



Received: 14 December 2016  
Accepted: 09 June 2017  
First Published: 14 June 2017

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Reviewing editor:  
Zudi Lu, University of Southampton,  
United Kingdom

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## STATISTICS | RESEARCH ARTICLE

# Multivariate copulas on the MCUSUM control chart

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**Abstract:** The copula approach is a popular method for multivariate modeling applied in several fields; it defines non-parametric measures of dependence between random variables. In this paper, three families are proposed from elliptical and Archimedean copulas on the multivariate cumulative sum (MCUSUM) control chart when observations are drawn from an exponential distribution. The performance of the control chart is based on the average run length (ARL)—via Monte Carlo simulations. A copula function for specifying the dependence between random variables is measured by Kendall's tau. The numerical results indicate that the observations can be fitted and that the copula can be used on the MCUSUM for cases of small and large dependencies.

**Subjects:** Science; Mathematics & Statistics; Statistics & Probability; Probability; Statistics; Multivariate Statistics; Technology; Engineering & Technology; Industrial Engineering & Manufacturing; Quality Control & Reliability

**Keywords:** copula; multivariate cumulative sum; average run length; Monte Carlo simulation

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### PUBLIC INTEREST STATEMENT

We have proposed and presented a copula approach for observation dependencies in quality control. This paper used the model of copula and generated an exponential distribution on control chart. We applied this approach to the problem of non-normal data.

## 1. Introduction

Quality control charts are applicable for processes that have one variable or more, which are referred to as univariate or multivariate control charts, respectively. Multivariate control charts are widely used to simultaneously monitor several quality characteristics for detecting the mean changes in manufacturing industries. They are a powerful tool in statistical process control (SPC) for identifying an out-of-control process.

Generally, multivariate detection procedures are based on a multi-normality assumption and independence; however, many processes exhibit non-normality and correlation. Most of these multivariate control charts are generalizations of their univariate counterparts (Mahmoud & Maravelakis, 2013), such as the Hotelling's  $T^2$  control chart, the multivariate exponentially weighted moving average (MEWMA) control chart proposed by Lowry, Woodall, Champ, and Rigdon (1992), and the multivariate cumulative sum (MCUSUM) control chart (Bersimis, Panaretos, & Psarakis, 2005; Bersimis, Psarakis, & Panaretos, 2007). Many practitioners have addressed the problem of correlated data in SPC (see Lowry & Montgomery, 1995), and multivariate control charts with the lack of related joint distribution and copula can reveal this characteristic.

Copulas modeling is a general approach to model multivariate non-normal data (Joe, 1997), with the dependence structure separated from the univariate margins. Copulas create a link between multivariate joint distributions and univariate marginal distributions; they have been widely studied and applied in areas that are concerned with the problem of dependence relations. Many researchers have developed the copula for use with control charts (see Dokouhaki & Noorossana, 2013; Fatahi, Dokouhaki, & Moghaddam, 2011; Fatahi, Noorossana, Dokouhaki, & Moghaddam, 2012; Hryniewicz, 2012; Hryniewicz & Szewi, 2010; Kuvattana, Sukparungsee, Busababodhin, & Areepong, 2016; Sukparungsee, Kuvattana, Busababodhin, & Areepong, in press; Verdier, 2013).

In manufacturing processes, the time is used to represent some attributes or variable measures. One type of time distribution for an event is known as an exponential distribution, which is a continuous distribution of monitoring for successive event occurrences. This article presents a study of the MCUSUM control chart when observations are generated by an exponential distribution (Khoo, Atta, & Phua, 2009; Mason, Champ, Tracy, Wierda, & Young, 1997) and a copula describe the dependence between exponentially distributed components of a trivariate observation with mean shifts.

## 2. Properties of the MCUSUM control chart

In the univariate case, the cumulative sum (CUSUM) procedure is often designed for monitoring and detecting small changes. CUSUM was first introduced by Page (1954) to detect changes in the mean of an independent and identically distributed (i.i.d.) observed sequence of random variables. The MCUSUM control chart is the multivariate extension of a univariate CUSUM chart (see Busaba, Sukparungsee, & Areepong, 2012; Busaba, Sukparungsee, Areepong, & Mititelu, 2012).

