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## APPLIED & INTERDISCIPLINARY MATHEMATICS | RESEARCH ARTICLE

# The martingale approach for vulnerable binary option pricing under stochastic interest rate

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**Abstract:** We consider the vulnerable option pricing problem when the stochastic interest rate is driven by a Hull-White model. Based on the firm value model, we suppose that the stock prices, assets and liabilities of a company follow the relevant O-U processes. We adopt the martingale approach to determine the equivalent martingale measure for pricing the vulnerable binary option, the analytical pricing formula of the vulnerable binary options is derived.

**Subjects:** Science; Mathematics & Statistics; Applied Mathematics; Statistics & Probability

**Keywords:** O-U process; vulnerable binary option; option pricing; change of measure; stochastic interest rate; martingale approach

### 1. Introduction

Credit risk is one of the major financial risk, it has become a major challenge facing the world's financial markets. On the exchange transaction market, because of the margin requirements, futures exchanging and options exchange derivatives almost no credit risk. Therefore, when option pricing happened in the exchange, we can assume credit risk does not exist. However, On OTC market, counter-party credit risk is a factor that must be considered. For derivatives on OTC market, since there is no guarantee, option bulls exposure to the credit risk and market risk. The assumption of no



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Quanxin Zhu is a professor of Nanjing Normal University, and he has obtained the Alexander von Humboldt Foundation of Germany. Prof Zhu is an associate editor of MPE, JAM, TJMAA, JAAM, RAM and he is a reviewer of Mathematical Reviews and Zentralblatt-Math. Also, Prof Zhu is a senior member of the IEEE and he is the Lead Guest Editor of "Recent Developments on the Stability and Control of Stochastic Systems" in MPE. Prof Zhu has obtained 2011 Annual Chinese One Hundred "The Most Influential International Academic Paper" Award and has been One of most cited Chinese researchers in 2014-2016, Elsevier. Prof Zhu is a reviewer of more than 50 other journals and he is the author or coauthor of more than 110 journal papers. His research interests include random processes, stochastic control, stochastic differential equations, stochastic partial differential equations, stochastic stability, nonlinear systems, Markovian jump systems, stochastic complex networks and finance mathematics.

### PUBLIC INTEREST STATEMENT

The option pricing problem is an important topic of finance field. At present, we mainly focus on the study of the option pricing problem (for example, the pricing of the vulnerable binary option in this paper), the option pricing formula of the price process of assets which is more reasonable in practice, and some corresponding real data analysis.

default risk no longer valid. Therefore, how to properly consider the counter-party risk factors and establish option pricing model containing credit risk are important to investors.

There are some articles about option pricing take credit risk theory into consideration. Firm value model is one of the most important model about vulnerable option pricing put forward by Merton (1974). Black, Fisher, and Cox (1976) studied the risk of bankrupt cost pricing of corporate bonds, improved Merton's idea about events of default. Johnson and Stulz (1987) extended Merton default model, discussing the option pricing problem under the structural model that has credit risk, putting forward the concept of vulnerable option. The seminal work that discussed option pricing containing credit risk is from Johnson and Stulz (1987). They used the term vulnerable option to define those options containing counter-party default risk, and pointed out a lot of features about them. Hull and White (1995) supposed the underlying asset and the counter-party firm's assets are independent of one another, obtained vulnerable options pricing formula. Jarrow and Tunbull (1995) supposed default time obey the strength of  $\lambda$  (non-negative constant) independent homogeneous poisson process, used discrete method to create a simple model of credit risk. Klein (1996) supposed breach of contract happened with a certain probability at any time, considered the pricing model of zero-coupon bonds can be liquidated at initial time zero.

In recent years, there is a considerable interest in the counter-party risk factors and establish option pricing model containing credit risk. Wang and Li (2003), Fu and Zhang (2002) and Xu and Li (2005) researched the problem of derivative pricing which have default risk. Deng and Kaleong (2007) and Wu, Lv, and Min (2007) consider stochastic interest rate and random counter-party liability case. They discussed the problem of option pricing with credit risk and getting its pricing formula. However, people always assume that stock price obey standard brown motion, which is not reality and has its limitation. Lin, Wang, and Feng (2000) proposed the O-U process model that stock price obeys, avoiding the limitations of standard brown motion. In the previous option pricing model, we assume interest rate is a constant. However, in reality, interest rate changes randomly. A lot of scholars conducted extensive research in this field. Zhou, Xinyu, and Gao (2011) gave European option pricing model under stochastic interest rate. Xue (2000) gave convertible bond pricing formula based on Hull-White Model which satisfy fractional Brownian motion drive.

This article mainly research the problem of vulnerable binary option pricing under the O-U process. Based on the firm value model, we assume the stock price, counter-party firm's assets and counter-party firm's debts obey O-U process, interest rate obeys Hull-White Model. By using martingale approach we derive an analytical pricing formula for such vulnerable binary option. This article is structured as follows. Section 2 describes firm value model and its assumption. Section 3 presents the use of the martingale approach for vulnerable binary option pricing. Moreover, we obtain the pricing formula of vulnerable binary option. Section 4 contains some conclusion.

## 2. Model and its assumptions

### 2.1. Firm value model

Considering continuous-time financial market, let  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, P)$  be a filtered probability space such that the filtration satisfies the usual conditions. Based on firm value model by Merton (1974), assume  $X(T)$  is the counter party firm promise to pay at the time  $T$ ,  $X$  is European style option, if the counter party firm always have the ability of pay before the due maturity date, then the owner of this option will get  $X(T)$  at the time  $T$ , otherwise the payoff depend on the ratio of company's assets and its liability. If we assume  $V(T)$  is the counter party firm's assets,  $D(T)$  is counter party firm's liability, when the counter party firm bankrupt, the payoff is  $X(T) \times [V(T)/D(T)]$ , so the owner of this vulnerable option's yield to maturity is  $Y(T) = X(T)I_{\{\delta(T) \geq 1\}} + \delta(T)X(T)I_{\{\delta(T) < 1\}}$ , where  $I_A$  is indicator function of the set  $A$ ,  $\delta(T)$  is the ratio of compensation,  $\delta(T) = V(T)/D(T)$ . According to martingale theory of

option pricing, the price of this option at the time  $0 \leq t \leq T$  can be represented as below under the equivalent martingale measurement  $\tilde{P}$ ,

$$C(t) = B(t)\tilde{E}\left[\frac{X(T)}{B(T)}(I_{\{\delta_T \geq 1\}} + \delta_{(T)}I_{\{\delta_T < 1\}}) \mid \mathcal{F}_t\right], \quad (1)$$

where  $B(t)$  is the price of default-free bonds, satisfy differential equation  $dB(t) = r(t)B(t)dt$ ,  $r(t)$  is risk-free interest rate,  $\mathcal{F}_t$  is information sets before the time  $t$ ,  $\tilde{E}(\cdot)$  is the mathematical expectation under equivalent martingale measurement  $\tilde{P}$ .

