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On locally nilpotent derivations of Boolean semirings

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Abstract: In this paper, we consider the composition of derivations in Boolean semirings and investigate the conditions that the composition of two derivations is a derivation. We also show that the nth derivation of a derivation d, denoted by d^n, on a Boolean semiring satisfies Leibniz rule. Finally, we show that any locally nipotent derivations on a zero-symmetric Boolean semiring must be zero.

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1. Introduction

The notion of the ring with derivation plays a significant role in the integration of analysis, algebraic geometry and algebra. By a derivation of a ring R, we mean any function $d:R \to R$ which satisfies the following conditions: (i) $d(a+b) = d(a) + d(b)$ and (ii) $d(ab) = d(a)b + ad(b)$, for all $a, b \in R$. The study of derivations of prime ring was initiated by Posner (1957). Posner considered the composition of derivations and showed that the composition of two nonzero derivations of a prime ring R cannot be a derivation provided that characteristic of R is different from 2. Bresar (1990) and Ashraf and Nadeem (2001) proved commutativity of prime and semiprime rings with derivations satisfying certain polynomial constraints. Chebotar (1995) and Bell and Argac (2001) obtained the necessary condition when the composition of derivations could be a derivation. Furthermore, Bell and Argac (2001)

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PUBLIC INTEREST STATEMENT

A derivation is a function on an algebra https://en.wikipedia.org/wiki/Algebra_over_a_field which generalizes certain features of the derivative https://en.wikipedia.org/wiki/Derivative operator. Derivations play a significant role in the integration of analysis, algebraic geometry and algebra. The study of derivations in rings though initiated long back, but got interested only after Posner who established two very striking results on derivations in prime rings in 1957. The notion of derivation has also been generalized in various directions, such as Jordan derivation, generalized derivation, generalized Jordan derivation etc. The objective of this paper is to study the composition of derivations on Boolean semiring. Also, we investigate the conditions that the composition of two derivations is a derivation. Moreover, we investigate the nth derivation satisfying Leibniz rule. Finally, we show that a locally nilpotent derivations on a Boolean semiring possesing some condition must be zero.

and Wang (1994) have investigated the invariance of certain ideals under derivations. Lee and Lee (1986), proved that if d is a derivation on a prime ring R with center Z such that $d^n(x) \in Z$ for all x, where n is a fixed integer, then either $d^n(x) \in Z$ for all x in R or R is a commutative integral domain.

Boolean semiring has been used in the mathematical literature with at least two different meanings. The first one was given by Subrahmanyam (1962). The second one was introduced by Galbiati and Veonesi (1980), which has also been investigated by Guzman (1992). In this paper we use the definition of Boolen semiring in the sense of Subrahmanyan.

In this paper our objective is to study the composition of derivations on Boolean semiring (Subrahmanyam, 1962). Also we investigate the conditions that the composition of two derivations is a derivation. Moreover, we investigate the nth derivation satisfying Leibniz rule.

A derivation d of a ring R is said to be locally nilpotent if for any $x \in R$, there exists a positive integer n such that $d^n(x) = 0$. Locally nilpotent derivations play an important role in commutative algebra and algebraic geometry, and several problems may be formulated using locally nilpotent derivations (see Essen, 1995; Ferrero, 1992). Finally, we show that locally nilpotent derivations on a Boolean semiring possesing some condition must be zero.

2. Preliminaries

In this section we recall the definition of Boolean semirings and some basic properties of Boolean semirings.

Definition 2.1 (Subrahmanyam, 1962) A Boolean semiring is a triple $(B, +, \cdot)$ satisfying the following axioms:

(1) $(B, +)$ is an abelian group,

(2) $(B, \cdot)$ is a semigroup,

(3) $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in B$,

(4) $a \cdot a = a$ for all $a \in B$,

(5) $a \cdot b \cdot c = b \cdot a \cdot c$ for all a, b, c in B.

We denote a Boolean semiring $(B, +, \cdot)$ by B without mentioning the operations, and denote $a \cdot b$ by ab.

Example 2.2 Let $B = \{0, a, b, c\}$. Define addition $(+)$ and multiplication $(\cdot)$ as follows:

+	0	a	b	c
0	0	a	b	c
a	a	b	c	0
b	b	c	0	a
c	c	0	a	b

·	0	a	b	c
0	0	a	b	c
a	0	a	b	c
b	0	a	b	c
c	0	a	b	c

Then $(B, +, \cdot)$ is a Boolean semiring.

Example 2.3 $(\mathbb{Z}_2, +, \cdot)$ is a Boolean semiring where $+$, and $\cdot$ are the addition, and multiplication modulo 2, respectively.

The following lemma summarises some basic properties of Boolean semiring. The proof is straightforward and hence omitted.

LEMMA 2.4 *For any a, b, c in a Boolean semiring B, we have*

(i) $-(-a) = a$,

(ii) $a0 = 0$,

(iii) $a(-b) = -(ab)$,

(iv) $a(b - c) = ab - ac$,

(v) $-(a + b) = -a - b$.