Crosier (1988) proposed the MCUSUM control chart; which is the extension of a univariate CUSUM control chart. The MCUSUM control chart may be expressed as follows:

$$C_t = \left[ (\mathbf{S}_{t-1} + \mathbf{X}_t - \mathbf{a})^T \boldsymbol{\Sigma}^{-1} (\mathbf{S}_{t-1} + \mathbf{X}_t - \mathbf{a}) \right]^{1/2}, \quad (2.1)$$

$t = 1, 2, \dots$ , where covariance matrix  $\boldsymbol{\Sigma}$  and  $\mathbf{S}_t$  are the cumulative sums as determined by:

$$\mathbf{S}_t = \begin{cases} \mathbf{0}, & \text{if } C_t \leq k \\ (\mathbf{S}_{t-1} + \mathbf{X}_t - \mathbf{a}) \left(1 - \frac{k}{C_t}\right), & \text{if } C_t > k \end{cases} \quad (2.2)$$

where the reference value and is the aim point or target value for the mean vector (see Khoo et al., 2009). The control chart statistics for MCUSUM chart are

$$Y_t = [S_t^T \Sigma^{-1} S_t]^{1/2}, \tag{2.3}$$

$t = 1, 2, \dots$  The signal gives an out-of-control indication if  $Y_t > h$ , where  $h$  is the control limit (see Alves, Samohyl, & Henning, 2010; Fricker, Knitt, & Hu, 2008).

### 3. Copula function

The copula function is a multivariate distribution with all univariate margins being  $U(0, 1)$ . The multivariate copula function is used for capturing the dependence between two or more random variables. Suppose that a random vector  $(X_1, \dots, X_d)$  has a joint distribution function  $H(x_1, \dots, x_d)$  with continuous marginal distribution function  $F_i(x_i) = u_i$ , where  $U_i$  has uniform distribution  $[0, 1]$  for  $i = 1, \dots, d$ ; then, there exists a unique  $d$ -dimensional copula  $C$ . In this section, the theory that is the central foundation of the copula is described and shown in Table 1 (see Genest & McKay, 1986; Joe, 2015; Sklar, 1959; Trivedi & Zimmer, 2005).

#### 3.1. Fréchet–Hoeffding bounds

Let  $\mathcal{F}(F_1, \dots, F_d)$  be the Fréchet–Hoeffding bounds for the  $d$ -variate joint cumulative distribution function (cdf)  $H(x_1, \dots, x_d)$  with univariate marginal cdfs  $F_1, \dots, F_d$ . The Fréchet–Hoeffding lower and upper bounds are determined by:

$$\max \left[ \sum_{i=1}^d F_i - d + 1, 0 \right] \leq H(x_1, \dots, x_d) \leq \min [F_1, \dots, F_d] \tag{3.1}$$

(see Joe, 1997).

#### 3.2. Sklar’s theorem

This theorem is central to the theory of copulas and is applied using statistical theory. Assuming  $R$  denotes the ordinary real time  $(-\infty, \infty)$ , and the unit square  $\mathbf{I}^2$  is the product  $\mathbf{I} \times \mathbf{I}$ , where  $\mathbf{I} = [0, 1]$ , Sklar’s theorem can be written as Theorem 1.

**THEOREM 1** *Sklar’s theorem in  $d$ -dimensions.*

Let  $F \in \mathcal{F}(F_1, \dots, F_d)$  with  $i^{\text{th}}$  univariate margin  $F_i$ . The copula associated with  $H$  is a distribution function  $C: [0, 1]^d \rightarrow [0, 1]$  with  $U(0, 1)$  that satisfies

$$H(x_1, \dots, x_d) = C(F_1(x_1), \dots, F_d(x_d)); x_1, \dots, x_d \in \mathbf{R}^d. \tag{3.2}$$

(i) If  $F$  is a continuous  $d$ -variate distribution function with univariate margins  $(F_1, \dots, F_d)$  and quantile functions  $F_1^{-1}, \dots, F_d^{-1}$ , then

$$C(u_1, \dots, u_d) = H(F_1^{-1}(u_1), \dots, F_d^{-1}(u_d)); u_1, \dots, u_d \in [0, 1]^d \tag{3.3}$$

is unique.

