \nsubseteq I\}$ is a prime hyperideal of $R$ for each $x \in r(I) \setminus I$.
(iii) $E_x = \{y \in R \mid yax \nsubseteq I\}$ is a prime hyperideal of $R$ for each $x \in (P \cup Q) \setminus I$.

Proof.

(i) $\Rightarrow$ (ii). This can be proved by using Theorems 2.4 and 2.6 in this article and Theorem 4 in Ghiasvand (2014).

(ii) $\Rightarrow$ (iii). Suppose that $x \in P \setminus Q$. It is clear that $yax \subseteq I$ if and only if $y \in Q$. Since $PoQ \subseteq I$, we conclude that $E_y = Q$ is a prime hyperideal of $R$. Let $z \in Q \setminus P$. By a similar argument as before we have $E_z = P$ is a prime of $R$. Since $E_z$ is a prime hyperideal of $R$ for each $d \in r(I) \setminus I$, we are done.

(iii) $\Rightarrow$ (i). Suppose that $xayz \nsubseteq I$. We may assume that $x \in (P \cup Q) \setminus I$. Thus $yaz \subseteq E_z$. Since $E_z$ is a prime hyperideal of $R$ by Theorem 2.4, we conclude that either $yax \subseteq I$ or $zay \subseteq I$, and hence $I$ is a 2-absorbing hyperideal of $R$.

Theorem 2.8 Suppose that $I$ is a nonzero proper hyperideal of a hyperring $R$. The following statements are equivalent:

(i) $I$ is a 2-absorbing hyperideal of $R$;
(ii) If $JoKoL \subseteq I$ for some hyperideals $J, K, L$ of $R$, then $JoK \subseteq I$ or $JoL \subseteq I$ or $JoL \subseteq I$.

Proof. Since (ii) $\Rightarrow$ (i) is trivial, we only need to show that (i) $\Rightarrow$ (ii). Suppose that $JoKoL \subseteq I$ for some hyperideals $J, K, L$ of $R$. By Theorem 4 in Ghiasvand (2014), we conclude that $r(I)$ is a prime hyperideal of $R$ or $r(I) = P \cap Q$ where $P$ and $Q$ are nonzero distinct prime hyperideals of $R$ that are minimal over $I$. If $I = r(I)$, then it is easily proved that $JoK \subseteq I$ or $JoL \subseteq I$ or $JoL \subseteq I$. Hence assume that $I \neq r(I)$. We consider two cases.

Case 1. Let $r(I)$ be a prime hyperideal of $R$. Then we may assume that $J \subseteq r(I)$ and $J \nsubseteq I$. Let $x \in J \setminus I$. Since $xJoKoL \subseteq I$, we conclude that $JoK \subseteq E_x$. Since $E_x$ is a prime hyperideal of $R$ by Theorem 2.6, we conclude that either $K \subseteq E_x$ or $L \subseteq E_x$. If $K \subseteq E_x$ and $L \subseteq E_x$ for each $d \in J \setminus I$, then $JoK \subseteq I$ (and $JoL \subseteq I$) and we are done. Hence suppose that $K \subseteq E_x$ and $L \nsubseteq E_x$ for some $y \in J \setminus I$. Since $\{E_w \mid w \in J \setminus I\}$ is a set of prime hyperideals of $R$ that are linearly ordered by Theorem 2.4 and $K \subseteq E_y$ and $L \subseteq E_y$, we conclude that $K \subseteq E_y$ for each $z \in J \setminus I$, and thus $JK \subseteq I$.

Case 2. Let $r(I) = P \cap Q$ where $P$ and $Q$ are nonzero distinct prime hyperideals of $R$ that are minimal over $I$. We may assume that $J \subseteq P$. If either $K \subseteq Q$ or $L \subseteq Q$, then either $JoK \subseteq I$ or $JoL \subseteq I$ because $PoQ \subseteq I$ by Theorem 4 in Ghiasvand (2014). Hence suppose that $J \nsubseteq r(I)$ and $J \nsubseteq I$. By using an argument similar to that one given in case 1 and Theorem 2.4, we are done. □

