Interval-valued Pythagorean fuzzy geometric aggregation operators and their application to group decision making problem


Abstract: There are many aggregation operators have been defined up to date, but in this work, we define the interval valued Pythagorean fuzzy weighted geometric (IPFWG) operator, the interval-valued Pythagorean fuzzy ordered weighted geometric (IPFOWG) operator, and the interval-valued Pythagorean fuzzy hybrid geometric operator. We also discuss some properties and give some examples also to develop these operators. At the last we apply the interval-valued IPFWG operator and the interval-valued IPFOWG operator to multiple attribute decision-making problem under the interval-valued Pythagorean fuzzy information.

Keywords: IVPFWG operator; IVPFOWG operator; IVPFHG operator; decision making problem

1. Introduction

Zadeh (1965) proposed a remarkable theory of fuzzy sets (FSs) characterized by a membership function. Since then, the FS theory has been successfully applied to various fields of multi-attribute...
decision-making. Moreover, extended FSs were developed, such as, interval valued fuzzy sets (IVFSs) (Turksen, 1986), intuitionistic fuzzy sets (IFSSs) (Atanassov, 1986), and hesitant fuzzy sets (HFSs) (Torra, 2010). Although FS theory has been developed and generalized, it cannot account for all possible uncertainties in a variety of physical problems. For instance, when an expert is asked a question, he or she may think that the possibility of a true answer is equal to 0.6, the possibility of a false answer is 0.4, and the degree of their uncertainty is 0.2. This issue is beyond the scope of FSs and IFSs. Hence, Smarandache proposed neutrosophic logic and neutrosophic sets (NSs) in Smarandache (1999). A NS is a set in which each element of the universe has respective degrees of truth, indeterminacy, and falsity, which lie in the nonstandard unit interval of \([0, 1]\). This method represents an extension of the standard interval \([0, 1]\) used for IFSs. The uncertainty presented here (i.e. the indeterminacy factor) is independent of the truth and falsity values. This extended IFS theory to account for uncertain information. Similarly, there are many cases where the decision-maker may provide the degree of membership and non-membership of a particular attribute in such a way that their sum is greater than one. For example, suppose a person expresses his preferences toward the alternative in such a way that degree of their satisfaction is 0.6 and degree of rejection is 0.8. Obviously its sum is greater than one. To solve these types of problems, Yager (2013, 2014) introduced the concept of another set called Pythagorean fuzzy set. Pythagorean fuzzy set is a more powerful tool to solve uncertain problems. Actually PFS is a generalization of IFS. There are many scholars Bustince, Herrera, and Montero (2007), Atanassov (1999), Jiang, Tang, Wang, and Tang (2009), Xu (2008) and Xu, Chen, and Wu (2008) have done works in the field of IFS and its applications. Particularly, information aggregation is a very crucial research area in AIFS theory that has been receiving more and more focus. Xu (2007a) developed some basic arithmetic aggregation operators, including IFWA operator, IFOWA operator, and IFHA operator. Xu and Yager (2006) defined some basic geometric aggregation operators, such as intuitionistic fuzzy weighted geometric (IFWG) operator, intuitionistic fuzzy ordered weighted geometric (IFOWG) operator, and intuitionistic fuzzy hybrid geometric (IFHG) operator, and applied them to multiple attribute decision-making (MADM) based on intuitionistic fuzzy information. In Rahman, Ali Khan, Abdullah, and Husain (2017), Rahman and Ali (2017), Rahman, Abdullah, Ali Khan, and Shakeel (2016) and Rahman, Abdullah, Husain, Ali Khan, and Shakeel (2017) introduced the notion of some properties of Pythagorean fuzzy hybrid averaging aggregation operator, Pythagorean fuzzy weighted geometric aggregation operator, Pythagorean fuzzy ordered weighted geometric aggregation operator, Pythagorean fuzzy hybrid geometric aggregation operator and applied them to group decision-making.

But, in some real decision-making problems, due to insufficiency in available information, it may be difficult for decision-makers to exactly quantify their opinions with a crisp number, but they can be represented by an interval number within \([0, 1]\). Therefore Atanassov and Gargov (1989) introduced the notion of interval-valued intuitionistic fuzzy sets (IVIFSs) which is a generalization of fuzzy sets and IFSs. Na and Zeshui (2014) introduced the concept of interval-valued hesitant fuzzy sets (IVHFSs) which is a generalization of HFSs. Wang, Smarandache, Zhang, and Sunderraman (2005) introduced the notion of interval-valued neutrosophic sets (IVNSs) which is a generalization of NSs and interval-valued intuitionistic fuzzy sets (IVIFSs). Peng and Yang (2015) introduced the concept of interval-valued Pythagorean fuzzy set which is a generalization of Pythagorean fuzzy sets and interval-valued intuitionistic fuzzy sets. Xu (2007b) and Xu and Chen (2007) Chen and Xu familiarized a series of a new types of aggregation operators, such as IIFHA operator, IIFOWA operator, IIFWA operator, IIFHG operator, IIFOWG operator, IIFWA operator and also applied them to group decision-making. Wang and Liu (2013a, 2013b) introduced the notion of IVIFEHW operator and IVIFEHWG operator and also applied them to group decision-making. Yang and Yuan (2014) introduced the notion of I-IVIFEHW operator and I-IVIFEHW operator and also applied them to group decision-making. There are two types of operations, such as sum and product. Rahman and Ali Khan (in press) used the sum and introduced the notion of arithmetic aggregating operators such as interval-valued Pythagorean fuzzy weighted averaging (IVPFWA) operator, interval-valued Pythagorean fuzzy ordered weighted averaging (IVPFOWA) operator, and interval-valued Pythagorean fuzzy hybrid averaging (IVPFHA) operator and applied them to decision-making. But in this paper we use the product and introduce the notion of geometric aggregation operators such as, interval-valued
Pythagorean fuzzy weighted geometric (IVPFWG) operator, interval-valued Pythagorean fuzzy ordered weighted geometric (IVPFOWG) operator, and interval-valued Pythagorean fuzzy hybrid geometric (IVPFHG) operator and applied them to decision-making. These both are new methods for decision-making problem under interval-valued Pythagorean fuzzy information but geometric aggregation operator is more reliable than arithmetic aggregating operators.

Thus keeping the advantages of the above-mentioned aggregation operators, in this article we introduce a series of interval-valued Pythagorean fuzzy geometric aggregation operators for interval-valued Pythagorean fuzzy numbers. Namely, interval-valued Pythagorean fuzzy weighted geometric (IVPFWG) operator, interval-valued Pythagorean fuzzy ordered weighted geometric (IVPFOWG) operator, and interval-valued Pythagorean fuzzy hybrid geometric (IVPFHG) operator and applied them to decision-making. Moreover, we introduce some of their basic properties and give some examples. At the last of the paper, we present an application of these proposed operators. By comparison with the existing method, it is decided that the method developed in this paper is a good complement to the existing method.

The remainder of this paper is structured as follows. In Section 2, we give some basic definitions and results which will be used in our later sections. In Section, we introduce some interval-valued Pythagorean fuzzy geometric aggregation operators for interval-valued Pythagorean fuzzy numbers. In Section 4, we apply these operators to MADM problem making. In Section 5, we develop a numerical example. In Section 6, we have conclusion.

2. Preliminaries

Definition 1 (Zadeh, 1965) Let $X$ be a fixed set, then a fuzzy set can be defined as:
\[
F = \{ \langle x, \mu(x) \rangle | x \in X \},
\]
where $\mu$ is a mapping from $X$ to $[0, 1]$, and $\mu(x)$ is said to be the degree membership of $x$ in $X$.