**Table 1. Kendall’s tau of copula function**

Copula	Kendall’s tau	Range of $\theta$
Normal	$\arcsin(\theta)/(\pi/2)$	$[0, 1]$
Clayton	$\theta/(\theta + 2)$	$(0, \infty)$
Frank	$1 + 4 \left( \frac{1}{\theta} \int_0^{\theta} \frac{t}{e^t - 1} dt - 1 \right) / \theta$	$(0, \infty)$

(ii) If  $F$  is a  $d$ -variate distribution function of discrete random variables, then the copula is unique only on the set

$$\text{Range}(F_1) \times \dots \times \text{Range}(F_d), \tag{3.4}$$

where  $F_i^{-1}(u_i)$  is the quantile function of  $F$ .

### 3.3. Copula families

For the purposes of the statistical method, it is desirable to parameterize the copula function. Let  $\theta$  denote the association parameter of the multivariate distribution; then, there exists a copula  $C$  (see Trivedi & Zimmer, 2005). This paper focuses on the normal copula and two families of Archimedean copulas, the Clayton and Frank copulas, because these copulas are well-known.

#### 3.3.1. Normal copula

The normal copula is an elliptical copula. From the multivariate normal distribution with zero means, unit variances and  $d \times d$  correlation matrix  $\Sigma$ , the trivariate normal copula is determined by

$$C(u_1, u_2, u_3; \Sigma) = \Phi(\Phi^{-1}(u_1), \Phi^{-1}(u_2), \Phi^{-1}(u_3); \Sigma); \tag{3.5}$$

$$-1 \leq \theta \leq 1, \Sigma = \begin{bmatrix} 1 & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & 1 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & 1 \end{bmatrix} \text{ and } \theta = (\mu_{X_1}, \mu_{X_2}, \mu_{X_3}, \sigma_{X_1}, \sigma_{X_2}, \sigma_{X_3}, \rho),$$

where  $\Phi(\cdot; \Sigma)$  is the trivariate normal cdf,  $\Phi$  is the univariate normal cdf and  $\Phi^{-1}$  is the univariate normal inverse cdf or quantile function (see Ganguli & Reddy, 2013; Joe, 2015).

#### 3.3.2. Archimedean copulas

Let  $\phi$  be a continuous strictly decreasing function from  $\mathbf{I}$  to  $[0, \infty]$  such that  $\phi(0) = \infty$  and  $\phi(1) = 0$ , and let  $\phi^{-1}$  denote the inverse of  $\phi$ . The  $d$ -dimensional function from  $\mathbf{I}^d$  to  $\mathbf{I}$  is given by:

$$C(u_1, \dots, u_d) = \phi^{[-1]}(\phi(u_1) + \dots + \phi(u_d)), \tag{3.6}$$

where  $u_i = F_i(x_i)$  is the marginal cdf of  $X_i$  and  $\phi^{[-1]}$  is the pseudo-inverse of an Archimedean generator  $\phi$  and is completely monotonic on  $[0, \infty]$ , i.e.,  $\phi^{[-1]} = \phi^{-1}$  (see Nelsen, 2006; Trivedi & Zimmer, 2005). The Archimedean copulas in this paper are generated as follows:

3.3.2.1. Clayton copula. Let  $\phi(t) = (t^\theta - 1)/\theta$  for  $\theta > 0$ , which generates a subfamily of the bivariate Clayton copula. Then:

$$C(u_1, u_2) = [\max(u_1^{-\theta} + u_2^{-\theta} - 1, 0)]^{-1/\theta}, \text{ and } \phi^{-1}(t) = (1 + t)^{-1/\theta}.$$

The trivariate Clayton copula can be generalized from the bivariate copula by:

$$C(u_1, u_2, u_3) = (u_1^{-\theta} + u_2^{-\theta} + u_3^{-\theta} - 2)^{-1/\theta}; \theta > 0. \tag{3.7}$$

3.3.2.2. Frank copula. Let  $\phi(t) = -\ln(\frac{e^{-\theta t} - 1}{e^{-\theta} - 1})$  for  $\theta > 0$ , which generates a subfamily of the bivariate Frank copula. Then:

$$C(u_1, u_2) = -\frac{1}{\theta} \ln \left( 1 + \frac{(e^{-\theta u_1} - 1)(e^{-\theta u_2} - 1)}{e^{-\theta} - 1} \right), \phi^{-1}(t) = -\frac{1}{\theta} \ln [1 - (1 - e^{-\theta})e^{-t}].$$