3. 2-absorbing hyperideal avoidance theorem

Let $I_1, I_2, \ldots, I_n$ be 2-absorbing hyperideals in a hyperring $R$. Let $I$ be any ideal of $R$. The idea is that if we can avoid the $I_i$, individually, in other words, for each $i$ we can find an element in $I$ but not in $I_i$, then we can avoid all the $I_i$ simultaneously, i.e. we can find a single element in $I$ that is in none of the $I_i$. We will state and prove the contrapositive. It is called 2-absorbing avoidance theorem for hyperideals. In this section, we assume that all prime hyperideals are $C$-ideal.
Lemma 3.1 Let $I_1, I_2, \ldots, I_n$ be 2-absorbing hyperideals of $R$ and let $I$ be a hyperideal of $R$ contained in $\bigcup_{i=1}^n I_i$. Then $r(I) \subseteq r(I_i)$ for some $1 \leq i \leq n$.

Proof. We suppose that $I$ is a 2-absorbing hyperideal of $R$, for all $i \geq 1$. we may assume $r(I_i) = P_i$ for $1 \leq i \leq j$, and $r(I_j) = P_j \cap Q_j$ for $j + 1 \leq i \leq n$ where $P_i, P_j, Q_j$ are prime hyperideals of $R$.

$r(I) \subseteq P_1 \cup \ldots \cup P_j \cup (P_{j+1} \cap Q_{j+1}) \cup \ldots \cup (P_n \cap Q_n)$

and so

$r(I) \subseteq P_1 \cup \ldots \cup P_j \cup Q_{j+1} \cup \ldots \cup Q_n$

By Proposition 2.19 in Dasgupta (2007), we have $r(I) \subseteq P_i$ where $1 \leq i \leq j$, or $r(I) \subseteq P_j$ or $r(I) \subseteq Q_n$ with $j + 1 \leq i \leq n$. If the first case happened, we are done. Let $r(I) \subseteq \bigcup_{i=j+1}^n P_i$, then we can assume that $r(I) \subseteq \bigcap_{i=j+1}^n P_i$ and $r(I) \not\subseteq \bigcap_{i=j+1}^n P_i$ with $j + 1 \leq k \leq n$. In this case, $r(I) \subseteq Q_{j+1} \cup \ldots \cup Q_k \cup P_{k+1} \cup \ldots \cup P_n$ implies that $r(I) \subseteq Q_n$ for some $j + 1 \leq t \leq k$. Thus $r(I) \subseteq P_r \cap Q_n = r(I_i)$. □

Theorem 3.2 (2-absorbing hyperideal avoidance theorem) 2-absorbing hyperideal avoidance theorem. Let $I_1, I_2, \ldots, I_n$ be 2-absorbing hyperideals of $R$ and let $I$ be a hyperideal of $R$ contained in $\bigcup_{i=1}^n I_i$, and let $r(I) \subseteq E^*_k$ for all $x \in r(I_i) \setminus I_k$ where $i \neq k$ (recall that $E^*_k = \{r \in R | r \cap I_k \neq I_k \}$). Then $I \subseteq I_k$ for some $1 \leq i \leq n$.

Proof. We assume $I \not\subseteq I_i$ for all $1 \leq i \leq n$. It means no $I_i$ is superfluous. Hence, $I = \bigcup_{i=1}^n (I_i \cap I)$ is an union in which none of the $I_i \cap I$ are excluded. Therefore $(\bigcap_{i=1}^n I_i) \cap I \subseteq I_k \cap I$. By using an argument similar to that of Lemma 3.1, we conclude that $I \subseteq r(I_k)$, for some $1 \leq i \leq n$. This leads to a contradiction. Let $I \subseteq r(I_i)$, for some $1 \leq i \leq n$. We can assume $x \in r(I_i) \setminus I_k$ such that $x \in I \setminus I_k$. Also, we have $r \in I \setminus E^*_k$ for all $i \neq k$. We consider $r \in r_1 \cup \ldots \cup r_{k-1} \cup r_{k+1} \cup \ldots \cup r_n$. Therefore $r \cap I_k \not\subseteq I_k$. Moreover, $r \cap I_k \not\subseteq I_k$. For otherwise, we have $r \in E^*_k$ which is a contradiction since $E^*_k$ is a prime hyperideal of $R$ by Theorems 2.3 and 2.4. □

4. 2-absorbing primary hyperideals in multiplicative hyperring

Definition 4.1 A nonzero proper hyperideal $I$ of $R$ is called to be 2-absorbing primary hyperideal if $aobc \subseteq I$ where $a, b, c \in R$, then $aob \subseteq I$ or $boc \subseteq r(I)$ or $aoc \subseteq r(I)$.