Definition 2 (Atanassov, 1986) Let $X$ be a fixed set, then an IFS can be defined as:
\[
I = \{ \langle x, \mu(x), \nu(x) \rangle | x \in X \},
\]
where $\mu(x)$ and $\nu(x)$ are mappings from $X$ to $[0, 1]$ also $0 \leq \mu(x) \leq 1$, $0 \leq \nu(x) \leq 1$, for all $x \in X$, and represent the degrees of membership and nonmembership of element $x \in X$ to set $I$, respectively. Let $\pi(x) = 1 - \mu(x) - \nu(x)$, then it is called the intuitionistic fuzzy index of $x \in X$ to set $I$, represent the degree of indeterminacy of $x$ to $I$. Also $0 \leq \pi(x) \leq 1$ for every $x \in X$.

Definition 3 (Yager, 2014) Let $X$ be a fixed set, then a Pythagorean fuzzy set can be defined as:
\[
P = \{ \langle x, \mu(x), \nu(x) \rangle | x \in X \},
\]
where $\mu(x)$ and $\nu(x)$ are mappings from $X$ to $[0, 1]$, such that $0 \leq \mu(x) \leq 1$, $0 \leq \nu(x) \leq 1$, and $0 \leq \mu^2(x) + \nu^2(x) \leq 1$, for all $x \in X$, and represent the degrees of membership and nonmembership of element $x \in X$ to set $P$. Let $\pi(x) = \sqrt{1 - \mu^2(x) - \nu^2(x)}$, then it is said to be Pythagorean fuzzy index of element $x \in X$ to set $P$, represent the degree of indeterminacy of $x$ to $P$. Also $0 \leq \pi(x) \leq 1$ for every $x \in X$.

Definition 4 Peng and Yang (2015) Let $X$ be a fixed set, then an interval-valued Pythagorean fuzzy set can be defined as:
\[
Q = \{ \langle x, \mu(x), \nu(x) \rangle | x \in X \},
\]
where
\[ \mu_Q(x) = [\mu_Q^L(x), \mu_Q^U(x)] \subset [0, 1], \] (5)

and
\[ \nu_Q(x) = [\nu_Q^L(x), \nu_Q^U(x)] \subset [0, 1], \] (6)

are the intervals, and \( \mu_Q^L(x) = \inf \mu_Q(x) \) and \( \mu_Q^U(x) = \sup \mu_Q(x) \), similarly \( \nu_Q^L(x) = \inf \nu_Q(x) \) and \( \nu_Q^U(x) = \sup \nu_Q(x) \) for all \( x \in X \). And also
\[ 0 \leq (\mu_Q^L(x))^2 + (\nu_Q^L(x))^2 \leq 1. \] (7)

If
\[ \pi_Q(x) = [\pi_Q^L(x), \pi_Q^U(x)], \] (8)

for all \( x \in X \), then it is called the interval-valued Pythagorean fuzzy index of \( x \) to \( Q \), where
\[ \pi_Q^L(x) = \sqrt{1 - (\mu_Q^L(x))^2 - (\nu_Q^L(x))^2}, \] (9)

and
\[ \pi_Q^U(x) = \sqrt{1 - (\mu_Q^U(x))^2 - (\nu_Q^U(x))^2}. \] (10)

There are two special cases of interval-valued Pythagorean fuzzy set.

(i) If \( \mu_Q^L(x) = \mu_Q^U(x) \) and \( \nu_Q^L(x) = \nu_Q^U(x) \), then an interval-valued Pythagorean fuzzy set reduces to Pythagorean fuzzy set.

(ii) If \( \mu_Q^L(x) + \nu_Q^U(x) \leq 1 \), then an interval-valued Pythagorean fuzzy set reduces to an interval-valued IFS.

Definition 5 Peng and Yang (2015) Let \( \alpha = ([\mu^L, \mu^U], [\nu^L, \nu^U]) \), \( \alpha_2 = ([\mu_2^L, \mu_2^U], [\nu_2^L, \nu_2^U]) \) be the three interval-valued Pythagorean fuzzy numbers and \( \delta > 0 \), then the following operational laws holds:

\[ \delta \alpha = \left[ \sqrt{1 - \left(1 - \left(\frac{\mu^L}{\delta}\right)^2\right)^2}, \sqrt{1 - \left(1 - \left(\frac{\mu^U}{\delta}\right)^2\right)^2} \right], \] (11)

\[ (\alpha)^\delta = \left[ \left(\frac{\mu^L}{\delta}\right)^2, \left(\frac{\mu^U}{\delta}\right)^2 \right], \] (12)

\[ \alpha_1 \otimes \alpha_2 = \left[ \sqrt{\left(\frac{\nu^L_1}{\nu^L_2}\right)^2 + \left(\frac{\nu^U_1}{\nu^U_2}\right)^2 - \left(\frac{\nu^L_1}{\nu^L_2}\right)^2 \left(\frac{\nu^U_1}{\nu^U_2}\right)^2}, \sqrt{\left(\frac{\nu^L_1}{\nu^L_2}\right)^2 + \left(\frac{\nu^U_1}{\nu^U_2}\right)^2 - \left(\frac{\nu^L_1}{\nu^L_2}\right)^2 \left(\frac{\nu^U_1}{\nu^U_2}\right)^2} \right]. \] (13)
\[ \bar{a}_1 \oplus \bar{a}_2 = \left( \frac{\sqrt{(\mu_{\bar{a}_1})^2 + (\mu_{\bar{a}_2})^2 - (\mu_{\bar{a}_1})^2 - (\mu_{\bar{a}_2})^2}}{\mu_{\bar{a}_1}} \right) \]

**Definition 6** Peng and Yang (2015) Let \( \alpha = ([\mu_\alpha^L, \mu_\alpha^U], [\nu_\alpha^L, \nu_\alpha^U]) \) be the interval-valued Pythagorean fuzzy number, then the score function of \( \alpha \) can be defined as:

\[ S(\alpha) = \frac{1}{2} \left[ (\mu_\alpha^L)^2 + (\mu_\alpha^U)^2 - (\nu_\alpha^L)^2 - (\nu_\alpha^U)^2 \right], \]

where \( S(\alpha) \in [-1, 1] \).

**Definition 7** Peng and Yang (2015) Let \( \alpha = ([\mu_\alpha^L, \mu_\alpha^U], [\nu_\alpha^L, \nu_\alpha^U]) \) be the interval-valued Pythagorean fuzzy number, then the accuracy function of \( \alpha \) can be defined as:

\[ H(\alpha) = \frac{1}{2} \left[ (\mu_\alpha^L)^2 + (\mu_\alpha^U)^2 + (\nu_\alpha^L)^2 + (\nu_\alpha^U)^2 \right], \]

where \( H(\alpha) \in [0, 1] \).

**Definition 8** Peng and Yang (2015) Let \( \alpha = ([\mu_\alpha^L, \mu_\alpha^U], [\nu_\alpha^L, \nu_\alpha^U]), \beta = ([\mu_\beta^L, \mu_\beta^U], [\nu_\beta^L, \nu_\beta^U]) \) be two interval-valued Pythagorean fuzzy numbers, then

\[ S(\alpha) = \frac{1}{2} \left[ (\mu_\alpha^L)^2 + (\mu_\alpha^U)^2 - (\nu_\alpha^L)^2 - (\nu_\alpha^U)^2 \right], \]  

and

\[ S(\beta) = \frac{1}{2} \left[ (\mu_\beta^L)^2 + (\mu_\beta^U)^2 - (\nu_\beta^L)^2 - (\nu_\beta^U)^2 \right], \]

be the scores of \( \alpha \) and \( \beta \), respectively, and

\[ H(\alpha) = \frac{1}{2} \left[ (\mu_\alpha^L)^2 + (\mu_\alpha^U)^2 + (\nu_\alpha^L)^2 + (\nu_\alpha^U)^2 \right], \]

and

\[ H(\beta) = \frac{1}{2} \left[ (\mu_\beta^L)^2 + (\mu_\beta^U)^2 + (\nu_\beta^L)^2 + (\nu_\beta^U)^2 \right], \]

be the accuracy degrees of \( \alpha \) and \( \beta \), respectively, then the following holds:

1. If \( S(\alpha) < S(\beta) \), then \( \alpha < \beta \)
2. If \( S(\alpha) = S(\beta) \), then we have the following two conditions.
   a. If \( H(\alpha) = H(\beta) \), then \( \alpha = \beta \),
   b. If \( H(\alpha) < H(\beta) \), then \( \alpha < \beta \)
   c. If \( H(\alpha) > H(\beta) \), then \( \alpha > \beta \).