The trivariate Frank copula can be generalized from the bivariate copula by:

$$C(u_1, u_2, u_3) = -\frac{1}{\theta} \ln \left( 1 + \frac{(e^{-\theta u_1} - 1)(e^{-\theta u_2} - 1)(e^{-\theta u_3} - 1)}{(e^{-\theta} - 1)^2} \right); \theta > 0. \tag{3.8}$$

The generators of Clayton and Frank copulas are valid only for  $\theta \in (0, \infty)$

#### 4. Measuring dependence

Spearman's rho is widely used to describe the association between random variables with linear dependence and it can be used for any dependence of a monotonic type. In the case of linear dependence Pearson's rho may be used, but it works well only for normal marginals. For association between random variables that are not linear, Kendall's tau is a measurement of the concordance. Kendall's tau is a non-parametric measurement of associations which is considered a copula-based dependence measurement (see Quessy, Said, & Favre, 2013). Kendall's tau is straightforward to calculate, and this measure is used for observation dependencies.

For the bivariate case, assuming  $X_1$  and  $X_2$  are continuous random variables whose copula is  $C$ , Kendall's tau is given by  $\tau_c = 4 \int \int_{\mathbf{I}^2} C(u_1, u_2) dC(u_1, u_2) - 1$ , where  $\tau_c$  is Kendall's tau of copula  $C$ , the unit square  $\mathbf{I}^2$  is the product  $\mathbf{I} \times \mathbf{I}$  with  $\mathbf{I} = [0, 1]$ , and the expected value is the function  $C(u_1, u_2)$  of uniform  $(0, 1)$  random variables  $U_1$  and  $U_2$ , whose joint distribution function is  $C$ , i.e.,  $\tau_c = 4E[C(U_1, U_2)] - 1$  (see Nelsen, 2006).

Genest and McKay (1986) considered two random variable of Archimedean copula  $C$  generated by  $\phi$ ; then,  $\tau_{Arch} = 4 \int_0^1 \frac{\phi(t)}{\phi'(t)} dt + 1$ , where  $\tau_{Arch}$  is Kendall's tau of  $C$ . In this study,  $\tau_c$  of the trivariate and bivariate cases is similarly estimated.

#### 5. Performance characteristics and numerical results

In this section, the performance characteristics for the SPC chart is the average run length (ARL). It is classified into  $ARL_0$  and  $ARL_1$ , where  $ARL_0$  is the average run length when the process is in control and  $ARL_1$  is the average run length when the process is out-of-control (see Busaba, Sukparungsee, & Areepong, 2012). Generally, an acceptable  $ARL_0$  should be sufficiently large when the process is in control, and  $ARL_1$  should be small when the process is out-of-control. Empirical studies are implemented in the R statistical software and all packages used are available from CRAN at <https://CRAN.R-project.org/> (see Hofert, Machler, & McNeil, 2012; Yan, 2007) with 50,000 simulation runs and a sample size of 1,000. Observations were drawn from the exponential distribution with parameter ( $\alpha$ ) equal to 1 for an in-control process ( $\mu_0 = 1$ ) and the shifts of the process level ( $\delta$ ) by  $\mu = \mu_0 + \delta$ . The process means are 1.1, 1.2, 1.3, ..., 2 for an out-of-control process. Copula estimations are restricted to the cases of positive dependence. For all copula models, the setting  $\theta$  corresponds to Kendall's tau and the level of dependence is measured by Kendall's tau values ( $-1 \leq \tau \leq 1$ ). The authors define 0.2 and 0.8 as small and large dependencies, respectively.

