AL-Hawary, *Cogent Mathematics* (2017), 4: 1343518
https://doi.org/10.1080/23311835.2017.1343518

cogent mathematics

PURE MATHEMATICS | RESEARCH ARTICLE

Fuzzy *W*-closed sets

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Abstarct: In this paper, we introduce the relatively new notion of fuzzy *W*-closed and fuzzy *W*-generalized closed sets. Several properties and connections to other well-known weak and strong fuzzy closed sets are discussed. Fuzzy *W*-generalized continuous and fuzzy *W*-generalized irresolute functions and their basic properties and relations to other fuzzy continuities are explored.

Subjects: Advanced Mathematics; Applied Mathematics; Foundations & Theorems

Keywords: Fuzzy *W*-open set; fuzzy *W*-closed set; fuzzy *W*-generalized closed; fuzzy *W*-generalized continuous function

AMS subject classifications: 54C08; 54H40

Received: 20 April 2017
Accepted: 13 June 2017
First Published: 26 June 2017

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Reviewing editor:
Hari M. Srivastava, University of Victoria, Canada

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1. Introduction

Fuzzy topological spaces were introduced by Chakrabarty and Ahsanullah (1992) and Chang (1968). Let $(X, \mathfrak{T})$ be a fuzzy topological space (simply, Fts). If λ is a fuzzy set (simply, F-set), then the closure of λ and the interior of λ will be denoted by $Cl_{\mathfrak{T}}(\lambda)$ and $\text{Int}_{\mathfrak{T}}(\lambda)$, respectively. If no ambiguity appears, we use $\bar{\lambda}$ and λ instead, respectively. A F-set λ is called *fuzzy open* (simply, FO) if $\lambda^o = \lambda$. The complement of an FO-set is called *fuzzy closed* (simply, FC). Clearly λ is a FC-set if and only if $\bar{\lambda} = \lambda$. A F-set λ is called *F-semi-open* (simply, *FSO*) Mahmoud and Fath (2004) if there exists a fuzzy open (simply, F-open) set μ such that $\mu \le \lambda \le Cl_{\mathfrak{T}}(\mu)$. Clearly λ is a FSO-set if and only if $\lambda \le Cl_{\mathfrak{T}}(\text{Int}_{\mathfrak{T}}(\lambda))$. A complement of a FSO-set is called *F-semi-closed* (simply, *FSC*). λ is called *fuzzy preopen* (simply, *FPO*) if $\lambda \le \text{Int}_{\mathfrak{T}}(Cl_{\mathfrak{T}}(\lambda))$. Finally, λ is called *fuzzy generalized closed* (simply, *FGC*) if for every FO-set μ with $\lambda \le \mu$, we have $Cl_{\mathfrak{T}}(\lambda) \le \mu$. The collection of all FSO (resp., FSC and FGC) subsets of *X* will be denoted by $FSO(X, \mathfrak{T})$ (resp., $FSC(X, \mathfrak{T})$ and $FGC(X, \mathfrak{T})$). A fuzzy space is called an *E.D. space* if the closoure of every FSO-set in it is FO. Equivalently, the interior of every FSC-set in it is SC. A fuzzy function $f:(X, \mathfrak{T}) \to (Y, \mathfrak{T}')$ is called *fuzzy continuous* (simply, FCts) if the inverse image of every FC-set is FC. *f* is called *fuzzy generalized continuous* (simply, FGCts) if the inverse image of every FC-set is FGC and is called *fuzzy open* (simply, FO) if the image of every FO-set is FO. For more on the preceding notions, the reader is referred to AL-Hawary (2008, 2017a, 2017b), Chakrabarty and Ahsanullah (1992), Chang (1968), Chaudhuri and Das (1993), Ekici (2007), Mahmoud and Fath (2004), Mursaleen et al. (2016), Nanda (1986), Pritha (2014) and Wong (1974).

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Talal Al-Hawary was born on June 11, 1969 in Jordan. He got his PhD from The University of Montana-USA on Dec. 1997. He published about 50 papers and a book in the fields of Combinatorics (Matroid Theory and Fuzzy Matroids), Fuzzy Graph Theory,Topology and Fuzzy Topology, Operations Research and category theory. He is now working as a full Prof. of Mathematics at Yarmouk University in jordan.