**Definition 9** (Xu & Chen, 2007) Let \( \Psi \) be the set of all interval-valued intuitionistic fuzzy numbers and \( \psi_j = ([\mu_{\psi_j}^L, \mu_{\psi_j}^U], [\nu_{\psi_j}^L, \nu_{\psi_j}^U]) \) \((j = 1, 2, \ldots, n)\) be a collection of interval-valued intuitionistic fuzzy numbers, and let \( \text{IVIFWG} : \Psi^n \rightarrow \Psi \), if
where \( w = (w_1, w_2, \ldots, w_n)^T \) is the weighted vector of \( a_j (j = 1, 2, \ldots, n) \) with \( w_j \in [0, 1] \) and \( \sum_{j=1}^n w_j = 1 \), then IVIFWG is called interval-valued IFWG operator.

**Definition 10** (Xu & Chen, 2007) An interval-valued intuitionistic fuzzy ordered weighted geometric (IVIFOWG) operator, interval-valued intuitionistic fuzzy sets

\[
\text{IVIFOWG}_w (a_1, a_2, a_3, \ldots, a_n)
= \left\{ \begin{array}{ll}
\prod_{j=1}^n \left( \mu_{a_j}^L \right)^{w_j}, & \prod_{j=1}^n \left( \mu_{a_j}^U \right)^{w_j}, \\
1 - \prod_{j=1}^n \left( 1 - \nu_{a_j}^L \right)^{w_j}, & 1 - \prod_{j=1}^n \left( 1 - \nu_{a_j}^U \right)^{w_j}
\end{array} \right. \right\}
\]

(21)

where \( a_j \) is the \( j \)th largest value of \( a_j \).

**Definition 11** (Xu & Chen, 2007) An interval-valued IFHG operator of dimension \( n \) is a mapping \( \Omega^n \to \Omega \), which has an associated vector \( w = (w_1, w_2, \ldots, w_n)^T \) with \( w_j \in [0, 1] \) \( \sum_{j=1}^n w_j = 1 \). Furthermore

\[
\text{IVIFHG}_{w,w} (a_1, a_2, a_3, \ldots, a_n)
= \left\{ \begin{array}{ll}
\prod_{j=1}^n \left( \mu_{a_j}^L \right)^{w_j}, & \prod_{j=1}^n \left( \mu_{a_j}^U \right)^{w_j}, \\
1 - \prod_{j=1}^n \left( 1 - \nu_{a_j}^L \right)^{w_j}, & 1 - \prod_{j=1}^n \left( 1 - \nu_{a_j}^U \right)^{w_j}
\end{array} \right. \right\}
\]

(22)

where \( \hat{a}_j \) is the \( j \)th largest of the weighted intuitionistic fuzzy values \( \hat{a}_j = (a_j)^{w_j} \), \( w = (w_1, w_2, \ldots, w_n)^T \) is the weighted vector of \( a_j \), where \( (j = 1, 2, \ldots, n) \) such that \( w_j \in [0, 1] \) \( \sum_{j=1}^n w_j = 1 \) and \( n \) is the balancing coefficient, which plays a role of balance. If the vector \( (w_1, w_2, \ldots, w_n)^T \) approaches \( \left( \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n} \right)^T \), then the vector \( (a_1)^{w_1}, (a_2)^{w_2}, \ldots, (a_n)^{w_n} \) approaches \( (a_1, a_2, \ldots, a_n)^T \).

### 3. Some geometric aggregation operators based on interval-valued Pythagorean fuzzy sets

In this section, we introduce the notion of some new aggregation operators such as, interval-valued Pythagorean fuzzy weighted geometric (IVPFWG) operator, interval-valued Pythagorean fuzzy
ordered weighted geometric (IVPFOWG) operator, and interval-valued Pythagorean fuzzy geometric (IVPFHG) operator. We also discuss some desirable properties and give some examples.

3.1. Interval-valued Pythagorean fuzzy weighted geometric operator

**Theorem 1** Let \( \alpha = \left( \left( \mu^l_1, \mu^u_1, v^l_1, v^u_1 \right), \left( \mu^l_2, \mu^u_2, v^l_2, v^u_2 \right) \right) \) and \( \alpha_2 = \left( \left( \mu^l_3, \mu^u_3, v^l_3, v^u_3 \right), \left( \mu^l_4, \mu^u_4, v^l_4, v^u_4 \right) \right) \) be the three interval-valued Pythagorean fuzzy numbers, then we have:

(i) If \( \mu^l_1 = \mu^u_1 = \mu^l_2 = \mu^u_2 = v^l_1 = v^u_1 = v^l_2 = v^u_2 \), then \( \alpha_1 = \alpha_2 \).

(ii) If \( \mu^l_1 \leq \mu^l_2, \mu^u_1 \leq \mu^u_2 \) and \( v^l_1 \geq v^l_2, v^u_1 \geq v^u_2 \), then \( \alpha_1 \leq \alpha_2 \).

(iii) If \( \mu^l_1 \leq \mu^l_2, \mu^u_1 \leq \mu^u_2 \) and \( v^l_1 = v^l_2, v^u_1 = v^u_2 \), then \( \alpha_1 = \alpha_2 \).

(iv) If \( \mu^l_1 \geq \mu^l_2, \mu^u_1 \geq \mu^u_2 \) and \( v^l_1 = v^l_2, v^u_1 = v^u_2 \), then \( \alpha_2 \leq \alpha_1 \).

(v) If \( \mu^l_1, \mu^u_1 < \mu^l_2, \mu^u_2 \), then the score function must be negative.

(vi) If \( \mu^l_1, \mu^u_1 > \mu^l_2, \mu^u_2 \), then the score function must be positive.

**Proof** Straightforward.

**Theorem 2** Let \( \alpha, \alpha_1, \alpha_2 \) be the three interval-valued Pythagorean fuzzy numbers, and \( \delta, \delta_1, \delta_2 > 0 \), then the following are hold:

(i) \( \alpha_1 \oplus \alpha_2 = \alpha_2 \oplus \alpha_1 \).

(ii) \( \alpha_1 \otimes \alpha_2 = \alpha_2 \otimes \alpha_1 \).

(iii) \( \delta(\alpha_1 \oplus \alpha_2) = \delta \alpha_1 \oplus \delta \alpha_2 \).

(iv) \( (\alpha_1 \otimes \alpha_2)^\delta = (\alpha_1)^\delta \otimes (\alpha_2)^\delta \).

(v) \( \delta_1(\alpha) \oplus \delta_2(\alpha) = (\delta_1 \oplus \delta_2)(\alpha) \).

(vi) \( (\alpha)^{\delta_1} \otimes (\alpha)^{\delta_2} = \alpha^{(\delta_1 \otimes \delta_2)} \).

