Jabari Sabeg et al., *Cogent Mathematics* (2017), 4: 1347017
https://doi.org/10.1080/23311835.2017.1347017

cogent mathematics

APPLIED & INTERDISCIPLINARY MATHEMATICS | RESEARCH ARTICLE

A new operational matrix for solving two-dimensional nonlinear integral equations of fractional order

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Received: 09 February 2017
Accepted: 13 June 2017
First Published: 27 June 2017

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Reviewing editor:
Xinguang Zhang, Curtin University, Australia

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Abstarct: In this paper, first, we derive the operational matrix of two-dimensional orthogonal triangular functions (2D-TFs) for two-dimensional fractional integrals. Then, we apply this operational matrix and properties of Two-dimensional orthogonal triangular functions to reduce two-dimensional fractional integral equations to a system of algebraic equations. Finally, in order to show the validity and efficiency, we present some numerical examples.

Subjects: Science; Mathematics & Statistics; Applied Mathematics; Computer Mathematics

Keywords: two-dimensional orthogonal triangular functions; two-dimensional fractional integral equations; operational matrix

1. Introduction

As a branch of mathematics, fractional calculus provides an excellent tool for describing and modeling such complex engineering and scientific phenomena as fluid-dynamic traffic model (He, 1999), model frequency-dependent damping behavior of viscoelastic materials (Bagley & Torvik, 1983, 1985), economics (Baillie, 1996), continuum and statistical mechanics (Mainardi, 1997), solid

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PUBLIC INTEREST STATEMENT

As we know, we can convert many initial and boundary value problems into problems of solving integral equations. So, it is important to develop numerical methods for solving integral equations. In this article, after deriving the operational matrix of two-dimensional orthogonal triangular functions for two-dimensional fractional integrals, we reduce two-dimensional fractional integral equations to a system of algebraic equations by applying this operational matrix. It is necessary to say that the introduced operational matrix in this paper can be applied for solving differential equations of fractional order, integro-differential equations of fractional order, etc.

mechanics (Rossikhin & Shitikova, 1997), and dynamics of interfaces between soft-nanoparticles and rough substrates (Chow, 2005). Several numerical methods to solve fractional differential equations and fractional integro-differential equations have been recently presented by many authors. In EI-Wakil, Elhanbaly, and Abdou (2006) authors used Adomian decomposition method for Fractional nonlinear differential equations. Saadatmandi and Dehghan in Saadatmandi and Dehghan (2010) used the Legendre operational matrix to solve fractional-order differential equations. In Saadatmandi (2014), Bernstein polynomials were used for solving partial differential equations. In Maleknejad and Asgari (2015) used triangular functions for multi-order fractional differential equations. In Chen, Liu, Turner, and Anh (2013), two-dimensional fractional percolation equation was solved. In Najafalizadeh and Ezzati (2016) we see two-dimensional block pulse operational matrix is used for two-dimensional nonlinear integral equations of fractional order.

Here, we try to extend the application of 2D-TFs to solve two-dimensional nonlinear integral equations of fractional order in. Our main aim is to obtain 2D-TFs operational matrix for two-dimensional fractional integral to reduce the original problem to a system of algebraic equations. In this paper first, we briefly review fractional calculus and one-dimensional triangular functions (1D-TFs). In Section 3, we present the approximation of function via 2D-TFs. Also, by using the properties of 2D-TFs, we derive the operational matrix of two-dimensional integration of fractional order. Section 4 is devoted to solving two-dimensional nonlinear fractional integral equations by applying the operational matrix of integration of fractional order introduced in previous section. In Section 5, we show the accuracy and the efficiency of the proposed method through several examples. Finally, a conclusion is given in Section 6.

2. Brief review for fractional calculus

The most commonly used definitions for fractional derivative and fractional integration are Caputo and Riemann–Liouville definitions, respectively.

Definition 2.1 The Riemann–Liouville fractional integral operator I^{α} of order $\alpha \geq 0$ is defined by:

$$I^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{(\alpha-1)} f(t)\,dt\,, \alpha > 0, \tag{2.1}$$

where $x > 0$ and $\Gamma(.)$ is the Euler gamma function (Monje, Chen, Vinagre, Xue, & Feliu, 2010).