PUBLIC INTEREST STATEMENT

We introduce the relatively new notion of fuzzy weak closed set and fuzzy weak generalized closed sets. Several properties and connections to other well-known weak and strong fuzzy closed sets are discussed. Fuzzy weak generalized continuous and fuzzy weak generalized irresolute functions and their basic properties and relations to other fuzzy continuities are explored.

In this paper, we introduce the relatively new notions of fuzzy W-closed sets, which is closely related to the class of fuzzy closed subsets. We investigate several characterizations of fuzzy W-open and fuzzy W-closed notions via the operations of interior and closure, for more on these notions for crisp sets see Al-Hawary (2004, 2007, 2013a, 2013b) and Al-Hawary and Al-Omari (2006, 2008, 2009). In Section 3, we introduce the notion of fuzzy W-generalized closed sets and study connections to other weak and strong forms of fuzzy generalized closed sets. Section 4 is devoted to introducing and studying fuzzy W-generalized continuous and fuzzy W-generalized irresolute functions and connections with other similar forms of fuzzy continuity.

2. Fuzzy W-closed sets

We begin this section by introducing the notions of fuzzy W-open and fuzzy W-closed subsets.

Definition 1 Let λ be a fuzzy subset of a Fts space $(X, \mathfrak{T})$. The fuzzy *W -interior* of λ is the union of all fuzzy open subsets of X whose closures are contained in $Cl(\lambda)$, and is denoted by $\mathrm{Int}_W(\lambda)$. λ is called fuzzy *W-open* (simply, *FWO*) if $\lambda = \mathrm{Int}_W(\lambda)$. The complement of a fuzzy *FWO* subset is called fuzzy fuzzy W-closed (simply, *FWC*). Alternatively, a fuzzy subset λ of X is fuzzy *FWC* if $\lambda = Cl_W(\lambda)$, where $Cl_W(\lambda) = \bigwedge_{\alpha\in\Delta}\{\lambda_\alpha:\lambda \le \lambda_\alpha, \lambda_\alpha \text{is FC-set in } X\}$.

Clearly $\mathrm{Int}_{\mathfrak{T}}(\lambda) \le \mathrm{Int}_W(\lambda) \le Cl_{\mathfrak{T}}(\lambda)$ and $\lambda \le Cl_{\mathfrak{T}}(\lambda) \le Cl_W(\lambda)$ and hence every *FWC*—set is a FC-set, but the converses needs not be true.

Example 1 Let $X = \{a, b, c\}$ and $\mathfrak{T} = \{0, 1, \chi_{\{a\}}, \chi_{\{b\}}, \chi_{\{a,b\}}\}$. Then $\chi_{\{a,b\}}$ is FC – *set*, but not FϵC since $Cl_\epsilon(\lambda) = 1$.

Even the intersection of two *FWO* subsets needs not be *FWO*.

Example 2 Let $X = \{a, b, c, d\}$ and $\mathfrak{T} = \{0, 1, \chi_{\{b\}}, \chi_{\{c\}}, \chi_{\{a,b\}}, \chi_{\{b,c\}}, \chi_{\{a,b,c\}}\}$. Then $\chi_{\{a,b\}}$ and $\chi_{\{a,c\}}$ are $F\epsilon O$–sets, but $\chi_{\{a,b\}} \wedge \chi_{\{a,c\}} = \chi_{\{a\}}$ is not a $F\epsilon O$–set.

Next, we show that arbitrary unions of *FWO* subsets are *FWO*.

THEOREM 1 *If* $(X, \mathfrak{T})$ *is a fuzzy space, then arbitrary union of FWO subsets are FWO.*

If $\{\lambda_\alpha:\alpha \in \Delta\}$ is a collection of *FWO* subsets of X, then for every $\alpha \in \Delta$, $\mathrm{Int}_W(\lambda\alpha) = \lambda_\alpha$. Hence

$$\begin{aligned}\mathrm{Int}_W(\bigvee_{\alpha\in\Delta}\lambda\alpha) &= \bigvee\{\mu \in \mathfrak{T}:Cl(\mu) \le Cl(\bigvee_{\alpha\in\Delta}\lambda\alpha)\}\\ &= \bigvee\{\mu \in \mathfrak{T}:Cl(\mu) \le \bigvee_{\alpha\in\Delta} Cl(\lambda\alpha)\}\\ &= \bigvee_{\alpha\in\Delta}(\mathrm{Int}_W(\lambda\alpha))\\ &= \bigvee_{\alpha\in\Delta}\lambda_\alpha.\end{aligned}$$

Hence $\bigvee_{\alpha\in\Delta}\lambda\alpha$ is *FWO*.