**Proof** The proof is easy so it is omitted here.

**Definition 12** Let \( \Psi \) be the set of all interval-valued Pythagorean fuzzy numbers and \( a_j = \left( \left( \mu^l_j, \mu^u_j, v^l_j, v^u_j \right), \left( \mu^l_j, \mu^u_j, v^l_j, v^u_j \right) \right) \) \( j = 1, 2, \ldots, n \) be a collection of interval-valued Pythagorean fuzzy numbers, and let IVPFWG: \( \Psi \to \Psi \), if

\[
\text{IVPFWG}_w(a_1, a_2, a_3, \ldots, a_n) = \left( \prod_{j=1}^{n} \left( \mu^l_j \right)^{w_j}, \prod_{j=1}^{n} \left( \mu^u_j \right)^{w_j} \right) \left\{ \sqrt{1 - \prod_{j=1}^{n} \left( 1 - \left( v^l_j \right)^3 \right)^{w_j}}, \sqrt{1 - \prod_{j=1}^{n} \left( 1 - \left( v^u_j \right)^3 \right)^{w_j}} \right\},
\]

where \( w = (w_1, w_2, \ldots, w_n)^T \) is the weighted vector of \( a_j (j = 1, 2, \ldots, n) \) with \( w_j \in [0, 1] \) and \( \sum_{j=1}^{n} w_j = 1 \), then IVPFWG is called interval-valued Pythagorean fuzzy weighted geometric operator. Specially if \( w = \left( \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n} \right)^T \), then interval-valued Pythagorean fuzzy weighted geometric operator is reduced to interval-valued Pythagorean fuzzy geometric operator.

**Example 1** Let

\( a_1 = ([0.5, 0.7], [0.3, 0.4]), a_2 = ([0.3, 0.6], [0.2, 0.6]) \)

\( a_3 = ([0.3, 0.5], [0.3, 0.6]), a_4 = ([0.2, 0.6], [0.4, 0.7]) \)
and let \( w = (0.1, 0.2, 0.3, 0.4)^T \) be the weighted vector of \( a_j (j = 1, 2, 3, 4) \), then we have

\[
\text{IVPFWG}_w(a_1, a_2, a_3, a_4) = \\
\left[ \prod_{j=1}^{4} \left( \mu_{a_j}^w \right)^{w_j} \right] \left( 1 - \left( \nu_{a_j}^w \right)^{2w_j} \right)^{w_j} \\
\prod_{j=1}^{4} \left( 1 - \left( \nu_{a_j}^w \right)^{2w_j} \right)^{w_j} \\
\sqrt{1 - \prod_{j=1}^{4} \left( 1 - \left( \nu_{a_j}^w \right)^{2w_j} \right)^{w_j}} \\
\sqrt{1 - \prod_{j=1}^{4} \left( \mu_{a_j}^w \right)^{w_j}} \\
\right]
\]

\[
= \left[ \begin{array}{c}
0.5^{0.1} \cdot 0.3^{0.2} \cdot 0.3^{0.3} \cdot 0.2^{0.4} \\
0.7^{0.1} \cdot 0.6^{0.2} \cdot 0.5^{0.3} \cdot 0.6^{0.4}
\end{array} \right],
\]

\[
= \left[ \begin{array}{c}
\sqrt{1 - (1 - 0.09)^{0.1} (1 - 0.04)^{0.2} (1 - 0.09)^{0.3} (1 - 0.16)^{0.4}} \\
\sqrt{1 - (1 - 0.16)^{0.1} (1 - 0.36)^{0.2} (1 - 0.36)^{0.3} (1 - 0.49)^{0.4}}
\end{array} \right]
\]

\[
= (0.2684, 0.5768), (0.3306, 0.6321).
\]

**Theorem 3** Let \( a_j (j = 1, 2, \ldots, n) \) be a collection of interval-valued Pythagorean fuzzy numbers, then their aggregated value using the interval-valued Pythagorean fuzzy weighted geometric (IVPFWG) operator is also interval-valued Pythagorean fuzzy number and also satisfies

\[
\text{IVPFWG}_w(a_1, a_2, a_3, \ldots, a_n) = \\
\left[ \prod_{j=1}^{n} \left( \mu_{a_j}^w \right)^{w_j} \right] \left( 1 - \left( \nu_{a_j}^w \right)^{2w_j} \right)^{w_j} \\
\prod_{j=1}^{n} \left( 1 - \left( \nu_{a_j}^w \right)^{2w_j} \right)^{w_j} \\
\sqrt{1 - \prod_{j=1}^{n} \left( 1 - \left( \nu_{a_j}^w \right)^{2w_j} \right)^{w_j}} \\
\sqrt{1 - \prod_{j=1}^{n} \left( \mu_{a_j}^w \right)^{w_j}} \\
\right]
\]

where \( w = (w_1, w_2, \ldots, w_n)^T \) is the weighted vector of \( a_j (j = 1, 2, \ldots, n) \) with \( w_j \in [0, 1] \) and \( \sum_{j=1}^{n} w_j = 1 \).

**Proof** This theorem can be proved by mathematical induction. First we show that Equation (25) holds, for \( n = 2 \). Since

\[
(a_1)^w = \\
\left[ \begin{array}{c}
\left( \mu_{a_1}^w \right)^{w_1}, \left( \nu_{a_1}^w \right)^{2w_1} \\
\sqrt{1 - \left( \nu_{a_1}^w \right)^{2w_1}}, \sqrt{1 - \left( \nu_{a_1}^w \right)^{2w_1}}
\end{array} \right]
\]

\[
(a_2)^w = \\
\left[ \begin{array}{c}
\left( \mu_{a_2}^w \right)^{w_2}, \left( \nu_{a_2}^w \right)^{2w_2} \\
\sqrt{1 - \left( \nu_{a_2}^w \right)^{2w_2}}, \sqrt{1 - \left( \nu_{a_2}^w \right)^{2w_2}}
\end{array} \right]
\]

Then
Thus Equation (25) holds for \( n = 2 \), now we show that Equation (25) holds for \( n = k \), i.e. 

\[
IVPFWG_w(a_1, a_2, a_3, \ldots, a_k) = \left( \prod_{j=1}^{k} \left( \mu_{\alpha_j}^{w_j} \right)^{w_j} \right)^{1/k} \left( \prod_{j=1}^{k} \left( \mu_{\gamma_j}^{w_j} \right)^{w_j} \right)^{1/k} \left( \prod_{j=1}^{k} \left( \mu_{\beta_j}^{w_j} \right)^{w_j} \right)^{1/k}, 
\]

\[
= \frac{1}{k} \left( \prod_{j=1}^{k} \left( 1 - (\mu_{\alpha_j}^{w_j})^2 \right)^{w_j} \right)^{1/k} \left( \prod_{j=1}^{k} \left( 1 - (\mu_{\gamma_j}^{w_j})^2 \right)^{w_j} \right)^{1/k} \left( \prod_{j=1}^{k} \left( 1 - (\mu_{\beta_j}^{w_j})^2 \right)^{w_j} \right)^{1/k}, 
\]

If Equation (25) holds for \( n = k \), then we show that Equation (25) holds for \( n = k + 1 \), i.e. 

\[
IVPFWG_w(a_1, a_2, a_3, \ldots, a_{k+1}) = \left( \prod_{j=1}^{k+1} \left( \mu_{\alpha_j}^{w_j} \right)^{w_j} \right)^{1/(k+1)} \left( \prod_{j=1}^{k+1} \left( \mu_{\gamma_j}^{w_j} \right)^{w_j} \right)^{1/(k+1)} \left( \prod_{j=1}^{k+1} \left( \mu_{\beta_j}^{w_j} \right)^{w_j} \right)^{1/(k+1)}, 
\]

\[
= \left( \frac{1}{k+1} \left( \prod_{j=1}^{k+1} \left( 1 - (\mu_{\alpha_j}^{w_j})^2 \right)^{w_j} \right)^{1/(k+1)} \right)^{1/k} \left( \prod_{j=1}^{k+1} \left( 1 - (\mu_{\gamma_j}^{w_j})^2 \right)^{w_j} \right)^{1/(k+1)} \left( \prod_{j=1}^{k+1} \left( 1 - (\mu_{\beta_j}^{w_j})^2 \right)^{w_j} \right)^{1/(k+1)} \left( \prod_{j=1}^{k+1} \left( 1 - (\mu_{\beta_{k+1}}^{w_{k+1}})^2 \right)^{w_{k+1}} \right)^{1/(k+1)}, 
\]

Hence Equation (25) holds for \( n = k + 1 \), thus (25) holds for all \( n \). \( \square \)

**Theorem 4** (Commutativity): Let \( a_j (j = 1, 2, \ldots, n) \) and \( a'_j (j = 1, 2, \ldots, n) \) be the collection of IVPFNs, then

\[
IVPFWG_w(a_1, a_2, a_3, \ldots, a_n) = IVPFWG_w(a'_1, a'_2, a'_3, \ldots, a'_n), 
\]

where \( \{a'_1, a'_2, a'_3, \ldots, a'_n\} \) is any permutation of \( \{a_1, a_2, a_3, \ldots, a_n\} \).