The Riemann–Liouville integral satisfy the following properties:

$$\begin{aligned} I^{\alpha} I^{\beta} f(x) &= I^{\alpha+\beta} f(x), \\ I^{\alpha} x^{\beta} &= \frac{\Gamma(\beta+1)}{\Gamma(\alpha+1+\beta)} x^{\alpha+\beta}. \end{aligned} \tag{2.2}$$

Definition 2.2 The left-sided mixed Riemann–Liouville integral of order α of f is defined as Abbas and Benchohra (2014):

$$I_{\theta}^{\alpha} f(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1} (y-t)^{r_2-1} f(s,t)\,dtds, \tag{2.3}$$

where $\alpha = (r_1, r_2) \in (0,\infty) \times (0,\infty)$ and $\theta = (0,0)$.

Some properties of the left-sided mixed Riemann–Liouville integral are the following:

$$I_{\theta}^{\theta} f(x,y) = f(x,y),$$

if $p, g \in (-1, \infty)$, then

$$I_{\theta}^{\alpha} x^p y^q = \frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+1+r_1)\Gamma(q+1+r_2)} x^{p+r_1} y^{q+r_2}. \tag{2.4}$$

2.1. One-dimensional triangular functions

Triangular functions are among orthogonal functions that are introduced by authors of Deb, Dasgupta, and Sarkar (2006), Deb, Sarkar, and Dasgupta (2007). Maleknejad and Asgari (2015) applied these functions to solve nonlinear integro-differential equations of fractional order. In Deb et al. (2006), an m-set of 1D-TFs over interval $[0, T)$ are defined as:

$$T1_i(t) = \begin{cases} 1 - \frac{t-ih}{h} & ih \le t < (i+1)h, \\ 0 & \text{otherwise} \end{cases} \qquad T2_i(t) = \begin{cases} \frac{t-ih}{h} & ih \le t < (i+1)h, \\ 0 & \text{otherwise} \end{cases} \tag{2.5}$$

where $i = 0, 1, 2, \cdots, m-1$. And $h = \frac{T}{m}$.

Clearly, we can define m-set of 1D-TF vectors as the following:

$$T1(t) = [T1_0(t), T1_1(t), \cdots, T1_{m-1}(t)]^T,$$

$$T2(t) = [T2_0(t), T2_1(t), \cdots, T2_{m-1}(t)]^T,$$

and

$$T(t) = [T1(t), T2(t)]^T.$$

The vector $T(t)$ is called 1D-TFs vector.

The operational matrix for fractional integration can be obtained as Maleknejad and Asgari (2015):

$$I^\alpha T1(t) = p_1^\alpha T1(t) + p_2^\alpha T2(t), \tag{2.6}$$

$$I^\alpha T2(t) = p_3^\alpha T1(t) + p_4^\alpha T2(t), \tag{2.7}$$

where

$$p_1^\alpha = \begin{pmatrix} 0 & \xi_1 & \xi_2 & \cdots & \xi_{m-1} \\ 0 & 0 & \xi_1 & \cdots & \xi_{m-2} \\ 0 & 0 & 0 & \cdots & \xi_{m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, p_2^\alpha = \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 & \cdots & \xi_m \\ 0 & \xi_1 & \xi_2 & \cdots & \xi_{m-1} \\ 0 & 0 & \xi_1 & \cdots & \xi_{m-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \xi_1 \end{pmatrix}$$

$$p_3^\alpha = \begin{pmatrix} 0 & \zeta_1 & \zeta_2 & \cdots & \zeta_{m-1} \\ 0 & 0 & \zeta_1 & \cdots & \zeta_{m-2} \\ 0 & 0 & 0 & \cdots & \zeta_{m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, p_4^\alpha = \begin{pmatrix} \zeta_1 & \zeta_2 & \zeta_3 & \cdots & \zeta_m \\ 0 & \zeta_1 & \zeta_2 & \cdots & \zeta_{m-1} \\ 0 & 0 & \xi_1 & \cdots & \xi_{m-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \zeta_1 \end{pmatrix}$$

and

$$\xi_r = \frac{h^\alpha}{\Gamma(\alpha+2)}((\alpha+1)r^\alpha - r^{\alpha+1} + (r-1)^{\alpha+1}),$$

$$\zeta r = \frac{h^\alpha}{\Gamma(\alpha+2)}(r^{\alpha+1} - (r-1)^{\alpha+1} - (\alpha+1)(r-1)^\alpha),$$

so

$$I^{\alpha}T(t)=P^{\alpha}T(t)$$

where P_{α} fractional integration operational matrix of $T(t)$, is

$$\begin{pmatrix} p_1^{\alpha} & p_2^{\alpha} \\ p_3^{\alpha} & p_4^{\alpha} \end{pmatrix}. \tag{2.8}$$