## 2.2. Model building

Assume stock price  $S_t$ , company's assets  $V_t$ , company's liability  $D_t$  all follow O-U process as follows

$$dS(t) = (\mu_S(t) - \alpha_S(t) \ln S(t))S(t) dt + \sigma_S(t)S(t) dB_S(t), \quad (2)$$

$$dV(t) = (\mu_V(t) - \alpha_V(t) \ln V(t))V(t) dt + \sigma_V(t)V(t) dB_V(t), \quad (3)$$

$$dD(t) = (\mu_D(t) - \alpha_D(t) \ln D(t))D(t) dt + \sigma_D(t)D(t) dB_D(t), \quad (4)$$

where  $B_S(t), B_V(t), B_D(t)$  is a standard Brown motion in probability space  $(\Omega, \mathcal{F}, \mathcal{F}_{0 \leq t \leq T}, P)$ ,  $\mu_\xi(t), \sigma_\xi(t)$ , ( $\xi = S, V, D$ ) is the corresponding expected return rate and volatility respectively, which are continuous functions about  $t$ .

O-U process is a Brown motion considering expected return rate depends on underlying assets's price,  $\alpha_\xi(\xi = S, V, D)$  can decrease underlying assets's price when the price up to a certain height. Assume  $\mathbf{B} = (B_S(t), B_V(t), B_D(t))$  is a 3-dimensional correlated  $\mathcal{F}_t$ -Brown motion with the correlation matrix  $\mathbf{C}$  given by

$$\mathbf{C} = \begin{pmatrix} 1 & \rho_{SV} & \rho_{SD} \\ \rho_{SV} & 1 & \rho_{VD} \\ \rho_{SD} & \rho_{VD} & 1 \end{pmatrix} \quad (5)$$

In the following, we shall introduce Girsanov's theorem and martingale representation theorem about multidimensional correlated Brown motion  $\mathbf{B}$ .

Let  $\Lambda(t) = (\lambda_S(t), \lambda_V(t), \lambda_D(t))$  is a 3-dimensional  $\mathcal{F}_t$ -adapted process, where  $\lambda_\xi(t) = [\mu_\xi - \alpha_\xi \ln \xi_t - r(t)]/\sigma_\xi$ , ( $\xi = S, V, D$ ), now define  $\lambda(t) = (\Lambda(t)\mathbf{C}\Lambda^*(t))^{\frac{1}{2}}$ ,  $\forall 0 \leq t \leq T$ , satisfy Novikov condition, where  $\Lambda^*(t)$  is the transposition of  $\Lambda(t)$ ,  $\mathbf{C}$  is the matrix defined by formula (5). Define

$$Z_\lambda(t) = \exp\left\{-\int_0^t \Lambda(s)d\mathbf{B}(s) - \frac{1}{2}\int_0^t \Lambda^2(s)ds\right\}, \quad \forall 0 \leq t \leq T, \quad (6)$$

then  $Z_\lambda(t)$  is a  $\mathcal{F}_t$ -martingale. We further assume that for each  $0 \leq t \leq T$ ,  $\lambda_\xi(t) \neq 0$ , and there exists a constant  $\rho_{\lambda\xi}$ ,  $\xi = S, V, D$  such that

$$\rho_{\lambda S}\lambda(t) = \lambda_S(t) + \rho_{SV}\lambda_V(t) + \rho_{SD}\lambda_D(t), \quad (7)$$

$$\rho_{\lambda V}\lambda(t) = \rho_{SV}\lambda_S(t) + \lambda_V(t) + \rho_{VD}\lambda_D(t), \quad (8)$$

$$\rho_{\lambda D}\lambda(t) = \rho_{SD}\lambda_S(t) + \rho_{VD}\lambda_V(t) + \lambda_D(t). \quad (9)$$

**THEOREM 1** On the measurable space  $(\Omega, \mathcal{F}_T)$ , we define new probability measure  $\tilde{P}$ , and for each  $A \in \mathcal{F}_T$   $\tilde{P}(A) = E_P[I_A Z(T)]$ , then  $\tilde{P}$  is the martingale measure equivalent to  $P$ , such that

$$\tilde{B}_S(t) = B_S(t) + \int_0^t \rho_{\lambda S} \lambda(s) ds, \tag{10}$$

$$\tilde{B}_V(t) = B_V(t) + \int_0^t \rho_{\lambda V} \lambda(s) ds, \tag{11}$$

$$\tilde{B}_D(t) = B_D(t) + \int_0^t \rho_{\lambda D} \lambda(s) ds, \tag{12}$$

for  $0 \leq t \leq T$ ,  $\tilde{B}_S(t)$ ,  $\tilde{B}_V(t)$ ,  $\tilde{B}_D(t)$  are all  $\mathcal{F}_t$  standard Brown motion, and for each  $\xi, \zeta = S, V, D$

$$\langle \tilde{B}_\xi, \tilde{B}_\zeta \rangle(t) = \langle B_\xi, B_\zeta \rangle(t) = \rho_{\xi\zeta} t,$$

where the cross-variations are computed under the appropriate measure  $P$  and  $\tilde{P}$ .

This result is due to Doob (1953), and its proof can be found in Deng and Kaleong (2007, Theorem 2.2).

LEMMA 1 (Steven, 1997) Assume  $\{B_1(t), t \geq 0\}$ ,  $\{B_2(t), t \geq 0\}$  are  $\mathcal{F}_t$ -standard Brown motions under the probability  $P$ , correlation coefficient is  $\rho_{12}$  and covariance is  $\langle B_1, B_2 \rangle(t) = \rho_{12}t$ . Denote

$$B(t) = \frac{\beta_1(t)B_1(t) - \beta_2(t)B_2(t)}{\Delta(t)},$$

$$\Delta(t) = \sqrt{\beta_1^2(t) + \beta_2^2(t) - 2\rho_{12}\beta_1(t)\beta_2(t)},$$

where  $\beta_1(t)$ ,  $\beta_2(t)$  are deterministic continuous function. Then  $B(t)$  is a standard  $\mathcal{F}_t$ -Brown motion under the probability  $P$ .

### 3. The martingale approach for vulnerable binary option pricing

Now we will discuss vulnerable binary option pricing under stochastic interest rate and liability.