Example 4.2 In Example 2.2, the hyperideal $< 2 > \cap < 3 >$ is a 2-absorbing primary hyperideal.

Example 4.3 In the multiplicative hyperring of integers $\mathbb{Z}_A$ with $A = \{8, 12\}$, the principal hyperideal $< 4 >$ is a primary hyperideal (by Proposition 4.6 in Dasgupta, 2007). Thus it is a 2-absorbing primary hyperideal.

Theorem 4.4 If $I$ is a 2-absorbing primary $C$-ideal of $R$, then $r(I)$ is a 2-absorbing ideal of $R$.

Proof. Let $a, b, c \in R$ such that $aobc \subseteq r(I)$, $aoc \not\subseteq r(I)$ and $boc \not\subseteq r(I)$. Then, for any $x \in aobc$ there exists $n \in \mathbb{N}$ such that $a^n \subseteq aobc$. Again, $a^n \subseteq (aobc)^n = a^n \cdot b^n \cdot c^n$ (since $R$ is commutative). So, $(a^n \cdot b^n \cdot c^n) \cap I \neq \emptyset$ and thus, $a^n \cdot b^n \cdot c^n \subseteq I$ (since, $I$ is a $C$-ideal). Since $aoc \not\subseteq r(I)$ implies $a^n \cdot c^n \cdot I \neq \emptyset$. Thus, for any $a_i \in a^n$ and $c_i \in c^n$, we have that $a_1 c_1 \not\subseteq I$ and $a_1 b_1 c_1 \not\subseteq a^n \cdot b^n \cdot c^n \subseteq I$. Also, $boc \not\subseteq r(I)$ implies $b^n \cdot c^n \cdot I \neq \emptyset$. Thus, we have that $b_1 c_1 \not\subseteq I$, for any $b_1 \in b^n$. Now we that $a_1 b_1 c_1 \not\subseteq d^n \cdot b^n \cdot c^n$. Thus, $a_1 b_1 c_1 \not\subseteq I$ (since $I$ is a 2-absorbing primary hyperideal). Since $a_1 b_1 c_1 \not\subseteq d^n \cdot b^n \cdot c^n$ we have $a_1 b_1 c_1 \not\subseteq I$. Thus $aobc \subseteq r(I)$. □
THEOREM 4.5 Suppose that $I$ is a 2-absorbing primary $C$-ideal of $R$. Then one of the following statements must hold.

(1) $r(I) = P$ is a prime hyperideal,

(2) $r(I) = P_1 \cap P_2$, where $P_1$ and $P_2$ are the only distinct prime hyperideals of $R$ that are minimal over $I$.

Proof. Suppose that $I$ is a 2-absorbing primary hyperideal of $R$. Then $r(I)$ is a 2-absorbing hyperideal by Theorem 4.4. Since $r(r(I)) = r(I)$, the claim follows from Proposition 3.3 in Dasgupta (2007).

THEOREM 4.6 Suppose that $I_1$ is a $P_1$-primary $C$-ideal of $R$ for some prime hyperideal $P_1$ of $R$ and $I_2$ is a $P_2$-primary $C$-ideal of $R$ for some prime hyperideal $P_2$ of $R$. Then the following statements hold.

(1) $I_1 \cap I_2$ is a 2-absorbing primary hyperideal of $R$.

(2) $I_1 \cap I_2$ is a 2-absorbing primary hyperideal of $R$.

Proof.