3. Two-dimensional triangular functions (2D-TFs)

In Babolian, Maleknejad, Roodaki, and Almasieh (2010), authors defined an $m_1 \times m_2$ -set of 2D-TFs on $[0,T_1)\times[0,T_2)$ as follows:

$$T^{1,1}_{i,j}(s,t)=\begin{cases}(1-\frac{s-ih_1}{h_1})(1-\frac{t-jh_2}{h_2}) & ih_1\le s<(i+1)h_1,\ jh_2\le t<(j+1)h_2,\\ 0 & \text{otherwise}\end{cases}$$

$$T^{1,2}_{i,j}(s,t)=\begin{cases}(1-\frac{s-ih_1}{h_1})(\frac{t-jh_2}{h_2}) & ih_1\le s<(i+1)h_1,\ jh_2\le t<(j+1)h_2,\\ 0 & \text{otherwise}\end{cases}$$

$$T^{2,1}_{i,j}(s,t)=\begin{cases}(\frac{s-ih_1}{h_1})(1-\frac{t-jh_2}{h_2}) & ih_1\le s<(i+1)h_1,\ jh_2\le t<(j+1)h_2,\\ 0 & \text{otherwise}\end{cases}$$

$$T^{2,2}_{i,j}(s,t)=\begin{cases}(\frac{s-ih_1}{h_1})(\frac{t-jh_2}{h_2}) & ih_1\le s<(i+1)h_1,\ jh_2\le t<(j+1)h_2,\\ 0 & \text{otherwise}\end{cases}$$

where $i=0,1,\cdots,m_1-1$, $j=0,1,\cdots,m_2-1$, and $h_1=\frac{T_1}{m_1}, h_2=\frac{T_2}{m_2}$. m_1 and m_2 are arbitrary positive integers. Also, they defined the following vectors:

$$\begin{aligned}
T11(s,t)&=[T^{11}_{0,0}(s,t),T^{11}_{0,1}(s,t),\cdots,T^{11}_{m_1-1,m_2-1}(s,t)]^T,\\
T12(s,t)&=[T^{12}_{0,0}(s,t),T^{12}_{0,1}(s,t),\cdots,T^{12}_{m_1-1,m_2-1}(s,t)]^T,\\
T21(s,t)&=[T^{21}_{0,0}(s,t),T^{21}_{0,1}(s,t),\cdots,T^{21}_{m_1-1,m_2-1}(s,t)]^T,\\
T22(s,t)&=[T^{22}_{0,0}(s,t),T^{22}_{0,1}(s,t),\cdots,T^{22}_{m_1-1,m_2-1}(s,t)]^T.
\end{aligned} \tag{3.1}$$

With the following properties:

$$\begin{aligned}
T^{1,1}_{i,j}(s,t)&=T1_i(s).T1_j(t),\\
T^{1,2}_{i,j}(s,t)&=T1_i(s).T2_j(t),\\
T^{2,1}_{i,j}(s,t)&=T2_i(s).T1_j(t),\\
T^{2,2}_{i,j}(s,t)&=T2_i(s).T2_j(t).
\end{aligned} \tag{3.2}$$

By considering above vectors, authors of Babolian et al. (2010) defined 2D-TF vector as the following form:

$$T(s,t)=[T11(s,t),T12(s,t),T21(s,t),T22(s,t)]^T. \tag{3.3}$$

According to this fact , to construct the operational matrix of 2D-TFs for the fractional integration in Section 4, we need to derive $T11(s,t)$, $T12(s,t)$, $T21(s,t)$, and $T22(s,t)$ by Kronecker product of $T1(t)$ and $T1(s)$. So, using (3.1) and (3.2), we can write

$$\begin{aligned}
T11(s,t)&=T1(s)\otimes T1(t),\\
T12(s,t)&=T1(s)\otimes T2(t),\\
T21(s,t)&=T2(s)\otimes T1(t),\\
T22(s,t)&=T2(s)\otimes T2(t),
\end{aligned} \tag{3.4}$$

where $\otimes$ denotes the Kronecker product defined for two arbitrary matrices A and B as

$$A \otimes B = (a_{ij}B),$$

and also it has the following two basic properties (Zhang & Ding, 2013):

$$\begin{aligned}&(A \otimes B)(C \otimes D) = (AC) \otimes (BD),\\&(A + B) \otimes C = A \otimes C + B \otimes C.\end{aligned} \tag{3.5}$$