Remark In Boolean semiring, $0a = 0$, $(-a)b = -(ab)$, and $(-a)(-b) = ab$ are not true in general, for instance Example 2.2, note that $0a = a$, $(-a)a \neq -(aa)$ and $(-a)(-a) \neq aa$.

3. Main results

In what follows, let B denote a Boolean semiring unless otherwise specified.

Definition 3.1 Let B be a Boolean semiring. A mapping $d{:}B \to B$ is called a derivation on B if

(i) $d(x + y) = d(x) + d(y)$, and

(ii) $d(xy) = xd(y) + d(x)y$,

for all $x, y \in B$.

Note that $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in B$ because $(B, +)$ is an abeliean group.

LEMMA 3.2 *Let d be a derivation on B. Then B satisfies the following partial distributive laws:*

$$(xd(y) + d(x)y)z = xd(y)z + d(x)yz$$

for all $x, y, z \in B$.

Proof. Let $x, y, z \in B$. Then

$$\begin{aligned} d(x(yz)) &= xd(yz) + d(x)(yz) \\ &= x(yd(z) + d(y)z) + d(x)(yz) \\ &= x(yd(z)) + x(d(y)z) + d(x)(yz), \end{aligned}$$

and

$$\begin{aligned} d((xy)z) &= (xy)d(z) + d(xy)z \\ &= (xy)d(z) + (xd(y) + d(x)y)z. \end{aligned}$$

Comparing the two expressions above, we have $(xd(y) + d(x)y)z = (xd(y))z + (d(x)y)z$. This completes the proof. □

Note that $(d(xy))z = xd(y)z + d(x)yz$.

In the following theorem, we obtain that the neccessary condition for the composition of deivation on a Boolean semiring could be a derivation.

THEOREM 3.3 *Let d_1 and d_2 be derivations on B. Then the composition of d_1 and d_2, $d_1 \circ d_2$ is a derivation on B if and only if $d_1(x)d_2(y) + d_2(x)d_1(y) = 0$ for all $x, y \in B$.*

Proof. Suppose that $d_1 \circ d_2$ is a derivation on B. Let x, $y \in B$. Then

$$(d_1 \circ d_2)(xy) = x(d_1 \circ d_2)(y) + (d_1 \circ d_2)(x)y,$$

and

$$\begin{aligned}(d_1 \circ d_2)(xy) &= d_1(xd_2(y) + d_2(x)y)\\ &= x(d_1 \circ d_2)(y) + d_1(x)d_2(y) + d_2(x)d_1(y) + (d_1 \circ d_2)(x)y.\end{aligned}$$

By comparing the two expressions yields $d_1(x)d_2(y) + d_2(x)d_1(y) = 0$ for all x, $y \in B$.

Conversely, assume that $d_1(x)d_2(y) + d_2(x)d_1(y) = 0$ for all x, $y \in B$. Then

$$\begin{aligned}(d_1 \circ d_2)(xy) &= x(d_1 \circ d_2)(y) + d_1(x)d_2(y) + d_2(x)d_1(y) + (d_1 \circ d_2)(x)y\\ &= x(d_1 \circ d_2)(y) + (d_1 \circ d_2)(x)y,\end{aligned}$$

for all $x, y \in B$. Clearly, $d_1 \circ d_2$ is an additive mapping on B. Thus $d_1 \circ d_2$ is a derivation on B. □

By Theorem 3.3, we obtain the following corollary.

COROLLARY 3.4 *Let d_1 and d_2 be derivations on B. If $d_1 \circ d_2$ is a derivation on B, so is $d_2 \circ d_1$.*

Definition 3.5 Let d be a derivation on B. For any positive interger n, the nth derivation of d is denoted by d^n and is obtained when d is composed with itself n times, and by $d^0(x)$ we mean the identity function defined by $d^0(x) = x$.

Next, we show that the nth derivation satisfying Leibniz rule.

LEMMA 3.6 *Let d be a derivation on B. For any positive integer n,*

$$d^n(xy) = \sum_{i=0}^{n} \binom{n}{i} d^i(x)d^{n-i}(y),$$

for all x, $y \in B$.

Proof. For $n = 1$, we have

$$d^1(xy) = xd(y) + d(x)y,$$

for all x, $y \in B$. Let $n > 1$ and assume that the theorem holds for $n - 1$. That is,