**Proof** As we know that

\[
IVPFWG_w(a_1, a_2, \ldots, a_n) = (a_1)^{w_1} \otimes (a_2)^{w_2} \otimes \ldots \otimes (a_n)^{w_n}, 
\]
and

$$\text{IVPFWG}_w(a_1^*, a_2^*, \ldots, a_n^*) = (a_1^*)^w \otimes (a_2^*)^w \otimes \cdots \otimes (a_n^*)^w.$$  

Since \((a_1^*, a_2^*, \ldots, a_n^*)\) is any permutation of \((a_1, a_2, \ldots, a_n)\), thus Equation (27) always holds. □

**Theorem 5** (Idempotency): Let \(a_j (j = 1, 2, \ldots, n)\) be a collection of IVPFNs, where \(a_j = a\) for all \(j\), then

$$\text{IVPFWG}_w(a_1, a_2, a_3, \ldots, a_n) = a.$$  

(28)

**Proof** As \(a_j = a\) for all \(j\), then we have

$$\text{IVPFWG}_w(a_1, a_2, a_3, \ldots, a_n) = (a)^w \otimes (a)^w \otimes \cdots \otimes (a)^w$$

$$= \left( \left[ \prod_{j=1}^{n} (\mu_j^L)^w, \prod_{j=1}^{n} (\mu_j^U)^w \right], \sqrt{1 - \prod_{j=1}^{n} \left( 1 - (\nu_j^L)^2 \right)^w}, \sqrt{1 - \prod_{j=1}^{n} \left( 1 - (\nu_j^U)^2 \right)^w} \right)$$

$$= \left( \left[ \mu_j^L, \mu_j^U \right], \left[ \sqrt{1 - (\nu_j^L)^2}, \sqrt{1 - (\nu_j^U)^2} \right] \right)$$

$$= (\mu_j^L, \mu_j^U), \left( (\nu_j^L), (\nu_j^U) \right) = a.$$  

This completes the proof. □

**Theorem 6** (Boundedness): Let \(a_j (j = 1, 2, \ldots, n)\) be a collection of interval-valued Pythagorean fuzzy numbers and let \(w = (w_1, w_2, \ldots, w_n)^T\) be the weighted vector of \(a_j\) such that \(\sum_{j=1}^{n} w_j = 1\), then

$$a_{\min \leq \text{IVPFWG}_w(a_1, a_2, a_3, \ldots, a_n) \leq a_{\max}}.$$  

(29)

where

$$a_{\min} = \min_j \left( a_j \right), \quad a_{\max} = \max_j \left( a_j \right).$$

**Proof** Let

$$\mu_j^L_{\min} = \min_j \left( \mu_j^L \right), \mu_j^U_{\min} = \min_j \left( \mu_j^U \right),$$

(30)

$$\nu_j^L_{\min} = \min_j \left( \nu_j^L \right), \nu_j^U_{\min} = \min_j \left( \nu_j^U \right),$$

(31)

$$\mu_j^L_{\max} = \max_j \left( \mu_j^L \right), \mu_j^U_{\max} = \max_j \left( \mu_j^U \right),$$

(32)
\[
\nu_{\text{max}}^\mu = \max_j (\nu_j^\mu), \quad \nu_{\text{max}}^\nu = \max_j (\nu_j^\nu).
\]

(33)

Let

\[
\text{IVPFWG}_w(a_1, a_2, \ldots, a_n) = \alpha = ([\mu^\nu, \mu^\mu], [\nu^\nu, \nu^\mu]).
\]

(34)

Then by the score function we have,

\[
([\mu_{\min}^\nu, \mu_{\min}^\mu], [\nu_{\max}^\nu, \nu_{\max}^\mu]) \leq ([\mu^\nu, \mu^\mu], [\nu^\nu, \nu^\mu]),
\]

(35)

\[
([\mu_{\max}^\nu, \mu_{\max}^\mu], [\nu_{\min}^\nu, \nu_{\min}^\mu]) \geq ([\mu^\nu, \mu^\mu], [\nu^\nu, \nu^\mu]),
\]

(36)

Thus from Equations (35) and (36), we have

\[
a_{\min} \leq \text{IVPFWG}_w(a_1, a_2, a_3, \ldots, a_n) \leq a_{\max}.\]

This completes the proof.

\[\blacksquare\]

**Theorem 7** (Monotonicity): Let \(a_j (j = 1, 2, \ldots, n)\) and \(\alpha_j (j = 1, 2, \ldots, n)\) be the collection of IVPFNs, where \(a_j \leq \alpha_j\) for all \(j\), then

\[
\text{IVPFWG}_w(a_1, a_2, \ldots, a_n) \leq \text{IVPFWG}_w(\alpha_1, \alpha_2, \ldots, \alpha_n).
\]

(37)

**Proof** As we know that

\[
\text{IVPFWG}_w(a_1, a_2, \ldots, a_n) = (a_1)^{\nu_1} \otimes (a_2)^{\nu_2} \otimes \ldots \otimes (a_n)^{\nu_n},
\]

and

\[
\text{IVPFWG}_w(\alpha_1, \alpha_2, \ldots, \alpha_n) = (\alpha_1)^{\nu_1} \otimes (\alpha_2)^{\nu_2} \otimes \ldots \otimes (\alpha_n)^{\nu_n}.
\]

As \(a_j \leq \alpha_j\) for all \(j\), thus

\[
\text{IVPFWG}_w(a_1, a_2, \ldots, a_n) \leq \text{IVPFWG}_w(\alpha_1, \alpha_2, \ldots, \alpha_n),
\]

This completes the proof.

\[\blacksquare\]

### 3.2. Interval-valued Pythagorean fuzzy ordered weighted geometric operator

**Definition 13** An interval-valued Pythagorean fuzzy ordered weighted geometric operator of dimension \(n\) is a mapping \(\text{IVPFOWG}: \Psi^n \rightarrow \Psi\) that has an associated weighted vector \(w = (w_1, w_2, \ldots, w_n)^T\) with \(w_j \in [0, 1]\) and \(\sum_{j=1}^n w_j = 1\), and is define to aggregate a collection of interval-valued Pythagorean fuzzy numbers \(a_j = ([\mu_j^\nu, \mu_j^\mu], [\nu_j^\nu, \nu_j^\mu]) (j = 1, 2, \ldots, n)\), according to the following expression:

\[
\text{IVPFOWG}_w(a_1, a_2, a_3, \ldots, a_n)
\]

\[
= \left[ \begin{array}{c}
\prod_{j=1}^n (\mu_j^\nu)^{w_j} \\
\prod_{j=1}^n (\mu_j^\mu)^{w_j} \\
\prod_{j=1}^n (\nu_j^\nu)^{w_j} \\
\prod_{j=1}^n (\nu_j^\mu)^{w_j} \\
\end{array} \right],
\]

\[
= \left[ \begin{array}{c}
\left(1 - \frac{1}{w_j} \prod_{j=1}^n (1 - (\mu_j^\nu)^2)^{w_j} \right)^{\frac{1}{w_j}} \\
\left(1 - \frac{1}{w_j} \prod_{j=1}^n (1 - (\mu_j^\mu)^2)^{w_j} \right)^{\frac{1}{w_j}} \\
\left(1 - \frac{1}{w_j} \prod_{j=1}^n (1 - (\nu_j^\nu)^2)^{w_j} \right)^{\frac{1}{w_j}} \\
\left(1 - \frac{1}{w_j} \prod_{j=1}^n (1 - (\nu_j^\mu)^2)^{w_j} \right)^{\frac{1}{w_j}} \\
\end{array} \right],
\]