(1) Assume that $aoboc \subseteq I_1 \cap I_2$ for some $a, b, c \in R$, $aoc \not\subseteq r(I_1 \cap I_2)$, and $boc \not\subseteq r(I_1 \cap I_2) = P_1 \cap P_2$. Since $r(I_1 \cap I_2) = P_1 \cap P_2$, we conclude that $r(I_1 \cap I_2)$ is a 2-absorbing hyperideal of $R$ and $aoc \not\subseteq r(I_1 \cap I_2)$. We show that $aob \subseteq I_1 \cap I_2$. Since $aob \not\subseteq r(I_1 \cap I_2) \subseteq P_2$, we may assume that $a \in P_2$. Since $a \not\subseteq r(I_1 \cap I_2)$ and $boc \not\subseteq r(I_1 \cap I_2) \subseteq P_2$, we conclude that $a \not\subseteq P_1$ and $b \in P_2$ since $b \in P_2$ and $b \not\subseteq r(I_1 \cap I_2)$. If $a \in I_1$ and $b \in I_2$, then $aob \not\subseteq I_1 \cap I_2$, and we are done. Thus assume that $a \not\subseteq I_1$. Since $b \in P_2$ and $boc \subseteq P_2$, we have $boc \subseteq r(I_1 \cap I_2)$, which is a contradiction. Thus $a \not\subseteq I_1$. Similarly, assume that $b \not\subseteq I_2$. Since $I_2$ is a $P_2$-primary hyperideal of $R$ and $b \not\subseteq I_2$, we have $aoc \subseteq P_2$. Since $aoc \subseteq P_2$ and $a \in P_2$, we have $aoc \subseteq r(I_1 \cap I_2)$, which is a contradiction. Thus $b \in I_2$. Hence $aoc \subseteq I_1 \cap I_2$.

(2) (similar to the proof in (1)). Let $K = I_1 \cap I_2$ and $r(K) = P_1 \cap P_2$. Suppose that $aoboc \subseteq K$ for some $a, b, c \in R$, $aoc \not\subseteq r(K)$, and $boc \not\subseteq r(K)$. Then $a, b, c \not\subseteq r(K) = P_1 \cap P_2$. Since $r(K) = P_1 \cap P_2$ is a 2-absorbing hyperideal of $R$ and $aoc \not\subseteq r(K)$, $boc \not\subseteq r(K)$. We show that $aob \not\subseteq K$. Since $aob \not\subseteq r(K) \subseteq P_2$, we may assume that $a \in P_2$. Since $a \not\subseteq r(K)$ and $aob \not\subseteq r(K) \subseteq P_2$, we conclude that $a \not\subseteq P_1$ and $b \in P_2$. Since $b \in P_2$ and $b \not\subseteq r(K)$, $boc \not\subseteq K$ if $a \in I_1$ and $b \in I_2$, then $aob \not\subseteq K$ and we are done. Thus assume that $a \not\subseteq I_1$. Since $I_1$ is a $P_1$-primary hyperideal of $R$ and $a \not\subseteq I_1$, we have $boc \subseteq P_1$. Since $b \in P_2$ and $boc \subseteq P_1$, we have $boc \subseteq r(K)$, which is a contradiction. Thus $a \not\subseteq I_1$. Similarly, assume that $b \not\subseteq I_2$. Since $I_2$ is a $P_2$-primary hyperideal of $R$ and $b \not\subseteq I_2$, we have $aoc \subseteq P_2$. Since $aoc \subseteq P_2$ and $I_2$ is a $P_2$-primary hyperideal of $R$ and $b \not\subseteq I_2$, we have $aoc \subseteq P_2$. Since $aoc \subseteq P_2$ and $a \in P_1$, we have $aoc \subseteq r(K)$, which is a contradiction. Thus $b \in I_2$. Hence $aob \not\subseteq K$.

Example 4.7 In the multiplicative hyperring of integers $\mathbb{Z}_A$ with $A = \{12, 24\}$, the principal hyperideals $< 4 >$ and $< 6 >$ are primary $C$-ideals (by Proposition 4.6 in Dasgupta, 2007). Thus $< 4 > \cdot < 6 >$ and $< 4 > \cap < 6 >$ are 2-absorbing primary hyperideals of $\mathbb{Z}_A$.

COROLLARY 4.8 Let $P_1$, $P_2$ be prime hyperideals of $R$. If $P_1^n$ is a $P_1$-primary hyperideal of $R$ for some positive integer $n \geq 1$ and $P_2^m$ is a $P_2$-primary hyperideal of $R$ for some positive integer $m \geq 1$, then $P_1^n P_2^m$ and $P_1^n \cap P_2^m$ are 2-absorbing primary hyperideals of $R$. In particular, $P_1 P_2$ is a 2-absorbing primary hyperideal of $R$.