In Babolian et al. (2010), it is proved that 2D-TFs are disjoint, orthogonal. Thus, for every $(4m_1m_2 \times 4m_1m_2)$-matrix B, we can write

$$T^T(s,t).B.T(s,t) \simeq \tilde{B}.T(s,t) \tag{3.6}$$

where $\tilde{B}$ is a $4m_1m_2$-vector with elements equal to the diagonal entries of matrix B.

Also,

$$T(s,t).T^T(s,t).X \simeq \tilde{X}.T(s,t) \tag{3.7}$$

where X is a $4m_1m_2$-vector and $\tilde{X} = diag(X)$

Regarding to orthogonality of 2D-TFs, a function $f(s, t)$ defined over $([0, T_1) \times [0, T_2))$ can be expanded by 2D-TFs as Babolian et al. (2010):

$$\begin{aligned}f(s,t) \simeq& \sum_{i=0}^{m_1-1}\sum_{j=0}^{m_2-1} c_{i,j}T_{i,j}^{1,1}(s,t) + \sum_{i=0}^{m_1-1}\sum_{j=0}^{m_2-1} d_{i,j}T_{i,j}^{1,2}(s,t)+\\&\sum_{i=0}^{m_1-1}\sum_{j=0}^{m_2-1} e_{i,j}T_{i,j}^{2,1}(s,t) + \sum_{i=0}^{m_1-1}\sum_{j=0}^{m_2-1} l_{i,j}T_{i,j}^{2,2}(s,t)\\&= C^TT11(s,t) + D^TT11(s,t) + E^TT21(s,t) + L^TT22(s,t)\\&= F^TT(s,t)\end{aligned} \tag{3.8}$$

where F is a $4m_1m_2$-vector given by:

$$F = [C^T, D^T, E^T, L^T]^T$$

and

$$\begin{aligned}c_{i,j} &= f(ih_1, jh_2),\\d_{i,j} &= f(ih_1, (j+1)h_2),\\e_{i,j} &= f((i+1)h_1, jh_2),\\l_{i,j} &= f((i+1)h_1, (j+1)h_2),\end{aligned} \tag{3.9}$$

the vector F is called the 2D-TFs coefficient vector.

Authors of Babolian et al. (2010) prove that

$$[f(s,t)]^p \simeq F_p^T T(s,t) \tag{3.10}$$

where F_p is a column vector whose elements are pth powers of the elements of the vector F and p is the positive integer. Also, for a function $k(x, y, s, t)$ defined on $([0, T_1) \times [0, T_2) \times [0, T_3) \times [0, T_4))$, we have Babolian et al. (2010):

$$k(x,y,s,t) \simeq T^T(x,y)KT(s,t), \tag{3.11}$$

where $T(x, y)$ and $T(s, t)$ are 2D-TFs vectors of dimension $4m_1m_2$ and $4m_3m_4$, respectively, and K is a $(4m_1m_2 \times 4m_3m_4)$ 2D-TFs coefficient matrix.

4. Operational matrix of 2D-TFs for the fractional integration

In this section, we construct operational matrix of 2D-TFs for the fractional integration.