COROLLARY 1 *Arbitrary intersection of FWC subsets are FWC, while finite unions of FWC subsets need not be FWC.*

In classical topology, the interior of a set is a subset of the set itself. But this is not the case for FWO-sets. Next we show that $\lambda \le \mathrm{Int}_W(\lambda)$ and $\mathrm{Int}_W(\lambda) \le \lambda$ need not be true.

Example 3 Consider the space in Example 1. Then $\chi_{\{c\}} \not\le \mathrm{Int}_\epsilon(\chi_{\{c\}}) = 0$ and $\mathrm{Int}_\epsilon(\chi_{\{a,b\}}) = 1 \not\le \chi_{\{a,b\}}$.

COROLLARY 2 *If* λ *is a fuzzy* ϵ*–dense subset of* X *(*$Cl_\epsilon(\lambda) = 1$*), then* $\mathrm{Int}_\epsilon(\lambda) = 1$.

Lemma 1 *The intersection of a FC-set with a FϵMC − set is FC.*

Let λ be a FC-set and μ be a FWC-set. For all $\gamma \leq Cl_W(\lambda \wedge \mu)$, then for every FO-set η such that $\gamma \leq \eta$, $\eta \wedge (\lambda \wedge \mu) \neq 0$. Hence $\eta \wedge \lambda \neq 0$ and $Cl(\eta) \wedge Cl(\mu) \neq 0$. Thus $\gamma \leq Cl(\lambda) \wedge Cl_W(\mu) = \lambda \wedge \mu$. Therefore, $\lambda \wedge \mu$ is FC.

Corollary 3 *The union of a FO-open set with a FWO-set is FO.*

Lemma 2 *If λ is a FSC subset of an E.D. fuzzy space, then $Cl_\epsilon(A) = Cl(A)$.*

We only need to show $Cl_W(\lambda) \leq Cl(\lambda)$ when λ is a FSC-set. For all $\mu \leq Cl_W(\lambda)$ and all η FO-set such that $\mu \leq \eta$, we have $Cl(\eta) \wedge Cl(\lambda) \neq 0$. As X is E.D., $Cl(\lambda) = \text{Int}(Cl(\lambda))$ and hence $Cl(\eta) \wedge \text{Int}(Cl(\lambda)) \neq 0$. Thus there exists $\gamma \leq Cl(\eta)$ and $\gamma \leq \text{Int}(Cl(\lambda))$ which is FO. Hence $\eta \wedge \text{Int}(Cl(\lambda)) \neq 0$ and as λ is FSC, $\eta \wedge \lambda \neq 0$. Therefore $Cl_W(\lambda) \leq Cl(\lambda)$.

We remark that X being an E. D. fuzzy space is necessary in Lemma 2.

Example 4 Consider the space in 2. Then $Cl_\epsilon(\chi_{\{b\}}) = 1 \neq \chi_{\{a,b,d\}} = Cl(\chi_{\{b\}})$.

Corollary 4 *In an E.D. fuzzy space, a F-set is FC if and only if it is FWC-set.*

3. Fuzzy *W*-generalized closed sets

In this section, we introduce the notion of fuzzy *W*-generalized closed set. Moreover, several interesting properties and constructions of these fuzzy subsets are discussed.

Definition 2 A fuzzy subset λ of a space X is called fuzzy *W-generalized closed* (simply, FWGC) if whenever μ is a FO subset such that $\lambda \leq \mu$, we have $Cl_W(\lambda) \leq \mu$. λ is fuzzy *W*-generalized-open (simply, FWGO) if $1 - \lambda$ is FWGC.

Theorem 2 *A fuzzy subset λ of $(X, \mathfrak{T})$ is FWGO-set if and only if $\mu \leq \text{Int}_W(\lambda)$, whenever $\mu \leq \lambda$ and μ is FC-set in $(X, \mathfrak{T})$.*

Let λ be a FWGO-set set and μ be a FC subset such that $\mu \leq \lambda$. Then $1 - \lambda \leq 1 - \mu$. As $1 - \lambda$ is FWGC and as $1 - \mu$ is FO, $Cl_W(1 - \lambda) \leq 1 - \mu$. So $\mu \leq 1 - Cl_W(1 - \lambda) = \text{Int}_W(\lambda)$.