Under the risk-neutral measure  $\tilde{P}$ , from Equation (1) we can get vulnerable binary option price can express as follow at the time  $0 \leq t \leq T$ ,

$$C(t) = \tilde{E} \left[ \exp \left\{ - \int_t^T r(u) du \right\} (X(T)I_{\{\delta(T) \geq 1\}} + \delta(T)X(T)I_{\{\delta(T) < 1\}}) | \mathcal{F}_t \right]. \tag{13}$$

For cash or nothing call option, payoff function on maturity date is  $X_T = RI_{\{S(T) \geq K\}}$ ;

For cash or nothing put option, payoff function on maturity date is  $X_T = RI_{\{S(T) < K\}}$ ;

For asset or nothing call option, payoff function on maturity date is  $X_T = S(T)I_{\{S(T) \geq K\}}$ ;

For asset or nothing put option, payoff function on maturity date is  $X_T = S(T)I_{\{S(T) < K\}}$ .

#### 3.1. Model solution

Assume the 3-dimensional  $\mathcal{F}_t$ -adapted process  $\Lambda(t) = (\lambda_S(t), \lambda_V(t), \lambda_D(t))$  satisfy the following SDEs:

$$\sigma_S(t)[\rho_{SV}\lambda_V(t) + \rho_{SD}\lambda_D(t)] = 0, \tag{14}$$

$$\sigma_V(t)[\rho_{SV}\lambda_S(t) + \rho_{VD}\lambda_D(t)] = 0, \tag{15}$$

$$\sigma_D(t)[\rho_{SD}\lambda_S(t) + \rho_{VD}\lambda_V(t)] = 0. \tag{16}$$

Let  $Z_\lambda(t)$  is the  $\mathcal{F}_t$ -martingale defined by Equation (6), then by the Theorem 1,  $\tilde{\mathbf{B}}(t) = (\tilde{B}_S(t), \tilde{B}_V(t), \tilde{B}_D(t))$  is a 3-dimensional correlated standard  $(\mathcal{F}_t, \tilde{P})$ -Brown motion, correlation matrix is  $\mathbf{C}$  defined by formula (5).

From Equations (7)–(9), (14)–(16), the definition of  $\lambda_\xi(t)$  ( $\xi = S, V, D$ ) and the Theorem 1, we can get counter party firm's assets  $V(t)$ , counter party firm's liability  $D(t)$ , stock price  $S(t)$  satisfy the following SDEs:

$$dS(t) = r(t)S(t) dt + \sigma_S(t)S(t) d\tilde{B}_S(t), \tag{17}$$

$$dV(t) = r(t)V(t) dt + \sigma_V(t)V(t) d\tilde{B}_V(t), \tag{18}$$

$$dD(t) = r(t)D(t) dt + \sigma_D(t)D(t) d\tilde{B}_D(t). \tag{19}$$

We assume short-term interest rate follows Hull-White model under the probability  $\tilde{P}$  as below:

$$dr(t) = (a(t) - b(t)r(t)) dt + \sigma_r(t) d\tilde{B}_r(t). \tag{20}$$

Under the probability  $\tilde{P}$ , from Ito formulas we can get:

$$\int_t^T r(u)du = G(t, T, r(t)) + \int_t^T \sigma_r(u)m(u, T)d\tilde{B}_r(u),$$

where  $G(t, T, r(t)) = r(t)m(t, T) + \int_t^T a(u)m(u, T)du$ ,  $m(u, v) = \int_u^v \exp(\eta(u) - \eta(s))ds$ ,  $\eta(s) = \int_0^s b(u)du$ .

Under the probability  $\tilde{P}$ , applying Ito formulas to (17)–(19) we can get:

$$\xi(T) = \xi(t) \exp \left\{ \int_t^T (r(u) - \sigma^2(\xi)/2)du + \int_t^T \sigma_\xi(u)d\tilde{B}_\xi(u) \right\}, \quad \xi = S, V, D.$$

Assume  $\tilde{B}_r(t)$  and  $\tilde{B}_S(t), \tilde{B}_V(t), \tilde{B}_D(t)$  are uncorrelated, then  $(\tilde{B}_r(t), \tilde{B}_S(t), \tilde{B}_V(t), \tilde{B}_D(t))$  is a 4-dimensional correlated standard Brown motion under probability  $\tilde{P}$ , the correlation matrix is

$$\tilde{\mathbf{c}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \rho_{SV} & \rho_{SD} \\ 0 & \rho_{SV} & 1 & \rho_{VD} \\ 0 & \rho_{SD} & \rho_{VD} & 1 \end{pmatrix}. \tag{21}$$

For the convenience of computing, we transform  $S(T), \delta(T)$  as follows:

$$\begin{aligned}
 S(T) &= S(t) \exp \left\{ \int_t^T (r(u) - \sigma_S^2(u)/2) du + \int_t^T \sigma_S(u) d\tilde{B}_S(u) \right\} \\
 &= S(t) \exp \left\{ G(t, T, r(t)) - \int_t^T \sigma_S^2(u)/2 du \right. \\
 &\quad \left. + \int_t^T (\sigma_r(u)m(u, T) d\tilde{B}_r(u) + \sigma_S(u) d\tilde{B}_S(u)) \right\} \\
 &= S(t) \exp \left\{ G(t, T, r(t)) - \int_t^T \frac{\sigma_S^2(u)}{2} du + \int_t^T \Delta_1 d\tilde{B}_S(u) \right\},
 \end{aligned} \tag{22}$$

where

$$\begin{aligned}
 d\tilde{B}_S(t) &= \frac{\sigma_r(t)m(t, T) d\tilde{B}_r(t) + \sigma_S(t) d\tilde{B}_S(t)}{\Delta_1}, \\
 \Delta_1 &= \sqrt{\sigma_r^2(t)m^2(t, T) + \sigma_S^2(t)}.
 \end{aligned} \tag{23}$$

From Lemma 1 we see that  $\tilde{B}_S(t)$  is a standard brown motion under the probability  $\tilde{P}$ .

Under the probability  $\tilde{P}$ , let  $\delta(t) = V(t)/D(t)$ , according to Itô's formula we have

$$\begin{aligned}
 \delta(T) &= \delta(t) \exp \left\{ \int_t^T \frac{1}{2} (\sigma_D^2(u) - \sigma_V^2(u)) du \right. \\
 &\quad \left. + \int_t^T (\sigma_V(u) d\tilde{B}_V(u) - \sigma_D(u) d\tilde{B}_D(u)) \right\} \\
 &= \delta(t) \exp \left\{ \int_t^T \frac{1}{2} (\sigma_D^2(u) - \sigma_V^2(u)) du + \int_t^T \Delta_2 d\tilde{B}_\sigma(u) \right\},
 \end{aligned} \tag{24}$$

where

$$\begin{aligned}
 d\tilde{B}_\sigma(t) &= \frac{\sigma_V(t) d\tilde{B}_V(t) - \sigma_D(t) d\tilde{B}_D(t)}{\Delta_2}, \\
 \Delta_2 &= \sqrt{\sigma_V^2(t) + \sigma_D^2(t) - 2\sigma_{VD}(t)}, \sigma_{VD}(t) = \rho_{VD}(t)\sigma_V(t)\sigma_D(t).
 \end{aligned} \tag{25}$$

By Lemma 1 again, we have that  $\tilde{B}_\sigma(t)$  is a standard brown motion under the probability  $\tilde{P}$ .