Tables 2–4 present the numerical results for empirical observations. The different values of exponential parameters are ( $\mu_1, \mu_2$ , and  $\mu_3$ ) for the variables ( $X_1, X_2, X_3$ ). For in-control processes, the MCUSUM control chart was chosen by setting the desired  $ARL_0 = 370$ , and  $\mu_1 = \mu_2 = \mu_3 = 1$  is fixed for each copula. Table 2 shows small and large positive dependencies in the case of all parameters shifts. Tables 3 and 4 show parameters shifts in the case of two parameters shifts. The results in Table 2 show the mean shifts of  $\tau = 0.2$  and 0.8. The  $ARL_1$  values of the Clayton copula are less than the other copulas for nearly all shifts except  $\tau = 0.8$  for large shifts ( $1.8 \leq \mu_1 = \mu_2 = \mu_3 \leq 2$ ). Table 3 shows the case of small dependence; the  $ARL_1$  values of the Clayton copula are less than the other

**Table 2. Comparison of ARL from MCUSUM chart with Kendall's tau values equal to 0.2 and 0.8 in the case of three exponential parameters shifts**

$\tau$	Parameters			ARL <sub>0</sub> and ARL <sub>1</sub>		
	$\mu_1$	$\mu_2$	$\mu_3$	Normal	Clayton	Frank
0.2	<b>1</b>	<b>1</b>	<b>1</b>	370.060	369.971	369.921
	1.1	1.1	1.1	217.481	210.289	214.199
	1.2	1.2	1.2	135.365	129.100	131.773
	1.3	1.3	1.3	91.264	84.685	87.340
	1.4	1.4	1.4	63.793	59.101	61.140
	1.5	1.5	1.5	46.623	43.731	44.582
	1.6	1.6	1.6	35.088	33.150	33.779
	1.7	1.7	1.7	27.443	26.153	26.434
	1.8	1.8	1.8	22.149	20.850	21.452
	1.9	1.9	1.9	18.188	17.301	17.575
	2	2	2	15.071	14.256	14.553
0.8	<b>1</b>	<b>1</b>	<b>1</b>	<b>369.991</b>	<b>369.856</b>	<b>369.831</b>
	1.1	1.1	1.1	244.250	233.546	242.530
	1.2	1.2	1.2	169.498	157.150	165.925
	1.3	1.3	1.3	124.159	113.134	120.537
	1.4	1.4	1.4	93.790	84.783	93.376
	1.5	1.5	1.5	72.202	65.819	73.478
	1.6	1.6	1.6	56.289	52.515	59.799
	1.7	1.7	1.7	44.559	43.301	48.432
	1.8	1.8	1.8	35.647	35.905	40.412
	1.9	1.9	1.9	29.355	30.207	34.210
	2	2	2	24.397	26.000	28.948

Note: Bold number means minimum ARL<sub>1</sub>.

**Table 3. Comparison of ARL from MCUSUM chart with Kendall's tau value equal to 0.2 in the case of two exponential parameters shifts**

Parameters			ARL <sub>0</sub> and ARL <sub>1</sub>		
$\mu_1$	$\mu_2$	$\mu_3$	Normal	Clayton	Frank
1	<b>1</b>	<b>1</b>	370.060	369.971	369.921
1	1.1	1.1	256.816	250.487	253.875
1	1.2	1.2	178.402	170.357	173.181
1	1.3	1.3	127.654	119.949	123.033
1	1.4	1.4	94.464	88.717	91.006
1	1.5	1.5	71.707	66.858	68.711
1	1.6	1.6	55.153	51.940	53.496
1	1.7	1.7	43.968	41.529	42.275
1	1.8	1.8	35.693	33.837	34.609
1	1.9	1.9	29.385	28.043	28.835
1	2	2	24.688	23.689	24.259
<b>1</b>	<b>1</b>	<b>1</b>	<b>370.060</b>	<b>369.971</b>	<b>369.921</b>
1.1	1	1.1	257.805	247.860	251.758

(Continued)

**Table 3. (Continued)**

Parameters			ARL <sub>0</sub> and ARL <sub>1</sub>		
$\mu_1$	$\mu_2$	$\mu_3$	Normal	Clayton	Frank
1.2	1	1.2	177.888	170.146	172.639
1.3	1	1.3	127.641	120.009	123.331
1.4	1	1.4	94.501	88.142	90.731
1.5	1	1.5	71.526	67.444	68.541
1.6	1	1.6	55.436	52.280	53.308
1.7	1	1.7	44.229	41.635	42.508
1.8	1	1.8	35.708	33.866	34.340
1.9	1	1.9	29.558	28.099	28.624
2	1	2	24.849	23.762	24.057
<b>1</b>	<b>1</b>	<b>1</b>	<b>370.060</b>	<b>369.971</b>	<b>369.921</b>
1.1	1.1	1	258.047	250.282	274.064
1.2	1.2	1	178.148	170.533	193.604
1.3	1.3	1	127.320	120.778	139.420
1.4	1.4	1	94.488	88.742	102.469
1.5	1.5	1	71.229	66.850	78.048
1.6	1.6	1	55.771	52.212	60.786
1.7	1.7	1	44.143	41.240	48.752
1.8	1.8	1	35.770	34.018	39.077
1.9	1.9	1	29.415	28.103	32.441
2	2	1	24.963	23.687	27.171

Note: Bold number means minimum ARL<sub>1</sub>.