Using Equations (2.3)–(3.3), we have:

$$\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}T(s,t)\,dtds$$
$$= \begin{pmatrix} \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}T11(s,t)\,dtds \\ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}T12(s,t)\,dtds \\ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}T21(s,t)\,dtds \\ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}T22(s,t)\,dtds \end{pmatrix}$$

Using Equation (3.4), we conclude that

$$\begin{pmatrix} \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}T1(s)\otimes T1(t)dtds \\ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}T1(s)\otimes T2(t)dtds \\ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}T2(s)\otimes T1(t)dtds \\ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}T2(s)\otimes T2(t)dtds \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{\Gamma(r_1)}\int_0^x (x-s)^{(r_1-1)}T1(s)\,ds \otimes \frac{1}{\Gamma(r_2)}\int_0^y (y-t)^{(r_2-1)}T1(t)\,dt \\ \frac{1}{\Gamma(r_1)}\int_0^x (x-s)^{(r_1-1)}T1(s)\,ds \otimes \frac{1}{\Gamma(r_2)}\int_0^y (y-t)^{(r_2-1)}T2(t)\,dt \\ \frac{1}{\Gamma(r_1)}\int_0^x (x-s)^{(r_1-1)}T2(s)\,ds \otimes \frac{1}{\Gamma(r_2)}\int_0^y (y-t)^{(r_2-1)}T1(t)\,dt \\ \frac{1}{\Gamma(r_1)}\int_0^x (x-s)^{(r_1-1)}T2(s)\,ds \otimes \frac{1}{\Gamma(r_2)}\int_0^y (y-t)^{(r_2-1)}T2(t)\,dt \end{pmatrix}$$

and also, by applying Equations (2.6)–(2.7), we obtain

$$\begin{pmatrix} (p_1^{r_1}T1(x)+p_2^{r_1}T2(x))\otimes((p_1^{r_2}T1(y)+p_2^{r_2}T2(y)) \\ (p_1^{r_1}T1(x)+p_2^{r_1}T2(x))\otimes((p_3^{r_2}T1(y)+p_4^{r_2}T2(y)) \\ (p_3^{r_1}T1(x)+p_4^{r_1}T2(x))\otimes((p_1^{r_2}T1(y)+p_2^{r_2}T2(y)) \\ (p_3^{r_1}T1(x)+p_4^{r_1}T2(x))\otimes((p_3^{r_2}T1(y)+p_4^{r_2}T2(y)) \end{pmatrix} =$$
$$\begin{pmatrix} (p_1^{r_1}T1(x)\otimes p_1^{r_2}T1(y))+(p_1^{r_1}T1(x)\otimes p_2^{r_2}T2(y))+(p_2^{r_1}T2(x)\otimes p_1^{r_2}T1(y))+(p_2^{r_1}T2(x)\otimes p_2^{r_2}T2(y)) \\ (p_1^{r_1}T1(x)\otimes p_3^{r_2}T1(y))+(p_1^{r_1}T1(x)\otimes p_4^{r_2}T2(y))+(p_2^{r_1}T2(x)\otimes p_3^{r_2}T1(y))+(p_2^{r_2}T2(x)\otimes p_4^{r_2}T2(y)) \\ (p_3^{r_1}T1(x)\otimes p_1^{r_2}T1(y))+(p_3^{r_1}T1(x)\otimes p_2^{r_2}T2(y))+(p_4^{r_1}T2(x)\otimes p_1^{r_2}T1(y))+(p_4^{r_1}T2(x)\otimes p_2^{r_2}T2(y)) \\ (p_3^{r_1}T1(x)\otimes p_3^{r_2}T1(y))+(p_3^{r_1}T1(x)\otimes p_4^{r_2}T2(y))+(p_4^{r_1}T2(x)\otimes p_3^{r_2}T1(y))+(p_4^{r_1}T2(x)\otimes p_4^{r_2}T2(y)) \end{pmatrix} =*$$

By applying Equation (3.5), we get

$$*= \begin{pmatrix} (p_1^{r_1}\otimes p_1^{r_2})(T1(x)\otimes T1(y))+(p_1^{r_1}\otimes p_2^{r_2})(T1(x)\otimes T2(y))+(p_2^{r_1}\otimes p_1^{r_2})(T2(x)\otimes T1(y))+(p_2^{r_1}\otimes p_2^{r_2})(T2(x)\otimes T2(y)) \\ (p_1^{r_1}\otimes p_3^{r_2})(T1(x)\otimes T1(y))+(p_1^{r_1}\otimes p_4^{r_2})(T1(x)\otimes T2(y))+(p_2^{r_1}\otimes p_3^{r_2})(T2(x)\otimes T1(y))+(p_2^{r_1}\otimes p_4^{r_2})(T2(x)\otimes T2(y)) \\ (p_3^{r_1}\otimes p_1^{r_2})(T1(x)\otimes T1(y))+(p_3^{r_1}\otimes p_2^{r_2})(T1(x)\otimes T2(y))+(p_4^{r_1}\otimes p_1^{r_2})(T2(x)\otimes T1(y))+(p_4^{r_1}\otimes p_2^{r_2})(T2(x)\otimes T2(y)) \\ (p_3^{r_1}\otimes p_3^{r_2})(T1(x)\otimes T1(y))+(p_3^{r_1}\otimes p_4^{r_2})(T1(x)\otimes T2(y))+(p_4^{r_1}\otimes p_3^{r_2})(T2(x)\otimes T1(y))+(p_4^{r_1}\otimes p_4^{r_2})(T2(x)\otimes T2(y)) \end{pmatrix}$$