(38)

where \(a_j^{(\cdot)}\) is the jth largest value of \(a_j\). If \(w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T\), then the interval-valued Pythagorean fuzzy ordered weighted geometric operator is reduced to the interval-valued Pythagorean fuzzy geometric operator.
Example 2 Let

\[ \alpha_1 = ([0.3, 0.7], [0.4, 0.6]), \alpha_2 = ([0.2, 0.7], [0.3, 0.6]), \]
\[ \alpha_3 = ([0.3, 0.5], [0.3, 0.8]), \alpha_4 = ([0.1, 0.3], [0.4, 0.9]), \]

and let \( w = (0.4, 0.3, 0.2, 0.1)^T \) be the weighted vector of \( \alpha_j (j = 1, 2, 3, 4) \), then first we calculate the score functions as follows:

\[ S(\alpha_1) = 0.03, S(\alpha_2) = 0.04, S(\alpha_3) = -0.19, S(\alpha_4) = -0.43. \]

Thus

\[ \alpha_{n1} = ([0.2, 0.7], [0.3, 0.6]), \alpha_{n2} = ([0.3, 0.7], [0.4, 0.6]) \]
\[ \alpha_{n3} = ([0.3, 0.5], [0.3, 0.8]), \alpha_{n4} = ([0.1, 0.3], [0.4, 0.9]) \]

Now applying the IVPFWG operator we have,

\[
\text{IVPFOWG}_w(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left[ \begin{array}{c}
\frac{1}{n} \prod_{j=1}^{n} (\mu_{\alpha_{ij}})^w, \frac{1}{n} \prod_{j=1}^{n} (\nu_{\alpha_{ij}})^w, \\
\sqrt{1 - \prod_{j=1}^{n} \left(1 - (\mu_{\alpha_{ij}})^w\right)^2}, \sqrt{1 - \prod_{j=1}^{n} \left(1 - (\nu_{\alpha_{ij}})^w\right)^2}\end{array} \right]
\]

\[
= \left[ \begin{array}{c}
(0.2)^{0.1}(0.3)^{0.2}(0.3)^{0.1}(0.1)^{0.4}, \\
(0.7)^{0.1}(0.7)^{0.2}(0.5)^{0.3}(0.3)^{0.4}, \\
\sqrt{1 - (1 - 0.09)^{0.1}(1 - 0.16)^{0.2}(1 - 0.09)^{0.1}(1 - 0.16)^{0.4}}, \\
\sqrt{1 - (1 - 0.36)^{0.1}(1 - 0.36)^{0.2}(1 - 0.64)^{0.3}(1 - 0.81)^{0.4}}\end{array} \right]
\]

\[ = \left( [0.2285, 0.6012], [0.3444, 0.7034] \right). \]

Theorem 8 Let \( \alpha_j (j = 1, 2, \ldots, n) \) be a collection of interval-valued Pythagorean fuzzy numbers, then their aggregated value by using the interval valued Pythagorean fuzzy ordered weighted geometric (IVPFOWG) operator is also interval-valued Pythagorean fuzzy number and also satisfies

\[
\text{IVPFOWG}_w(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left[ \begin{array}{c}
\frac{1}{n} \prod_{j=1}^{n} (\mu_{\alpha_{ij}})^w, \frac{1}{n} \prod_{j=1}^{n} (\nu_{\alpha_{ij}})^w, \\
\sqrt{1 - \prod_{j=1}^{n} \left(1 - (\mu_{\alpha_{ij}})^w\right)^2}, \sqrt{1 - \prod_{j=1}^{n} \left(1 - (\nu_{\alpha_{ij}})^w\right)^2}\end{array} \right]
\]

where \( w = (w_1, w_2, \ldots, w_n)^T \) is the weighted vector of \( \alpha_j (j = 1, 2, \ldots, n) \) with \( w_j \in [0, 1] \) and \( \sum_{j=1}^{n} w_j = 1. \)

Proof Proof is similar to Theorem 1. □

Theorem 9 (Commutativity): Let \( \alpha_j (j = 1, 2, \ldots, n) \) and \( \alpha_j^* (j = 1, 2, \ldots, n) \) be the collection of IVPFNs, then

\[
\text{IVPFOWG}_w(\alpha_1, \alpha_2, \ldots, \alpha_n) = \text{IVPFOWG}_w(\alpha_1^*, \alpha_2^*, \ldots, \alpha_n^*),
\]

where \((\alpha_1^*, \alpha_2^*, \ldots, \alpha_n^*)\) is any permutation of \((\alpha_1, \alpha_2, \ldots, \alpha_n)\).

Proof Proof is similar to Theorem 2. □
Theorem 10 (Idempotency): Let \( a_j (j = 1, 2, ..., n) \) be a collection of IVPFNs, where \( a_j = a \), for all \( j \), then
\[
\text{IVPFOWG}_w(a_1, a_2, \ldots, a_n) = a.
\tag{41}
\]
Proof Proof is similar to Theorem 3.

Theorem 11 (Boundedness): Let \( a_j (j = 1, 2, ..., n) \) be a collection of interval-valued Pythagorean fuzzy numbers and let \( w = (w_1, w_2, \ldots, w_n) \) be the weighted vector of \( a_j \) such that \( \sum_{j=1}^{n} w_j = 1 \), then
\[
a_{\min} \leq \text{IVPFOWG}_w(a_1, a_2, \ldots, a_n) \leq a_{\max},
\tag{42}
\]
where
\[
a_{\min} = \min_j \left( a_{\omega(j)} \right)
\]
\[
a_{\max} = \max_j \left( a_{\omega(j)} \right).
\]
Proof Proof is similar to Theorem 4.

Theorem 12 (Monotonicity): Let \( a_j (j = 1, 2, ..., n) \) and \( a'_j (j = 1, 2, ..., n) \) be the collection of interval-valued Pythagorean fuzzy numbers, where \( a_j \leq a'_j \) for all \( j \), then
\[
\text{IVPFOWG}_w(a_1, a_2, \ldots, a_n) \leq \text{IVPFOWG}_w(a'_1, a'_2, \ldots, a'_n).
\tag{43}
\]
Proof Proof is similar to Theorem 5.