Conversely if $1 - \lambda \leq \mu$ where μ is FO, then the FC-set $1 - \mu \leq \lambda$. Thus $1 - \mu \leq \text{Int}_W(\lambda) = 1 - Cl_W(1 - \lambda)$ and so $Cl_W(1 - \lambda) \leq \mu$.

Next we show the class of FWGC-sets is properly placed between the classes of FC- and FWC-sets. Moreover, the class FGC-sets is properly placed between the classes of FC-sets and FWGC−sets. A FC-set is trivially FGC and clearly every FWC-set is FC and every FWGC-set is FGC as $Cl(\lambda) \leq Cl_W(\lambda)$ for every F-set λ in space X. In Example 1, $\lambda = \chi_{\{a,c\}}$ is a FC-set that is not FWC. In Example 2, $\lambda = \chi_{\{a,b,d\}}$ is not FWC, but as the only super set of λ is 1, λ is FWGC.

Example 5 Cosider the space $X = \{a, b, c, d\}$ and $\mathfrak{T} = \{0, 1, \chi_{\{a,b,d\}}, \chi_{\{c,d\}}, \chi_{\{d\}}\}$. Then $\chi_{\{a,b\}}$ is FC and hence FGC, but it is not FϵGC as $Cl_\epsilon(\chi_{\{a,c\}}) = 1$.

The following is an immediate result from Lemma 2:

Theorem 3 *If λ is a FSC subset of a fuzzy E.D. space X, the following are equivalent:*

(1) λ is a FWGC − *set*;
(2) λ is a FGC−set.

Its clear that if $\lambda \leq \mu$, then $\text{Int}_W(\lambda) \leq \text{Int}_W(\mu)$ and $Cl_W(\lambda) \leq Cl_W(\mu)$.

LEMMA 3 *If λ and μ are F subsets of a space X, then $Cl_\epsilon(\lambda \vee \mu) = Cl_\epsilon(\lambda) \vee Cl_\epsilon(\mu)$ and $Cl_\epsilon(\lambda \wedge \mu) \leq Cl_\epsilon(\lambda) \wedge Cl_\epsilon(\mu)$.*

Since λ and μ are F-subsets of $\lambda \vee \mu, Cl_W(\lambda) \vee Cl_W(\mu) \leq Cl_W(\lambda \vee \mu)$. On the other hand, if $\eta \leq Cl_W(\lambda \vee \mu)$ and γ is a FO-set such that $\eta \leq \gamma$, then $Cl(\gamma) \wedge \text{Int}(\lambda \vee \mu) \neq 0$. Hence either $Cl(\gamma) \wedge Cl(\lambda) \neq 0$ or $Cl(\gamma) \wedge \text{Int}(\mu) \neq 0$. Thus $\eta \leq Cl_W(\lambda) \vee Cl_W(\mu)$.

Finally since $\lambda \wedge \mu$ is a F-subset of λ and μ, $Cl_W(\lambda \wedge \mu) \leq Cl_W(\lambda) \wedge Cl_W(\mu)$.

COROLLARY 5 *Finite unions of FWGC-sets are FWGC.*

While the finite intersections of FWGC-sets need not be FWGC.

Example 6 Cosider the space $X = \{a, b, c, d, e\}$ and $\mathfrak{T} = \{0, 1, \chi_{\{a,b\}}, \chi_{\{c\}}, \chi_{\{a,b,c\}}\}$. Then $\lambda = \chi_{\{a,c,d\}}$ and $\mu = \chi_{\{b,c,e\}}$ are FϵGC -sets as the only supper fuzzy set of them is 1, but $\lambda \wedge \mu = \chi_{\{c\}}$ is not a FϵGC-set.