### 3.2. The pricing formula of vulnerable binary option

**THEOREM 2** Under the firm value model, assume underlying asset price is  $S_t$  at the time  $t$ , exercise price is  $K$ , the ratio of compensation is  $\delta(T) = V(T)/D(T)$ , the price of vulnerable cash or nothing call option at the time  $t(0 \leq t \leq T)$  is:

$$C(t) = GRN(d_1, d_2; \rho) + GK\delta(t)b_2N\left(d_1 + a_1, -(d_2 + \sqrt{\int_t^T \Delta_2^2 du}); -\rho\right). \tag{26}$$

where

$$d_1 = \frac{\ln \frac{S(t)}{K} + G(t, T, r(t)) - \int_t^T \sigma_S^2(u)/2 du}{\sqrt{\int_t^T \Delta_1^2 du}},$$

$$d_2 = \frac{\ln \delta(t) + \int_t^T \frac{1}{2}(\sigma_D^2(u) - \sigma_V^2(u)) du}{\sqrt{\int_t^T \Delta_2^2 du}},$$

$$a_1 = \frac{\int_t^T (\sigma_{SV}(u) - \sigma_{VD}(u)) du}{\sqrt{\int_t^T \Delta_1^2 du}},$$

$$b_2 = \exp\left\{\int_t^T (\sigma_D^2(u) - \sigma_{VD}(u)) du\right\},$$

$$G = \exp\left\{-G(t, T, r(t)) + \frac{1}{2} \int_t^T \sigma_r^2(u)m^2(u, T) du\right\},$$

$$\rho = \frac{\int_t^T \Delta_1 \Delta_2 \rho_{\bar{B}_S \bar{B}_\sigma} du}{\sqrt{\int_t^T \Delta_1^2 du} \sqrt{\int_t^T \Delta_2^2 du}},$$

$$\rho_{\bar{B}_S \bar{B}_\sigma} = \frac{(\sigma_S(t)\sigma_V(t)\rho_{SV} + \sigma_S(t)\sigma_D(t)\rho_{SD})}{\sqrt{(\sigma_r^2(t)m^2(t, T) + \sigma_S^2(t))} \sqrt{(\sigma_r^2(t)m^2(t, T) + 2\sigma_V(t)\sigma_D(t)m(t, T)\rho_{VD})}}$$

$\Delta_1, \Delta_2$  is defined by Equations (23) and (25).

*Proof* Denote  $A_1 = (S(T) > K, \delta(T) \geq 1)$ ,  $A_2 = (S(T) > K, \delta(T) < 1)$ , according to Equation (13), under the risk neutral probability, the price of vulnerable cash or nothing call option is:

$$\begin{aligned} C(t) &= \tilde{E}\left[\exp\left\{-\int_t^T r(u) du\right\} (X(T)I_{\{\delta(T) \geq 1\}} + \delta(T)X(T)I_{\{\delta(T) < 1\}}) | \mathcal{F}_t\right] \\ &= \tilde{E}\left[\exp\left\{-\int_t^T r(u) du\right\} (RI_{A_1} + \delta(T)RI_{A_2}) | \mathcal{F}_t\right] \\ &= \tilde{E}\left[\exp\left\{-G(t, T, r(t)) - \int_t^T \sigma_r(u)m(u, T) d\bar{B}_r(u)\right\} (RI_{A_1} + \delta(T)RI_{A_2}) | \mathcal{F}_t\right] \\ &= \tilde{E}\left[\exp\left\{-G(t, T, r(t)) + \frac{1}{2} \int_t^T \sigma_r^2(u)m^2(u, T) du\right\} (RI_{A_1} + \delta(T)RI_{A_2}) | \mathcal{F}_t\right] \\ &= G\tilde{E}\left[(RI_{A_1} + \delta(T)RI_{A_2}) | \mathcal{F}_t\right] \\ &= G\tilde{E}\left[RI_{A_1} | \mathcal{F}_t\right] + G\tilde{E}\left[\delta(T)RI_{A_2} | \mathcal{F}_t\right] \\ &= D_1 + D_2, \end{aligned}$$

where  $G = \exp \left\{ -G(t, T, r(t)) + \frac{1}{2} \int_t^T \sigma_r^2(u) m^2(u, T) du \right\}$ ,  $D_1 = G\tilde{E}[RI_{A_1} | \mathcal{F}_t]$ ,  $D_2 = G\tilde{E}[\delta(T)RI_{A_2} | \mathcal{F}_t]$ .

Denote the joint distribution function of two dimensional standardized normal random vector is:

$$N(z_1, z_2; \rho) = \int_{z_1}^{-\infty} \int_{z_2}^{-\infty} f(x_1, x_2; \rho) dx_1 dx_2, \quad (\forall -\infty < z_1, z_2 < +\infty),$$

$$f(x_1, x_2; \rho) = \frac{1}{2\pi \sqrt{1-\rho^2}} \exp \left\{ -\frac{1}{2(1-\rho^2)} (x_1^2 - 2\rho x_1 x_2 + x_2^2) \right\}, \quad (\forall -\infty < x_1, x_2 < +\infty, 1-\rho^2 \neq 0).$$

Evaluation of term  $D_1$ :

$$D_1 = GR\tilde{P}(A_1 | \mathcal{F}_t) = GR\tilde{P}((S(T) > K, \delta(T) \geq 1) | \mathcal{F}_t),$$

according to  $S(T) > K$  and  $S(T)$  follows from (22) that

$$S(T) = S(t) \exp \left\{ G(t, T, r(t)) - \int_t^T \sigma_S^2(u)/2 du + \int_t^T \Delta_1 d\tilde{B}_S(u) \right\} > K$$

then,

$$G(t, T, r(t)) - \int_t^T \sigma_S^2(u)/2 du + \int_t^T \Delta_1 d\tilde{B}_S(u) > \ln \frac{K}{S(t)}$$

therefore,

$$\frac{-\int_t^T \Delta_1 d\tilde{B}_S(u)}{\sqrt{\int_t^T \Delta_1^2 du}} < \frac{\ln \frac{S(t)}{K} + G(t, T, r(t)) - \int_t^T \frac{\sigma_S^2(u)}{2} du}{\sqrt{\int_t^T \Delta_1^2 du}}.$$