3.3. Interval-valued Pythagorean fuzzy hybrid geometric operator

Definition 14 An interval-valued Pythagorean fuzzy hybrid geometric operator of dimension \( n \) is a mapping \( \text{IVPFHG} : \Omega^n \rightarrow \Omega \), which has an associated vector \( w = (w_1, w_2, \ldots, w_n) \), such that \( w_j \in [0, 1] \) and \( \sum_{j=1}^{n} w_j = 1 \). Furthermore
\[
\text{IVPFHG}_{w,w}(a_1, a_2, a_3, \ldots, a_n)
= \left[ \frac{\prod_{j=1}^{n} (\mu_{\omega(j)})^{w_j}, \prod_{j=1}^{n} (\nu_{\omega(j)})^{w_j}}{\sqrt{1 - \prod_{j=1}^{n} \left[ 1 - (\frac{\mu_{\omega(j)}}{\nu_{\omega(j)}})^2 \right]^w_j}}, \sqrt{1 - \prod_{j=1}^{n} \left[ 1 - (\frac{\mu_{\omega(j)}}{\nu_{\omega(j)}})^2 \right]^w_j} \right],
\tag{44}
\]
where \( a_{\omega(j)} \) is the \( j \)th largest of the weighted Pythagorean fuzzy values \( a_j (a_j = (a_j)_{\text{row}}) \), \( w = (w_1, w_2, \ldots, w_n) \) is the weighted vector of \( a_j (j = 1, 2, \ldots, n) \) such that \( w_j \in [0, 1] \) and \( \sum_{j=1}^{n} w_j = 1 \), and \( n \) is the balancing coefficient, which plays a role of balance. If the vector \( (w_1, w_2, \ldots, w_n) \) approaches \( (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}) \), then the vector \( ((a_1)_{\text{row}}, (a_2)_{\text{row}}, \ldots, (a_n)_{\text{row}})^T \) approaches \( (a_1, a_2, \ldots, a_n)^T \).

Theorem 13 Let \( a_j (j = 1, 2, \ldots, n) \) be a collection of IVPFNs, then their aggregated value by using the IVPFHG operator is also IVPFN and also satisfies
\[
\text{IVPFHG}_{w,w}(a_1, a_2, a_3, \ldots, a_n)
= \left[ \frac{\prod_{j=1}^{n} (\mu_{\omega(j)})^{w_j}, \prod_{j=1}^{n} (\nu_{\omega(j)})^{w_j}}{\sqrt{1 - \prod_{j=1}^{n} \left[ 1 - (\frac{\mu_{\omega(j)}}{\nu_{\omega(j)}})^2 \right]^w_j}}, \sqrt{1 - \prod_{j=1}^{n} \left[ 1 - (\frac{\mu_{\omega(j)}}{\nu_{\omega(j)}})^2 \right]^w_j} \right],
\tag{45}
\]
where \( w = (w_1, w_2, \ldots, w_n)^\top \) is the weighted vector of \( a_j (j = 1, 2, \ldots, n) \) with \( w_j \in [0, 1] \) and \( \sum_{j=1}^{n} w_j = 1 \).

**Proof** Proof is similar to Theorem 1. \( \square \)

**Theorem 14** (Idempotency): Let \( a_j (j = 1, 2, \ldots, n) \) be a collection of IVPFNs, if \( a_j = \bar{a} \), for all \( j \), then

\[
\text{IVPFHG}_{w, w}(a_1, a_2, a_3, \ldots, a_n) = \bar{a}.
\]

**Proof** As \( a_j = \bar{a} \) for all \( j \), then we have

\[
\text{IVPFHG}_{w, w}(a_1, a_2, a_3, \ldots, a_n) = (\bar{a})^{w_1} \otimes (\bar{a})^{w_2} \otimes \cdots \otimes (\bar{a})^{w_n} = (\bar{a})^n = \bar{a}.
\]

The proof is complete. \( \square \)

**Theorem 15** (Boundedness): Let \( a_j (j = 1, 2, \ldots, n) \) be a collection of interval-valued Pythagorean fuzzy numbers and let \( w = (w_1, w_2, \ldots, w_n)^\top \) be the weighted vector of \( a_j \) such that \( \sum_{j=1}^{n} w_j = 1 \), then

\[
\bar{a}_{\min} \leq \text{IVPFHG}_{w, w}(a_1, a_2, a_3, \ldots, a_n) \leq \bar{a}_{\max},
\]

where

\[
\bar{a}_{\min} = \min_j \left( a_j \right),
\]

\[
\bar{a}_{\max} = \max_j \left( a_j \right).
\]

**Proof** Proof is similar to Theorem 4. \( \square \)

**Example 3** Let

\[
\begin{align*}
a_1 &= (0.3, 0.4], [0.4, 0.7]), \\
a_2 &= (0.2, 0.4], [0.3, 0.6]), \\
a_3 &= (0.3, 0.5], [0.3, 0.7]), \\
a_4 &= (0.1, 0.3], [0.4, 0.8]),
\end{align*}
\]

let

\[
w = (0.1, 0.2, 0.3, 0.4)^\top
\]
be the weighted vector of
\( a_j (j = 1, 2, 3, 4), \)

then
\[
\begin{align*}
\hat{a}_1 &= ([0.6178, 0.6931], [0.2595, 0.4859]) \\
\hat{a}_2 &= ([0.2759, 0.4804], [0.2695, 0.5479]) \\
\hat{a}_3 &= ([0.2358, 0.4352], [0.3271, 0.7444]) \\
\hat{a}_4 &= ([0.0251, 0.1456], [0.4933, 0.8972]) \\
\end{align*}
\]

Now we can find the score functions \( \dot{a}_j (j = 1, 2, 3, 4). \)

\[
\begin{align*}
S(\hat{a}_1) &= 0.2793, \\
S(\hat{a}_2) &= -0.0329, \\
S(\hat{a}_3) &= -0.2080, \\
S(\hat{a}_4) &= -0.5132.
\end{align*}
\]

Hence
\[
S(\hat{a}_1) > S(\hat{a}_2) > S(\hat{a}_3) > S(\hat{a}_4)
\]

So
\[
\begin{align*}
\hat{a}_{n1} &= ([0.6178, 0.6931], [0.2595, 0.4859]), \\
\hat{a}_{n2} &= ([0.2759, 0.4804], [0.2695, 0.5479]), \\
\hat{a}_{n3} &= ([0.2358, 0.4352], [0.3271, 0.7444]), \\
\hat{a}_{n4} &= ([0.0251, 0.1456], [0.4933, 0.8972]).
\end{align*}
\]

Thus
\[
\text{IVPFHG}_{w,u}(a_1, a_2, a_3, a_4) = \left[ \sqrt{1 - \prod_{j=1}^{4} \left( 1 - \left( \frac{\mu_{\hat{a}_{n1}}}{\mu_{a_j}} \right)^2 \right)^{w_j}}, \frac{\prod_{j=1}^{4} \left( \mu_{\hat{a}_{n1}}^{w_j} \right)}{\prod_{j=1}^{4} \left( \mu_{a_j}^{w_j} \right)} \right],
\]

\[
= \left[ \left[ \begin{array}{c}
(0.6178)^{0.1} (0.2595)^{0.2} (0.2358)^{0.3} (0.0251)^{0.4}, \\
(0.6931)^{0.1} (0.4804)^{0.2} (0.4352)^{0.3} (0.1456)^{0.4}
\end{array} \right], \right.
\]

\[
= \left[ \begin{array}{c}
1 - \left(1 - (0.2595)^2\right)^{0.1} \left(1 - (0.2695)^2\right)^{0.2}, \\
\left(1 - (0.3271)^2\right)^{0.3} \left(1 - (0.4933)^2\right)^{0.4}, \\
1 - \left(1 - (0.4859)^2\right)^{0.1} \left(1 - (0.5479)^2\right)^{0.2}, \\
\left(1 - (0.7444)^2\right)^{0.3} \left(1 - (0.8972)^2\right)^{0.4},
\end{array} \right]
\]

\[
= ([0.1093, 0.3000], [0.7055, 0.7938]).
\]
Since
\[
\hat{a}_{\min} = ([0.0251, 0.1456], [0.4933, 0.8972]),
\]
\[
\hat{a}_{\max} = ([0.6178, 0.6931], [0.2595, 0.4859]).
\]
By calculating the score functions we have
\[
S(\hat{a}_{\min}) = \frac{1}{2} \left[ (0.0251)^2 + (0.1456)^2 - (0.4933)^2 - (0.8972)^2 \right] = -0.513
\]
\[
S(\hat{a}_{\max}) = \frac{1}{2} \left[ (0.6178)^2 + (0.6931)^2 - (0.2595)^2 - (0.4859)^2 \right] = 0.27
\]
\[
S(\text{IVPFHG}) = \frac{1}{2} \left[ (0.1093)^2 + (0.3000)^2 - (0.7055)^2 - (0.7938)^2 \right] = -0.512
\]
Hence
\[
S(\hat{a}_{\min}) < S(\text{IVPFHG}) < S(\hat{a}_{\max}).
\]
Thus from the above example we can say that the boundedness property holds in IVPFHG operator.