Now, by employing Equation (3.4) we have

$$\begin{pmatrix} (p_1^{r_1} \otimes p_1^{r_2})T11(x,y) + (p_1^{r_1} \otimes p_2^{r_2})T12(x,y) + (p_2^{r_1} \otimes p_1^{r_2})T21(x,y) + (p_2^{r_1} \otimes p_2^{r_2})T22(x,y) \\ (p_1^{r_1} \otimes p_3^{r_2})T11(x,y) + (p_1^{r_1} \otimes p_4^{r_2})T12(s,t) + (p_2^{r_1} \otimes p_3^{r_2})T21(x,y) + (p_2^{r_1} \otimes p_4^{r_2})T22(x,y) \\ (p_3^{r_1} \otimes p_1^{r_2})T11(x,y) + (p_3^{r_1} \otimes p_2^{r_2})T12(x,y) + (p_4^{r_1} \otimes p_1^{r_2})T21(x,y) + (p_4^{r_1} \otimes p_2^{r_2})T22(x,y) \\ (p_3^{r_1} \otimes p_3^{r_2})T11(x,y) + (p_3^{r_1} \otimes p_4^{r_2})T12(x,y) + (p_4^{r_1} \otimes p_3^{r_2})T21(x,y) + (p_4^{r_1} \otimes p_4^{r_2})T22(x,y) \end{pmatrix}$$

$$= p^{r_1,r_2} \begin{pmatrix} T11(x,y) \\ T12(x,y) \\ T21(x,y) \\ T22(x,y) \end{pmatrix},$$

where $P^{r_1,r_2}_{4m_1m_2 \times 4m_1m_2}$, operational matrix of fractional integration of T(x,y), is

$$p^{r_1,r_2} = \begin{pmatrix} p_1^{r_1} \otimes p_1^{r_2} & p_1^{r_1} \otimes p_2^{r_2} & p_2^{r_1} \otimes p_1^{r_2} & p_2^{r_1} \otimes p_2^{r_2} \\ p_1^{r_1} \otimes p_3^{r_2} & p_1^{r_1} \otimes p_4^{r_2} & p_2^{r_1} \otimes p_3^{r_2} & p_2^{r_1} \otimes p_4^{r_2} \\ p_3^{r_1} \otimes p_1^{r_2} & p_3^{r_1} \otimes p_2^{r_2} & p_4^{r_1} \otimes p_1^{r_2} & p_4^{r_1} \otimes p_2^{r_2} \\ p_3^{r_1} \otimes p_3^{r_2} & p_3^{r_1} \otimes p_4^{r_2} & p_4^{r_1} \otimes p_3^{r_2} & p_4^{r_1} \otimes p_4^{r_2} \end{pmatrix}.$$

Hence,

$$\frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} T(s,t)\, dtds \; = p^{r_1,r_2} \begin{pmatrix} T11(x,y) \\ T12(x,y) \\ T21(x,y) \\ T22(x,y) \end{pmatrix}. \tag{4.1}$$