By (24) and  $\delta(T) \geq 1$ , we have

$$\delta(T) = \delta(t) \exp \left\{ \int_t^T \frac{1}{2} (\sigma_D^2(u) - \sigma_V^2(u)) du + \int_t^T \Delta_2 d\tilde{B}_\sigma(u) \right\} \geq 1.$$

So

$$-\frac{\int_t^T \Delta_2 d\tilde{B}_\sigma(u)}{\sqrt{\int_t^T \Delta_2^2 du}} \leq \frac{\ln \delta_t + \int_t^T \frac{1}{2} (\sigma_D^2(u) - \sigma_V^2(u)) du}{\sqrt{\int_t^T \Delta_2^2 du}}.$$



Substituting this into  $D_1$ , we obtain

$$\begin{aligned}
 D_1 &= GR\tilde{P}((S(T) > K, \delta(T) \geq 1) | \mathcal{F}_t) \\
 &= GR\tilde{P} \left[ \frac{\int_t^T \Delta_1 d\tilde{B}_S(u)}{\sqrt{\int_t^T \Delta_1^2 du}} \right. \\
 &\quad \left. < \frac{\ln \frac{S(t)}{K} + G(t, T, r(t)) - \int_t^T \sigma_S^2(u)/2 du}{\sqrt{\int_t^T \Delta_1^2 du}}, \right. \\
 &\quad \left. - \frac{\int_t^T \Delta_2 d\tilde{B}_\sigma(u)}{\sqrt{\int_t^T \Delta_2^2 du}} \leq \frac{\ln \delta(t) + \int_t^T \frac{1}{2}(\sigma_D^2(u) - \sigma_V^2(u)) du}{\sqrt{\int_t^T \Delta_2^2 du}} \middle| \mathcal{F}_t \right] \\
 &= GRN(d_1, d_2; \rho),
 \end{aligned}$$

where

$$\begin{aligned}
 d_1 &= \frac{\ln \frac{S(t)}{K} + G(t, T, r(t)) - \int_t^T \sigma_S^2(u)/2 du}{\sqrt{\int_t^T \Delta_1^2 du}}, \\
 d_2 &= \frac{\ln \delta(t) + \int_t^T \frac{1}{2}(\sigma_D^2(u) - \sigma_V^2(u)) du}{\sqrt{\int_t^T \Delta_2^2 du}} | \mathcal{F}_t |, \\
 \rho &= \text{Corr} \left( \frac{\int_t^T \Delta_1 d\tilde{B}_S(u)}{\sqrt{\int_t^T \Delta_1^2 du}}, \frac{\int_t^T \Delta_2 d\tilde{B}_\sigma(u)}{\sqrt{\int_t^T \Delta_2^2 du}} \right) \\
 &= \frac{\int_t^T \Delta_1 \Delta_2 \rho_{\tilde{B}_S \tilde{B}_\sigma} du}{\sqrt{\int_t^T \Delta_1^2 du} \sqrt{\int_t^T \Delta_2^2 du}}, \\
 \rho_{\tilde{B}_S \tilde{B}_\sigma} &= \frac{(\sigma_S(t)\sigma_V(t)\rho_{SV} + \sigma_S(t)\sigma_D(t)\rho_{SD})}{\sqrt{(\sigma_S^2(t)m^2(t, T) + \sigma_S^2(t))} \sqrt{(\sigma_S^2(t)m^2(t, T) + 2\sigma_V(t)\sigma_D(t)m(t, T)\rho_{VD})}}.
 \end{aligned}$$

Evaluation of term  $D_2$ : firstly we define a new probability measure, since  $\tilde{B}_\sigma(t), \tilde{B}_S(t)$  are standard brown motions under the measure  $\tilde{P}$ . Define Rodon-Nikodym derivative:

$$\frac{d\hat{P}}{d\tilde{P}} | \mathcal{F}_t = \exp \left\{ \int_t^T \Delta_2 d\tilde{B}_S(u) - \frac{1}{2} \int_t^T \Delta_2^2 du \right\} Z_1(t).$$

Let

$$\begin{aligned}
 d\hat{B}_\sigma(t) &= -\Delta_2 dt + d\tilde{B}_\sigma(t), \\
 d\hat{B}_S(t) &= -\rho_{S\sigma} \Delta_2 dt + d\tilde{B}_S(t) = -\frac{\sigma_{SV}(t) - \sigma_{SD}(t)}{\Delta_1} dt + d\tilde{B}_S(t),
 \end{aligned}$$

where  $\rho_{S\sigma} = \frac{\sigma_{SV}(t) - \sigma_{SD}(t)}{\Delta_1 \Delta_2}, \sigma_{SV}(t) = \rho_{SV} \sigma_S(t) \sigma_V(t), \sigma_{SD}(t) = \rho_{SD} \sigma_S(t) \sigma_V(t).$

According to the Lemma 1, we see that  $\hat{B}_\sigma(t), \hat{B}_S(t)$  are standard brown motions under the measure  $\hat{P}$ .

Under the measure  $\hat{P}$ , we have

$$S(T) = S(t) \exp \left\{ G(t, T, r(t)) - \int_t^T (\sigma_S^2(u)/2 - \sigma_{SV}(u) + \sigma_{SD}(u)) du + \int_t^T \Delta_1 d\hat{B}_S(t) \right\},$$

$$\delta(T) = \delta(t) \exp \left\{ \int_t^T \left[ \frac{1}{2} (\sigma_D^2(u) - \sigma_V^2(u)) + \Delta_2^2 \right] du + \int_t^T \Delta_2 d\hat{B}_\sigma(u) \right\}.$$