THEOREM 4.9 Let $I$ be an hyperideal of $R$. If $r(I)$ is a prime hyperideal of $R$, then $I$ is a 2-absorbing primary hyperideal of $R$. In particular, if $P$ is a prime hyperideal of $R$, then $P^n$ is a 2-absorbing primary hyperideal of $R$ for every positive integer $n \geq 1$. 
Proof. Assume that $aoboc \subseteq I$ and $aob \not\subseteq I$. Since $(aoc)oboc = aoboc^2 \subseteq I \subseteq r(I)$ and $r(I)$ is a prime hyperideal of $R$, we have $boc \subseteq r(I)$ or $aoc \subseteq r(I)$. Hence $I$ is a 2-absorbing primary hyperideal of $R$. □

**Definition 4.10** Let $I$ be a 2-absorbing primary $C$-ideal of $R$. Then $P = r(I)$ is a 2-absorbing hyperideal by Theorem 4.4. We say that $I$ is a $P$-2-absorbing primary hyperideal of $R$.

**Theorem 4.11** Let $I_1, I_2, \ldots, I_n$ be $P$-2-absorbing primary $C$-ideals of $R$ for some 2-absorbing hyperideal $P$ of $R$. Then $I = \bigcap_{i=1}^n I_i$ is a $P$-2-absorbing primary hyperideal of $R$.

**Proof.** It is clear. □

In the following result, we show that a proper hyperideal $I$ of a ring $R$ is a 2-absorbing primary hyperideal of $R$ if and only if whenever $I_1, I_2, I_3 \subseteq I$ for some hyperideals $I_1, I_2, I_3$ of $R$, then $I_1I_2 \subseteq I$ or $I_1I_3 \subseteq r(I)$ or $I_2I_3 \subseteq r(I)$.

**Lemma 4.12** Let $I$ be a 2-absorbing primary hyperideal of a hyperring $R$ and suppose that $aoboc \subseteq I$ for some elements $a, b \in R$ and some hyperideal $J$ of $R$. If $aob \not\subseteq I$, then $aobJ \subseteq r(I)$ or $bocJ \subseteq r(I)$.

**Proof.** Suppose that $aob \not\subseteq I$ or $boc \not\subseteq I$. Then $aobJ \not\subseteq I$ and $bocJ \not\subseteq I$ for some $j_1, j_2 \in J$. Since $aoboc \subseteq I$ and $aob \not\subseteq I$ and $boc \not\subseteq I$, we have $boc \not\subseteq I$. Since $aoboc \subseteq I$ and $aob \not\subseteq I$ and $boc \not\subseteq I$, we have $aoboc \not\subseteq I$. Now, since $aoboc \not\subseteq I$, we have $aoboc \not\subseteq I$ and $bocJ \not\subseteq I$ and $aobJ \not\subseteq I$. Suppose that $aoboc \subseteq I$ and $bocJ \not\subseteq I$ and $aobJ \not\subseteq I$. Since $aoboc \subseteq I$, we have $aoboc \subseteq I$, a contradiction. Suppose that $bocJ \subseteq I$ and $bocJ \subseteq I$. Since $bocJ \subseteq I$, we have $bocJ \subseteq I$, a contradiction again. Thus $aoboc \subseteq I$ or $bocJ \subseteq I$.

**Theorem 4.13** Let $I$ be a proper hyperideal of $R$. Then $I$ is a 2-absorbing primary hyperideal if and only if whenever $I_1, I_2, I_3 \subseteq I$ for some hyperideals $I_1, I_2, I_3$ of $R$, then $I_1I_2 \subseteq I$ or $I_1I_3 \subseteq r(I)$ or $I_2I_3 \subseteq r(I)$.