5. Numerical solution of two-dimensional nonlinear fractional integral equations

In this section, we present an effective method to solve two-dimensional nonlinear integral equations of fractional order. For this purpose, we apply two-dimensional triangular functions to approximate known and unknown functions, whose properties of these functions were shown in Section 3. Consider the following two-dimensional nonlinear fractional integral equation

$$f(x,y) - \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} k(x,y,s,t)[f(s,t)]^p \, dtds = g(x,y) \tag{5.1}$$

where $r_1 > 0, r_2 > 0$, the functions $k(x, y, s, t)$ and $g(x, y)$ are known and $f(x, y)$ is the unknown function to be determined. Also, $p \geq 1$ is a positive integers. Using the methods mentioned in Section 4, the functions $f(x, y)$, $g(x, y)$, $[f(x,y)]^p$, and $k(x, y, s, t)$ can be approximated by:

$$\begin{aligned} f(x,y) &= T(x,y)^T F, \\ g(x,y) &= T(x,y)^T G, \\ [f(x,y)]^p &= T(x,y)^T F_p, \\ k(x,y,s,t) &= T(x,y)^T K T(s,t), \end{aligned} \tag{5.2}$$

where $T(x, y)$ is defined in Equation (3.3), the vectors F, G, F_p and matrix K are 2D-TFs coefficients of $f(x, y)$, $g(x, y)$, $[f(x,y)]^p$, and $k(x, y, s, t)$, respectively. Now, by substituting Equation (5.2) in Equation (5.1), we have

$$T^T(x,y)F - \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} T^T(x,y) K T(s,t) T^T(s,t) F_p \, dtds = T^T(x,y)G.$$

Using Equation (3.7), we conclude that

$$T^T(x,y)F - \frac{T^T(x,y)K\tilde{F}_p}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}T(s,t)\,dtds = T^T(x,y)G. \tag{5.3}$$

Substituting Equation (4.1) in Equation (5.3), we have

$$T^T(x,y)F - T^T(x,y)K\tilde{F}_p P^{r_1,r_2}T(x,y) = T^T(x,y)G \tag{5.4}$$

Clearly, by assuming $H = K\tilde{F}_p P^{r_1,r_2}$, we will get

$$T^T(x,y)F - T^T(x,y)HT(x,y) = T^T(x,y)G.$$

Now, using (3.7), we have:

$$T^T(x,y)F - T^T(x,y)\tilde{H} = T^T(x,y)G$$

Since $\tilde{F}_p$ is a diagonal matrix, we conclude that

$$\tilde{H} = BF_p,$$

where B is $(4m_1m_2 \times 4m_1m_2)$- matrix with components $B_{i,j} = K_{i,j}P_{i,j}^{r_1,r_2}$. Hence, we will get the following nonlinear algebraic system:

$$F - BF_p = G. \tag{5.5}$$

Clearly, this system can be solved by known methods such as Newton's method. After solving (5.5), we can obtain the approximate solution of (5.1) using (3.8).

6. Illustrative examples

To illustrate the effectiveness of the proposed method in the Section 5, we present three test examples. In these examples, we assume that $T_1 = T_2 = 1$, $m_1 = m_2$. Also, in this section, we apply the following error function

$$e(x,y) = |f(x,y) - \tilde{f}_{m_1,m_2}(x,y)|$$

where $f(x, y)$ and $\tilde{f}_{m_1,m_2}(x,y)$ are the exact and the approximate solutions of the two-dimensional fractional integral Equation (5.1), respectively.

Example 6.1 (Najafalizadeh & Ezzati, 2016) Consider the following two-dimensional fractional integral equation:

$$f(x,y) - \frac{1}{\Gamma(\frac{3}{2})\Gamma(\frac{5}{2})} \int_0^x \int_0^y (x-s)^{\frac{1}{2}}(y-t)^{\frac{3}{2}} \sqrt{xyt}[f(s,t)]^2\,dtds = \sqrt{y}\left(\frac{-1}{180}x^3y^{\frac{7}{2}} + \sqrt{\frac{x}{3}}\right),$$

whose exact solution is given by $f(x,y) = \frac{\sqrt{3xy}}{3}$.

The approximate solution of $f(x, y)$ is obtained using 2D-TFs method described in Section 5. Table 1 shows a comparison of the proposed method and the method of Najafalizadeh and Ezzati (2016). The displayed results show that the proposed method is more accurate than the proposed method in Najafalizadeh et al. (2016).