Substituting this into  $D_2$ , we get

$$\begin{aligned} D_2 &= G\tilde{E}(\delta(T)RI_{A_2} | \mathcal{F}_t) \\ &= GR\tilde{E} \left[ \delta(t) \exp \left\{ \int_t^T \left[ \frac{1}{2} (\sigma_D^2(u) - \sigma_V^2(u)) + \Delta_2^2 \right] du + \int_t^T \Delta_2 d\hat{B}_\sigma(u) \right\} I_{A_2} | \mathcal{F}_t \right] \\ &= GR\delta(t) \exp \left\{ \int_t^T \left[ \frac{1}{2} (\sigma_D^2(u) - \sigma_V^2(u)) + \Delta_2^2 \right] du - \frac{1}{2} \int_t^T \Delta_2^2 du \right\} \tilde{E}[Z_1(t)Z_1^{-1}(t)I_{A_2} | \mathcal{F}_t] \\ &= GR\delta(t) \exp \left\{ \int_t^T (\sigma_D^2(u) - \sigma_{VD}(u)) \right\} \hat{E}[I_{A_2} | \mathcal{F}_t] \\ &= GR\delta(t) \exp \left\{ \int_t^T (\sigma_D^2(u) - \sigma_{VD}(u)) \right\} \hat{E}[S(T) > K, \delta(T) < 1 | \mathcal{F}_t] \\ &= GR\delta(t) \hat{P} \left[ \left[ -\frac{\int_t^T \Delta_1 d\hat{B}_S(u)}{\sqrt{\int_t^T \Delta_1^2 du}} < \frac{\ln \frac{S(t)}{K} + G(t, T, r(t)) - \int_t^T (\sigma_S^2(u)/2 - \sigma_{SV}(u) + \sigma_{SD}(u)) du}{\sqrt{\int_t^T \Delta_1^2 du}}, \right. \right. \\ &\quad \left. \left. -\frac{\int_t^T \Delta_2 d\hat{B}_\sigma(u)}{\sqrt{\int_t^T \Delta_2^2 du}} > \frac{\ln \delta(t) + \int_t^T \left[ \frac{1}{2} (\sigma_D^2(u) - \sigma_V^2(u)) + \Delta_2^2 \right] du}{\sqrt{\int_t^T \Delta_2^2 du}} \right] | \mathcal{F}_t \right] \\ &= GR\delta(t) b_2 N \left( d_1 + a_1, - \left( d_2 + \sqrt{\int_t^T \Delta_2^2 du}; -\rho \right) \right), \end{aligned}$$

where

$$\begin{aligned}
 b_2 &= \exp \left\{ \int_t^T (\sigma_D^2(u) - \sigma_{VD}(u)) du \right\}; \\
 a_1 &= \frac{\int_t^T (\sigma_{SV}(u) - \sigma_{SD}(u)) du}{\sqrt{\int_t^T \Delta_1^2 du}}, \\
 \rho &= \frac{\int_t^T \Delta_1 \Delta_2 \rho_{\hat{B}_S, \hat{B}_\sigma} du}{\sqrt{\int_t^T \Delta_1^2 du} \sqrt{\int_t^T \Delta_2^2 du}} \\
 &= \text{Corr} \left( \frac{\int_t^T \Delta_1 d\hat{B}_S(u)}{\sqrt{\int_t^T \Delta_1^2 du}}, \frac{\int_t^T \Delta_2 d\hat{B}_\sigma(u)}{\sqrt{\int_t^T \Delta_2^2 du}} \right).
 \end{aligned}$$

Then, by the result of  $D_1, D_2$ , we complete the proof of the Theorem 2.

**COROLLARY 1** Under the firm value model, assume the price of underlying asset at time  $t$  is  $S_t$ , exercise price is  $K$ , the ratio of compensation is  $\delta(T) = V(T)/D(T)$ , the price of vulnerable cash or nothing put option at the time  $t(0 \leq t \leq T)$  is as follows:

$$C(t) = GRN(-d_1, d_2; -\rho) + GR\sigma(t)b_2N\left(-d_1 + a_1, -\left(d_2 + \sqrt{\int_t^T \Delta_2^2 du}\right); \rho\right), \tag{27}$$

where  $G, d_1, d_2, a_1, b_2, \rho$  are given in the Theorem 2.

**THEOREM 3** Under the firm value model, assume the price of underlying asset at time  $t$  is  $S_t$ , exercise price is  $K$ , the ratio of compensation is  $\delta(T) = V(T)/D(T)$ , the price of vulnerable asset or nothing call option at the time  $t(0 \leq t \leq T)$  is as follows:

$$\begin{aligned}
 C(t) &= S(t)N(d_1 + \sqrt{\int_t^T \Delta_1^2 du}, d_2 + a_2; \rho) \\
 &\quad + S(t)\delta(t)b_1N\left(d_1 + a_1 + 2\sqrt{\int_t^T \Delta_1^2 du}, -(d_2 + 2a_2 + \sqrt{\int_t^T \Delta_2^2 du}); -\rho\right).
 \end{aligned} \tag{28}$$

where

$$\begin{aligned}
 a_2 &= \frac{\int_t^T (\sigma_{SV}(u) - \sigma_{SD}(u)) du}{\sqrt{\int_t^T \Delta_2^2 du}}, \\
 b_1 &= \exp \int_t^T (-\sigma_V^2(u) - \sigma_S^2(u) + \sigma_{VD}(u) + \sigma_{SV}(u) - \sigma_{SD}(u)) du,
 \end{aligned}$$

$a_1, d_1, d_2, \rho$  are given in the Theorem 2.

*Proof* Let  $A_1 = (S(T) > K, \delta(T) \geq 1)$ ,  $A_2 = (S(T) > K, \delta(T) < 1)$ , according to (13), under the risk neutral probability, the price of vulnerable cash or nothing call option is as follows:

$$\begin{aligned}
 C(t) &= \tilde{E} \left[ \exp \left\{ - \int_t^T r(u) du \right\} (X(T)I_{\{\delta(T) \geq 1\}} + \sigma(T)X(T)I_{\{\delta(T) < 1\}}) | \mathcal{F}_t \right] \\
 &= \tilde{E} \left[ \exp \left\{ - \int_t^T r(u) du \right\} (S(T)I_{A_1} + \delta(T)S(T)I_{A_2}) | \mathcal{F}_t \right] \\
 &= \tilde{E} \left[ \exp \left\{ -G(t, T, r(t)) - \int_t^T \sigma_r(u)m(u, T) d\tilde{B}_r(u) \right\} (S(T)I_{A_1} + \delta(T)S(T)I_{A_2}) | \mathcal{F}_t \right] \\
 &= \tilde{E} \left[ \exp \left\{ -G(t, T, r(t)) + \frac{1}{2} \int_t^T \sigma_r^2(u)m^2(u, T) du \right\} (S(T)I_{A_1} + \delta(T)S(T)I_{A_2}) | \mathcal{F}_t \right] \\
 &= G\tilde{E}[(S(T)I_{A_1} + \delta(T)S(T)I_{A_2}) | \mathcal{F}_t] \\
 &= G\tilde{E}[S(T)I_{A_1} | \mathcal{F}_t] + G\tilde{E}[\delta(T)S(T)I_{A_2} | \mathcal{F}_t] \\
 &= V_1 + V_2,
 \end{aligned}$$

where  $G = \exp \left\{ -G(t, T, r(t)) + \frac{1}{2} \int_t^T \sigma_r^2(u)m^2(u, T) du \right\}$ .

Evaluation of  $V_1$ : we can define a new probability measure equivalent to  $\tilde{P}$ .