**Proof.** Assume that whenever $I_1I_2I_3 \subseteq I$ for some hyperideals $I_1, I_2, I_3$ of $R$, then $I_1I_2 \subseteq I$ or $I_1I_3 \subseteq r(I)$ or $I_2I_3 \subseteq r(I)$. Then clearly $I$ is a 2-absorbing primary hyperideal of $R$. Conversely, suppose that $I$ is a 2-absorbing primary hyperideal of $R$ and $I_1I_2I_3 \subseteq I$ for some hyperideals $I_1, I_2, I_3$ of $R$, such that $I_1I_2 \subseteq I$. We show that $I_1I_2 \subseteq r(I)$ or $I_1I_3 \subseteq r(I)$. Suppose that neither $I_1I_2 \subseteq r(I)$ nor $I_1I_3 \subseteq r(I)$. Then there are $q_1, q_2 \in I$ such that neither $q_1I_2 \subseteq r(I)$ nor $q_2I_3 \subseteq r(I)$. Since $q_2I_2 \subseteq I$ and either $q_1I_2 \subseteq r(I)$ or $q_2I_3 \subseteq r(I)$ or $q_2I_3 \subseteq r(I)$, we have $q_2I_3 \subseteq I$ by Lemma 4.12. Since $I_2I_3 \subseteq I$, we have $q_2I_3 \subseteq r(I)$ for some $a, b, c \in I$. Since $q_2I_3 \subseteq r(I)$ and $boc \not\subseteq I$, we have $q_2I_3 \subseteq r(I)$ or $q_2I_3 \subseteq r(I)$ by Lemma 4.12. We consider three cases.

**Case 1.** Assume that $aoboc \subseteq I$ and $bococ \subseteq I$ or $aoboc \subseteq I$. Since $q_1bococ \subseteq I$ and neither $bococ \subseteq I$ nor $aoboc \subseteq I$, we conclude that $q_1ob \subseteq I$ by Lemma 4.12. Since $a + q_1bococ \subseteq I$ and $aoboc \subseteq I$, but $aoboc \not\subseteq I$ or $bococ \not\subseteq I$, we conclude that $a + q_1bococ \not\subseteq r(I)$ or $aoboc \not\subseteq r(I)$. Since $aoboc \not\subseteq r(I)$ or $aoboc \not\subseteq r(I)$, we conclude that $a + q_1bococ \not\subseteq r(I)$, a contradiction.

**Case 2.** Assume that $boI_2 \subseteq r(I)$ but $aoboc \not\subseteq I$. Since $aoboc \not\subseteq I$ and neither $aoboc \not\subseteq I$ nor $aoboc \not\subseteq I$, we conclude that $aqoc \not\subseteq I$. Since $aoboc \not\subseteq I$ and neither $aoboc \not\subseteq I$ nor $aoboc \not\subseteq I$, we conclude that $aqoc \not\subseteq r(I)$ or $aqoc \not\subseteq r(I)$. Since $aqoc \not\subseteq r(I)$ or $aqoc \not\subseteq r(I)$, we conclude that $aqoc \not\subseteq I$, a contradiction.

**Case 3.** Assume that $aoboc \subseteq I$ and $bococ \not\subseteq I$. Since $aoboc \not\subseteq I$, we conclude that $b + aqoc \not\subseteq I$. Since $aqoc \not\subseteq I$, we conclude that $b + aqoc \not\subseteq r(I)$, a contradiction.
by Lemma 4.12. Since \( q_1 o q_2 \subseteq I \) and \( a o q_2 + q_1 o q_2 \subseteq I \), we conclude that \( a o q_2 \subseteq I \). Now, since \( (a + q_1) o (b + q_2) \subseteq I \) and neither \( (a + q_1) o I \subseteq r(I) \) nor \( (b + q_2) o I \subseteq r(I) \), we conclude that \( (a + q_1) o (b + q_2) \subseteq a o b + a o q_2 + b o q_1 + q_1 o q_2 \subseteq I \) by Lemma 4.12. Since \( a o q_2, b o q_1, q_1 o q_2 \subseteq I \), we have \( a o q_2 + b o q_1 + q_1 o q_2 \subseteq I \). Since \( a o b + a o q_2 + b o q_1 + q_1 o q_2 \subseteq I \) and \( a o q_2 + b o q_1 + q_1 o q_2 \subseteq I \), we conclude that \( a o b \subseteq I \), a contradiction. Hence \( I_1 o I_3 \subseteq r(I) \) or \( I_2 o I_3 \subseteq r(I) \).

\[ \square \]

**Funding**

The author received no direct funding for this research.

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**Citation information**

Cite this article as: On 2-absorbing and 2-absorbing primary hyperideals of a multiplicative hyperring, M. Anbarloei, Cogent Mathematics (2017), 4: 1354447.

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