THEOREM 4 *The intersection of a FWGC-set set with a FWC-set is FWGC.*

Proof Let λ be a *FWGC*-set and μ be a *FWC*-set. Let γ be a FO-set such that $\lambda \wedge \mu \leq \gamma$. Then $\lambda \leq \gamma \vee (1 - \mu)$. Since $1 - \mu$ is *FWO*, by Corollary 3, $\gamma \vee (1 - \mu)$ is FO and since λ is *FWGC*, $Cl_W(\lambda \wedge \mu) \leq Cl_W(\lambda) \wedge Cl_W(\mu) = Cl_W(\lambda) \wedge \mu \leq (\gamma \vee 1 - \mu) \wedge \mu = \gamma \wedge \mu \leq \gamma$.

4. Fuzzy W-g-continuous and Fuzzy W-g-irresolute functions

Definition 3 A fuzzy function $f:(X, \mathfrak{T}) \rightarrow (Y, \mathfrak{T}')$ is called

(1) fuzzy *W-g-continuous* (simply, FWGcts) if $f^{-1}(\lambda)$ is a *FWGC*-set in $(X, \mathfrak{T})$ for every FC-set λ of $(Y, \mathfrak{T}')$,
(2) fuzzy *W-g-irresolute* (simply, FWGI) if $f^{-1}(V)$ is a *FWGC* -set in $(X, \mathfrak{T})$ for every *FWGC*-set λ of $(Y, \mathfrak{T}')$.

LEMMA 4 *Let $f:(X, \mathfrak{T}) \rightarrow (Y, \mathfrak{T}')$ be FϵGcts. Then f is FGcts.*
Follows from the fact that every *FWGC*-set is FGC. The converse of the preceding Lemma needs not be true.

Example 7 Cosider the space $(X, \mathfrak{T})$ in Example 5 and the identity function $f:(X, \mathfrak{T}) \rightarrow (X, \mathfrak{T}')$ where $\mathfrak{T}' = \{0, 1, \chi_{\{c,d\}}\}$. Since $f^{-1}(\chi_{\{a,b\}}) = \chi_{\{a,b\}} \neq Cl_\epsilon(\chi_{\{a,b\}})$, *f* is not F$\epsilon$ Gcts, but *f* is FCts and hence FGCts.

Even the composition of FWGCts functions needs not be *FWGCts*.

Example 8 Let *f* be the function in Example 6. Let $\mathfrak{T}^e = \{0, 1, \chi_{\{a,b,d,e\}}\}$. Let $g:(X, \mathfrak{T}') \rightarrow (X, \mathfrak{T}^e)$ be the identity function. It is easily observed that *g* is also FϵGCts as the only super set of $\chi_{\{c\}}$ is 1. But the composition function $g \circ f$ is not Fϵ GCts as $\chi_{\{c\}}$ is a FC-set in $(X, \mathfrak{T}^e)$ and it is not a FϵGC-set in $(X, \mathfrak{T})$.

We end this section by giving a necessary condition for a FWGCts function to be FWGI.

THEOREM 5 *If $f : (X, \mathfrak{T}) \rightarrow (Y, \mathfrak{T}')$ is a bijective, FO- and FWGCts function, then f is FWGI.*

Proof Let λ be a *FWGC* subset of *Y* and let $f^{-1}(\lambda) \leq \eta$, where $\eta \in \mathfrak{T}$. Clearly, $\lambda \leq f(\eta)$. Since $f(\eta) \in \mathfrak{T}'$ and since λ is *FWGC*, $Cl_W(\lambda) \leq f(\eta)$ and thus $f^{-1}(Cl_W(\lambda)) \leq \eta$. Since *f* is *FWGCts* continuous and since $Cl_W(\lambda)$ is FC in *Y*, $f^{-1}(Cl_W(\lambda)$ is *FWGC*. $f^{-1}(Cl_W(\lambda) \leq Cl_W(f^{-1}(Cl_W(\lambda))) = f^{-1}(Cl_W(\lambda)) \leq \eta$. Therefore, $f^{-1}(\lambda)$ is *FWGC* and hence, *f* is *FWGI*. □

5. Conclusion

Several characterizations of fuzzy *W*-open and fuzzy *W*-closed notions via the operations of interior and closure are explored and the notion of fuzzy *W*-generalized closed sets is studied. Finally, fuzzy *W*-generalized continuous and fuzzy *W*-generalized irresolute functions are discussed and connections with other similar forms of fuzzy continuity are established.

Funding
The authors received no direct funding for this research.

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Citation information
Cite this article as: Fuzzy *W*-closed sets, Talal AL-Hawary, *Cogent Mathematics* (2017), 4: 1343518.

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