$\tilde{B}_s(t), \tilde{B}_\sigma(t)$  are standard brown motions under the measure  $\tilde{P}$ . Denote  $d\tilde{B}_s(t) d\tilde{B}_\sigma(t) = \rho_{S\sigma} dt$ .

Define the Rodon-Nikodym derivative:

$$\frac{d\tilde{P}}{d\tilde{P}} | \mathcal{F}_t = \exp \left\{ \int_t^T \Delta_1 d\tilde{B}_s(u) - \frac{1}{2} \int_t^T \Delta_1^2 du \right\} Z_2(t).$$

Let  $d\tilde{B}_s(t) = -\Delta_1 dt + d\tilde{B}_s(t), d\tilde{B}_\sigma(t) = -\rho_{S\sigma} \Delta_1 dt + d\tilde{B}_\sigma(t) = -\frac{\sigma_{SV}(t) - \sigma_{SD}(t)}{\Delta_2} dt + d\tilde{B}_\sigma(t)$ , according to the Lemma 1, we see that  $\tilde{B}_s(t), \tilde{B}_\sigma(t)$  are standard brown motions under the measure  $\tilde{P}$ .

Under the measure  $\tilde{P}$ , we obtain

$$\begin{aligned}
 S(T) &= S(t) \exp \left\{ G(t, T, r(t)) - \int_t^T (\sigma_S^2(u)/2 - \Delta_1^2) du + \int_t^T \Delta_1 d\tilde{B}_s(u) \right\}, \\
 \delta(T) &= \delta(t) \exp \left\{ \int_t^T \left[ \frac{1}{2} (\sigma_D^2(u) - \sigma_V^2(u)) + (\sigma_{SV}(u) + \sigma_{SD}(u)) \right] du + \int_t^T \Delta_2 d\tilde{B}_\sigma(u) \right\}.
 \end{aligned}$$

Substituting thus into  $V_1$ , we have

$$\begin{aligned}
 V_1 &= G\tilde{E}(S(T)I_{A_2} | \mathcal{F}_t) \\
 &= G\tilde{E} \left[ S(t) \exp \left\{ G(t, T, r(t)) - \int_t^T \frac{\sigma_S^2(u)}{2} du - \Delta_1^2 \right\} \right. \\
 &\quad \left. + \int_t^T \Delta_1 d\tilde{B}_S(u) \right] I_{A_1} | \mathcal{F}_t \\
 &= GS(t) \exp \left\{ G(t, T, r(t)) - \int_t^T \frac{\sigma_S^2(u)}{2} du \right. \\
 &\quad \left. + \frac{1}{2} \int_t^T \Delta_1^2 du \right\} \tilde{E}[Z_2(t)Z_2^{-1}(t)I_{A_1} | \mathcal{F}_t] \\
 &= S(t)\tilde{E}[I_{A_1} | \mathcal{F}_t] \\
 &= S(t)\tilde{P}[(S(T) > K, \delta(T) \geq 1) | \mathcal{F}_t] \\
 &= S(t)\tilde{P} \left[ \left( -\frac{\int_t^T \Delta_1 d\tilde{B}_S(u)}{\sqrt{\int_t^T \Delta_1^2 du}} < \frac{\ln \frac{S(t)}{K} + G(t, T, r(t)) - \int_t^T (\sigma_S^2(u)/2 - \Delta_1^2) du}{\sqrt{\int_t^T \Delta_1^2 du}}, -\frac{\int_t^T \Delta_2 d\tilde{B}_\sigma(u)}{\sqrt{\int_t^T \Delta_2^2 du}} \right. \right. \\
 &\quad \left. \left. < \frac{\ln \delta(t) + \int_t^T \left[ \frac{1}{2}(\sigma_D^2(u) - \sigma_V^2(u)) + \int_t^T (\sigma_{SV}(u) - \sigma_{SD}(u)) du \right]}{\sqrt{\int_t^T \Delta_2^2 du}} \right) \middle| \mathcal{F}_t \right] \\
 &= S(t)N \left( d_1 + \sqrt{\int_t^T \Delta_1^2 du}, d_2 + a_2; \rho \right),
 \end{aligned}$$

where

$$\begin{aligned}
 a_2 &= \frac{\int_t^T (\sigma_{SV}(u) - \sigma_{SD}(u)) du}{\sqrt{\int_t^T \Delta_2^2 du}}, \\
 \rho &= \frac{\int_t^T \Delta_1 \Delta_2 \rho_{\tilde{B}_S, \tilde{B}_\sigma} du}{\sqrt{\int_t^T \Delta_1^2 du} \sqrt{\int_t^T \Delta_2^2 du}} \\
 &= \text{Corr} \left( -\frac{\int_t^T \Delta_1 d\tilde{B}_S(u)}{\sqrt{\int_t^T \Delta_1^2 du}}, -\frac{\int_t^T \Delta_2 d\tilde{B}_\sigma(u)}{\sqrt{\int_t^T \Delta_2^2 du}} \right).
 \end{aligned}$$

Evaluation of  $V_2$ , under the measure  $\tilde{P}$ , we get

$$\begin{aligned} \delta(T)S(T) &= S(t)\delta(t) \exp \left\{ G(t, T, r(t)) + \int_t^T \left[ \frac{1}{2}(\sigma_D^2(u) - \sigma_V^2(u) - \sigma_S^2(u)) \right] du \right. \\ &\quad \left. + \int_t^T (\Delta_1 d\tilde{B}_S(u) + \int_t^T \Delta_2 d\tilde{B}_\sigma(u)) \right\} \\ &= S(t)\delta(t) \exp \left\{ G(t, T, r(t)) + \int_t^T \left[ \frac{1}{2}(\sigma_D^2(u) - \sigma_V^2(u) - \sigma_S^2(u)) \right] du \right. \\ &\quad \left. + \int_t^T \Delta_3 d\tilde{B}(u) \right\}, \end{aligned}$$

where  $d\tilde{B}(t) = \frac{\Delta_1 d\tilde{B}_S(t) + \Delta_2 d\tilde{B}_\sigma(t)}{\Delta_3}$ ,  $\Delta_3 = \sqrt{\Delta_1^2 + \Delta_2^2 - 2\rho_{S\sigma}\Delta_1\Delta_2}$ , then according to Lemma 1, we see that  $\tilde{B}(t)$  is standard brown motion under the measure  $\tilde{P}$ .