Table 1. Numerical results for Example 6.1

Absolute error			
	Proposed method	Proposed method	Method of Najafalizadeh and Ezzati (2016)
$x = y$	$m = 6$	$m = 8$	$m = 16$
0.0	0	0	0.01845
0.1	0.020069	0.008448	0.25361
0.2	0.00764	0.002634	0.01714
0.3	0.000789	0.005404	0.0256
0.4	0.003448	0.000709	0.01582
0.5	0.006956	0.007308	0.0266
0.6	0.000093	0.000053	0.01451
0.7	0.000335	0.000044	0.0256
0.8	0.001909	0.001421	0.0132
0.9	0.003807	0.003041	0.0186
0.99	0.00549	0.005405	0.01195
Max error	2.006×10^{-2}	8.45×10^{-3}	2.66×10^{-2}

Example 6.2 Consider the following two-dimensional fractional integral equation:

$$f(x,y) - \frac{1}{\Gamma(\frac{7}{2})\Gamma(\frac{7}{2})} \int_0^x \int_0^y (x-s)^{\frac{5}{2}} (y-t)^{\frac{5}{2}} xy \sqrt{t} f(s,t)\, dtds = \frac{1}{2}xy - \frac{1}{9450} x^{\frac{11}{2}} y^6.$$

The exact solution of this example is $f(x,y) = \frac{1}{2}xy$. Table 2 illustrates the numerical results for this example.

Example 6.3 As a last example, we present the following two-dimensional fractional integral equation:

$$f(x,y) - \frac{1}{\Gamma(\frac{9}{2})\Gamma(\frac{3}{2})} \int_0^x \int_0^y (x-s)^{\frac{7}{2}} (y-t)^{\frac{1}{2}} 5\sqrt{s}(x-y)[f(s,t)]^2\, dtds$$

$$= x\left(y - 1 - \frac{x^6 y^{\frac{3}{2}} (x-y)(8y^2 - 28y + 35)}{14112}\right),$$

Table 2. The numerical results for Example 6.2

$x = y$	$m = 4$	$m = 6$	$m = 8$	Exact solution
0.0	0	0	0	0
0.1	0.00516632	0.00522855	0.0051126	0.005
0.2	0.0202946	0.0201507	0.0201363	0.02
0.3	0.0450777	0.0451359	0.0450622	0.045
0.4	0.080057	0.0800271	0.0800127	0.08
0.5	0.124982	0.124982	0.124983	0.125
0.6	0.179968	0.179968	0.179954	0.18
0.7	0.244952	0.244949	0.244948	0.245
0.8	0.319932	0.319933	0.319932	0.32
0.9	0.404914	0.404914	0.404914	0.405
0.99	0.489946	0.489946	0.489946	0.49005
Max error	2.94×10^{-4}	2.28×10^{-4}	2.11×10^{-4}	0

Table 3. The numerical results for Example 6.3

$x = y$	$m = 4$	$m = 6$	$m = 10$	Exact solution
0.0	0	0	0	0
0.1	−0.086608	−0.081108	−0.089999	−0.09
0.2	−0.146444	−0.159541	−0.16	−0.16
0.3	−0.209838	−0.203193	−0.21	−0.21
0.4	−0.238619	−0.239545	−0.24	−0.24
0.5	−0.25	−0.25	−0.25	−0.25
0.6	−0.239979	−0.239862	−0.24	−0.24
0.7	−0.209905	−0.210004	−0.21	−0.21
0.8	−0.160028	−0.15996	−0.16	−0.16
0.9	−0.090042	−0.090035	−0.09	−0.09
0.99	−.009906	−0.009905	−0.009904	−0.0099
Max error	1.35×10^{-2}	8.89×10^{-3}	4.28×10^{-6}	0

with the exact solution $f(x, y) = xy - x$. Table 3 illustrates the numerical results for this example.

7. Conclusion

A general formulation for 2D-TFs operational matrix for two-dimensional fractional integral equations has been derived. This matrix is used to approximate the numerical solution of the two-dimensional nonlinear fractional integral equations. The properties of 2D-TFs and the operational matrix are used to reduce the problem to a system of algebraic equations that can be solved by known methods. Finally, we presented three numerical examples to demonstrate the validity and applicability of the proposed method.

Funding
The authors received no direct funding for this research.

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Citation information
Cite this article as: A new operational matrix for solving two-dimensional nonlinear integral equations of fractional order, D. Jabari Sabeg, R. Ezzati & K. Maleknejad, *Cogent Mathematics* (2017), 4: 1347017.

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