$$d^{n-1}(xy) = \sum_{i=0}^{n-1} \binom{n-1}{i} d^i(x)d^{n-1-i}(y),$$

for all x, $y \in B$. Then

$$
\begin{aligned}
d^n(xy) = d(d^{n-1}(xy)) &= d\left(\sum_{i=0}^{n-1}\binom{n-1}{i}d^i(x)d^{n-1-i}(y)\right)\\
&= \sum_{i=0}^{n-1}\binom{n-1}{i}\left(d^i(x)d^{n-i}(y) + d^{i+1}(x)d^{n-1-i}(y)\right)\\
&= xd^n(y) + d(x)d^{n-1}(y) + \binom{n-1}{1}\left(d(x)d^{n-1}(y) + d^2(x)d^{n-2}(y)\right)\\
&\quad + \cdots + \binom{n-1}{i-1}\left(d^{i-1}(x)d^{n-i+1}(y) + d^i(x)d^{n-i}(y)\right)\\
&\quad + \binom{n-1}{i}\left(d^i(x)d^{n-i}(y) + d^{i+1}(x)d^{n-i-1}(y)\right)\\
&\quad + \cdots + d^{n-1}(x)d(y) + d^n(x)y\\
&= xd^n(y) + \cdots + \left[\binom{n-1}{i-1} + \binom{n-1}{i}\right]d^i(x)d^{n-i}(y) + \cdots + d^n(x)y\\
d^n(xy) &= xd^n(y) + \cdots + \binom{n}{i}d^i(x)d^{n-i}(y) + \cdots + d^n(x)y\\
&= \sum_{i=0}^{n}\binom{n}{i}d^i(x)d^{n-i}(y).
\end{aligned}
$$

The proof is completed. □

The characteristic of a Boolean semiring B is the smallest positive integer n such that

$$\underbrace{a + a + \cdots + a}_{n \text{ times}} = 0 \qquad \text{for each } a \in B.$$

COROLLARY 3.7 *Let B be a Boolean semiring of characteristic prime $p \geq 2$ and let d be a derivation on B. Then d^p is a derivation on B.*

Proof. Let $x, y \in B$. By Theorem 3.6, we obtain

$$d^p(xy) = \sum_{i=0}^{p}\binom{p}{i}d^i(x)d^{p-i}(y).$$

Since the characteristic of B is equal to p and p divides $\binom{p}{i}$ for all $1 \leq k \leq p-1$, we have

$$d^p(xy) = \binom{p}{0}xd^p(y) + \binom{p}{p}d^p(x)y.$$

Clearly, d^p is a additive mapping on B. Hence d^p is a derivation on B. □

Definition 3.8 A derivation d on a Boolean semiring B is said to be locally nilpotent if for any $b \in B$, there exists a poistive integer n such that $d^n(b) = 0$.

Definition 3.9 A Boolean semiring B is called zero-symmtric if $0a = 0$ for all $a \in B$.

LEMMA 3.10 *Let B be a zero-symmtric Boolean semiring, $x \in B$, and d a derivation on B. If $d^2(x) = 0$, then $d(x) = 0$.*

Proof. Assume that $d^2(x) = 0$. Then

$$0 = d^2(x) = d^2(xx) = xd^2(x) + 2d(x)d(x) + d^2(x)x = 2d(x),$$

and by Lemma 3.2.,

$$\begin{aligned} d(x) &= d(xx)d(x) = (xd(x) + d(x)x)d(x) \\ &= (xd(x))d(x) + (d(x)x)d(x) = x(d(x) + d(x)) = 0. \end{aligned}$$

Therefore, $d(x) = 0$. □

LEMMA 3.11 *Let B be a zero-symmtric Boolean semiring, and let $x \in B$. If d is a derivation on B satisfying $d^n(x) = 0$ for some positive integer $n \geq 2$, then $d^{n-1}(x) = 0$.*

Proof. Assume that $d^n(x) = 0$ for some positive integer $n \geq 2$. Then

$$\begin{aligned} 0 = d^n(x) &= d(d(d^{n-2}(x)d^{n-2}(x))) \\ &= d^{n-2}(x)d^n(x) + d^{n-1}(x)d^{n-1}(x) + d^{n-1}(x)d^{n-1}(x) \\ &+ d^n(x)d^{n-2}(x) \end{aligned}$$

Hence $d^{n-1}(x)d^{n-1}(x) + d^{n-1}(x)d^{n-1}(x) = 0$. We then have

$$\begin{aligned} 0 &= d^{n-1}(x)d(d^{n-2}(x)d^{n-2}(x)) + d(d^{n-2}(x)d^{n-2}(x))d^{n-1}(x) \\ &= d^{n-1}(x)d^{n-2}(x)d^{n-1}(x) + d^{n-1}(x)d^{n-1}(x)d^{n-2}(x) \\ &\quad + d^{n-2}(x)d^{n-1}(x)d^{n-1}(x) + d^{n-1}(x)d^{n-2}(x)d^{n-1}(x) \\ &= d^{n-2}(x)(d^{n-1}(x)d^{n-1}(x) + d^{n-1}(x)d^{n-1}(x)) \\ &\quad + d^{n-1}(x)d^{n-2}(x) + d^{n-2}(x)d^{n-1}(x). \end{aligned}$$

Hence $d^{n-1}(x)d^{n-2}(x) + d^{n-2}(x)d^{n-1}(x) = 0$. It follows that

$$d^{n-1}(x) = d(d^{n-2}(x)d^{n-2}(x)) = d^{n-1}(x)d^{n-2}(x) + d^{n-2}(x)d^{n-1}(x) = 0$$

Therefore, $d^{n-1}(x) = 0$. □

THEOREM 3.12 *Let B be a zero-symmtric Boolean semiring. If d is a locally nilpotent derivation on B, then $d(B) = 0$.*

The proof is immediately obtained by Lemma 3.11.

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