Define the Rodon-Nikodym derivative:

$$\frac{d\tilde{P}}{dP} | \mathcal{F}_t = \exp \left\{ \int_t^T \Delta_3 d\tilde{B}(u) - \frac{1}{2} \int_t^T \Delta_3^2 du \right\} Z_3(t).$$

Let  $d\check{B}(t) = -\Delta_3 dt + d\tilde{B}(t)$ ,  $d\check{B}_S(t) = -\frac{\sigma_{SV}(t) - \sigma_{SD}(t) + \Delta_1^2}{\Delta_1} dt + d\tilde{B}_S(t)$ ,  $d\check{B}_\sigma(t) = -\frac{\sigma_{SV}(t) - \sigma_{SD}(t) + \Delta_2^2}{\Delta_2} dt + d\tilde{B}_\sigma(t)$ , according to the Lemma 1, we can get that  $(\check{B}(t), \check{B}_S(t), \check{B}_\sigma(t))$  is standard brown motion under the measure  $\check{P}$ .

Under the measure  $\check{P}$ , we get

$$\begin{aligned} S(T) &= S(t) \exp \left\{ G(t, T, r(t)) - \int_t^T (\sigma_S^2(u)/2 - \sigma_{SV}(u) \right. \\ &\quad \left. + \sigma_{SD}(u) - 2\Delta_1^2) du + \int_t^T \Delta_1 d\check{B}_S(u) \right\}, \\ \delta(T) &= \delta(t) \exp \left\{ \int_t^T \left[ \frac{1}{2}(\sigma_D^2(u) - \sigma_V^2(u)) + 2(\sigma_{SV}(u) \right. \right. \\ &\quad \left. \left. - \sigma_{SD}(u) + \Delta_2^2) \right] du + \int_t^T \Delta_2 d\check{B}_\sigma(u) \right\}. \end{aligned}$$

Substituting this into  $V_2$ , we have

$$\begin{aligned}
 V_2 &= G\tilde{E}(S(T)\delta(T)I_{A_2}|\mathcal{F}_t) \\
 &= G\tilde{E}\left[S(t)\delta(t)\exp\left\{G(t, T, r(t)) - \int_t^T \frac{1}{2}(\sigma_D^2(u) - \sigma_V^2(u) - \sigma_S^2(u))du + \int_t^T \Delta_3 d\tilde{B}_S(u)\right\}I_{A_2}|\mathcal{F}_t\right] \\
 &= S(t)\delta(t)\exp\left\{\int_t^T (-\sigma_V^2(u) - \sigma_S^2(u) + \sigma_{VD}(u) - \sigma_{SV}(u) + \sigma_{SD}(u))du\right\}\tilde{E}[Z_3(t)Z_3^{-1}(t)I_{A_2}|\mathcal{F}_t] \\
 &= S(t)\delta(t)b_1\tilde{E}[I_{A_2}|\mathcal{F}_t] \\
 &= S(t)\delta(t)b_1\tilde{P}[(S(T) > K, \delta(T) < 1)|\mathcal{F}_t] \\
 &= S(t)\delta(t)b_1\tilde{P}\left[\left(-\frac{\int_t^T \Delta_1 d\tilde{B}_S(u)}{\sqrt{\int_t^T \Delta_1^2 du}} < \frac{\ln \frac{S(t)}{K}}{\sqrt{\int_t^T \Delta_1^2 du}}\right.\right. \\
 &\quad \left.\left. + \frac{G(t, T, r(t)) - \int_t^T \left(\frac{\sigma_S^2(u)}{2} - \sigma_{SV}(u) - \sigma_{SD}(u) - 2\Delta_1^2\right)du}{\sqrt{\int_t^T \Delta_1^2 du}} - \frac{\int_t^T \Delta_2 d\tilde{B}_\sigma(u)}{\sqrt{\int_t^T \Delta_2^2 du}}\right.\right. \\
 &\quad \left.\left. > \frac{\ln \delta(t) + \int_t^T \left[\frac{1}{2}(\sigma_D^2(u) - \sigma_V^2(u) + 2\sigma_{SV}(u) - 2\sigma_{SD}(u) + \Delta_2^2)du\right]}{\sqrt{\int_t^T \Delta_2^2 du}}\right]|\mathcal{F}_t\right] \\
 &= S(t)\delta(t)b_1N\left(d_1 + a_1 + 2\sqrt{\int_t^T \Delta_1^2 du}, -\left(d_2 + 2a_2 + \sqrt{\int_t^T \Delta_2^2 du}\right); -\rho\right),
 \end{aligned}$$

where

$$\begin{aligned}
 b_1 &= \exp\left\{\int_t^T (-\sigma_V^2(u) - \sigma_S^2(u) + \sigma_{VD}(u) - \sigma_{SV}(u) + \sigma_{SD}(u))du\right\}, \\
 \rho &= \frac{\int_t^T \Delta_1 \Delta_2 \rho_{\tilde{B}_S, \tilde{B}_\sigma} du}{\sqrt{\int_t^T \Delta_1^2 du} \sqrt{\int_t^T \Delta_2^2 du}} \\
 &= \text{Corr}\left(\frac{\int_t^T \Delta_1 d\tilde{B}_S(u)}{\sqrt{\int_t^T \Delta_1^2 du}}, \frac{\int_t^T \Delta_2 d\tilde{B}_\sigma(u)}{\sqrt{\int_t^T \Delta_2^2 du}}\right).
 \end{aligned}$$

Then by the result of  $D_1, D_2$ , we complete the proof of Theorem 3.

**COROLLARY 2** Under the firm value model, assume the price of underlying asset at time  $t$  is  $S_t$ , exercise price is  $K$ , the ratio of compensation is  $\delta(T) = V(T)/D(T)$ , the price of vulnerable asset or nothing put option at the time  $t(0 \leq t \leq T)$  is as follows:

$$C(t) = -S(t)N\left(-\left(d_1 + \sqrt{\int_t^T \Delta_1^2 du}\right), d_2 + a_2; -\rho\right) + S(t)\delta(t)b_1$$

$$N\left(-\left(d_1 + a_1 + 2\sqrt{\int_t^T \Delta_1^2 du}\right), -\left(d_2 + 2a_2 + \sqrt{\int_t^T \Delta_2^2 du}\right); \rho\right), \quad (29)$$

where  $b_1$ ,  $a_2$ ,  $\rho$  are given in the Theorem 3.

#### 4. Conclusion

The analytical pricing formula of the vulnerable binary options is derived in this paper by using the martingale method under the assumptions that the stock prices, assets and liabilities of a company follow the relevant O-U processes and the interest rate follows a Hull-White model. Comparing with the models of the pricing vulnerable European option in Klein and Inglis (2001), Ammann (2002), and Ting and Deng (2007), the volatility is corrected to be a function with respect to time rather than a constant, the geometric Brownian motion is corrected to be an O-U process. And also the interest rate is assumed to be random one following the Hull-White model rather than only a function with respect to time in Ding and Chan(2007). All the corrections make the model more realistic.

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