**Table 4. Comparison of ARL from MCUSUM chart with Kendall's tau value equal to 0.8 in the case of two exponential parameters shifts**

Parameters			ARL <sub>0</sub> and ARL <sub>1</sub>		
$\mu_1$	$\mu_2$	$\mu_1$	Normal	Clayton	Frank
1	1	1	369.991	369.856	369.831
1	1.1	1.1	266.164	266.571	269.593
1	1.2	1.2	181.295	187.729	192.194
1	1.3	1.3	125.663	138.298	140.802
1	1.4	1.4	90.369	104.374	106.990
1	1.5	1.5	65.008	80.979	82.385
1	1.6	1.6	47.922	63.232	63.753
1	1.7	1.7	35.814	49.949	50.603
1	1.8	1.8	27.873	40.450	40.171
1	1.9	1.9	21.836	33.319	32.539
1	2	2	17.808	27.725	26.893
<b>1</b>	<b>1</b>	<b>1</b>	<b>369.991</b>	<b>369.856</b>	<b>369.831</b>
1.1	1	1.1	263.852	265.699	269.720
1.2	1	1.2	181.293	188.003	191.940
1.3	1	1.3	127.951	138.216	141.017
1.4	1	1.4	90.359	104.787	106.054

(Continued)

**Table 4. (Continued)**

Parameters			ARL <sub>0</sub> and ARL <sub>1</sub>		
$\mu_1$	$\mu_2$	$\mu_1$	Normal	Clayton	Frank
1.5	1	1.5	64.983	80.888	82.560
1.6	1	1.6	48.113	63.464	63.837
1.7	1	1.7	35.692	50.224	50.145
1.8	1	1.8	27.732	40.471	39.923
1.9	1	1.9	21.950	33.520	32.556
2	1	2	17.790	27.803	26.863
<b>1</b>	<b>1</b>	<b>1</b>	<b>369.991</b>	<b>369.856</b>	<b>369.831</b>
1.1	1.1	1	263.517	266.568	268.499
1.2	1.2	1	181.281	188.995	192.164
1.3	1.3	1	126.077	139.426	141.516
1.4	1.4	1	89.942	104.859	106.496
1.5	1.5	1	65.635	80.260	81.926
1.6	1.6	1	47.364	63.097	63.787
1.7	1.7	1	35.630	50.036	50.300
1.8	1.8	1	27.486	40.315	40.069
1.9	1.9	1	21.850	33.214	32.597
2	2	1	17.774	27.613	26.816

Note: Bold number means minimum ARL<sub>1</sub>.

copulas for all shifts. Table 4 shows the case of large dependence; the ARL<sub>1</sub> values of the normal copula are less than the other copulas for all shifts.

### 6. Conclusion

Families of well-known copulas are considered in this work, and elliptical and Archimedean copulas are compared. This paper shows the MCUSUM control chart for three families of copulas and the level of dependence is measured by Kendall’s tau values. Combining copula with control chart is interesting and can apply for further research. The results revealed that it is necessary to detect the observation dependence to indicate the copula and use that detection to fit the observation on the MCUSUM chart for small and large dependencies.

#### Funding

The authors were supported by the Ministry of Science and Technology, Thailand and the Graduate College, King Mongkut’s University of Technology North Bangkok. This work was supported by the King Mongkut’s University of Technology North Bangkok [grant number KMUTNB-ART-60-52].

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#### Citation information

Cite this article as: Multivariate copulas on the MCUSUM control chart, Saowanit Sukparungsee, Sasigarn Kuvattana, Piyapatr Busababodhin & Yupaporn Areepong, *Cogent Mathematics* (2017), 4: 1342318.

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