**THEOREM 16** (Monotonicity): Let \( a_j = 1, 2, \ldots, n \) and \( a_j^* = 1, 2, \ldots, n \) be the collection of interval-valued Pythagorean fuzzy numbers, where \( a_j \leq a_j^* \) for all \( j \), then

\[
\text{IVPFHG}_{w,w}(a_1, a_2, \ldots, a_n) \leq \text{IVPFHG}_{w,w}(a_1^*, a_2^*, \ldots, a_n^*). \tag{50}
\]

**Proof** Proof is similar to Theorem 5.

**THEOREM 17** The Pythagorean fuzzy weighted geometric operator is a specials case of the interval-valued Pythagorean fuzzy hybrid geometric operator.

**Proof** Let \( w = \left( \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n} \right)^T \), then

\[
\text{IVPFHG}_{w,w}(a_1, a_2, a_3, \ldots, a_n) = \left( \hat{a}_{(n1)} \right)_{w_1} \otimes \left( \hat{a}_{(n2)} \right)_{w_2} \otimes \left( \hat{a}_{(n3)} \right)_{w_3} \otimes \cdots \otimes \left( \hat{a}_{(nm)} \right)_{w_n}
\]

\[
= \left( \hat{a}_{(n1)} \right)_{\frac{1}{2}} \otimes \left( \hat{a}_{(n2)} \right)_{\frac{1}{2}} \otimes \left( \hat{a}_{(n3)} \right)_{\frac{1}{2}} \otimes \cdots \otimes \left( \hat{a}_{(nm)} \right)_{\frac{1}{2}}
\]

\[
= \left( \left( a_1 \right)_{w_1} \otimes \left( a_2 \right)_{w_2} \otimes \cdots \otimes \left( a_n \right)_{w_n} \right)^{\frac{1}{2}}
\]

\[
= \text{PFWG}_{w}(a_1, a_2, a_3, \ldots, a_n)
\]

**THEOREM 18** Pythagorean fuzzy ordered weighted geometric operator is a specials case of the interval-valued Pythagorean fuzzy hybrid geometric operator.
Proof  Let $w = \left( \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n} \right)^T$, and $\alpha_j = \left( a_j \right)^{\alpha_j} = a_j$, then

$\text{IVPFHG}_{w,w}(a_1, a_2, a_3, \ldots, a_n)
= (a_{n(1)})^w \otimes (a_{n(2)})^w \otimes \ldots \otimes (a_{n(n)})^w
= (a_{n(1)})^w \otimes (a_{n(2)})^w \otimes \ldots \otimes (a_{n(n)})^w
= \text{PFWG}_w(a_1, a_2, a_3, \ldots, a_n)$.

4. An application of the interval-valued Pythagorean fuzzy weighted geometric operator and interval-valued Pythagorean fuzzy ordered weighted geometric operator to multiple attribute decision making

In this section, we discuss an application of the interval-valued Pythagorean fuzzy weighted geometric (IVPFWG) and the interval-valued Pythagorean fuzzy ordered weighted geometric (IVPFOWG) operator to multiple criteria decision-making problem. Now we are using Pythagorean fuzzy information to develop the interval-valued Pythagorean fuzzy multi-criteria decision-making problem. In decision-making problem, the main point of decision-maker is that to select the best alternative. In this method, we develop an algorithm to rank the alternatives. This method contains the following steps.

Algorithm: Let $A = \{A_1, A_2, \ldots, A_n\}$ be a set of $n$ alternatives, and $F = \{F_1, F_2, \ldots, F_m\}$ be the set of $m$ attributes and $w = (w_1, w_2, \ldots, w_m)^T$ be the weighted vector of the attributes $F_i (i = 1, 2, \ldots, m)$ such that $w_i \in [0, 1]$ and $\sum_{i=1}^{m} w_i = 1$.

This method having the following steps:

Step 1: The decision-makers provide information in the form of matrix.
Step 2: Compute $\alpha_j (j = 1, 2, \ldots, n)$ using the interval-valued Pythagorean fuzzy weighted geometric (IVPFWG) aggregation operator.
Step 3: Compute the scores of $\alpha_j (j = 1, 2, \ldots, n)$. If there is no difference between two or more than two scores, then have we must to calculate the accuracy degrees.
Step 4: Arrange the scores function of the all alternatives in the form of descending order and select that alternative, which has the highest score function value.

5. Illustrative example

Suppose a customer wants to buy a computer from different computers, let $A_1, A_2, A_3, A_4$ represent the four computer of different companies. Let $F_1, F_2, F_3, F_4$ be the criteria of these computers. In the process of choosing one of the computers, four factors are consider. $F_1$: price of each computer. $F_2$: model of each computer. $F_3$: design of each computer. $F_4$: be tree of each computer. Suppose the weight vector of $F_i (i = 1, 2, 3, 4)$ is $w = (0.1, 0.2, 0.3, 0.4)^T$, and the interval-valued Pythagorean fuzzy numbers of the alternative $A_j (j = 1, 2, 3, 4)$ are represented by the following decision matrix.

For IVPFWG Operator

Step 1: The decision-maker gives his decision in Table 1.

<table>
<thead>
<tr>
<th>Table 1. Interval-valued Pythagorean fuzzy decision matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>$F_1$</strong></td>
</tr>
<tr>
<td>(0.4, 0.8)</td>
</tr>
<tr>
<td>(0.3, 0.7)</td>
</tr>
<tr>
<td>(0.2, 0.5)</td>
</tr>
<tr>
<td>(0.4, 0.6)</td>
</tr>
</tbody>
</table>
Step 2: Compute $\alpha_j (j = 1, 2, 3, 4)$

\[
\begin{align*}
\alpha_1 &= ([0.3067, 0.6029], [0.3306, 0.5939]) \\
\alpha_2 &= ([0.2861, 0.5932], [0.3444, 0.5825]) \\
\alpha_3 &= ([0.2698, 0.7256], [0.3306, 0.5485]) \\
\alpha_4 &= ([0.3861, 0.6903], [0.3174, 0.6033])
\end{align*}
\]

Step 3: In this step, we can find the scores of $\alpha_j$, where $j = 1, 2, 3, 4$.

\[
\begin{align*}
S(\alpha_1) &= -0.0022, S(\alpha_2) = -0.0121 \\
S(\alpha_3) &= 0.0946, S(\alpha_4) = 0.0804
\end{align*}
\]

Step 4: Arrange the scores of the all alternatives in the form of descending order and select that alternative, which has the highest score function. Since $\alpha_3 > \alpha_1 > \alpha_2 > \alpha_4$. Hence $A_3 > A_1 > A_2 > A_4$. Thus $A_3$ is the best option for the customer.

For IVPFOWG Operator

Step 1: First we ordered the original decision-making matrix, with the help of their score function and got the following ordered decision matrix (Table 2).

Step 2: Compute $\alpha_j (j = 1, 2, 3, 4)$

\[
\begin{align*}
\alpha_1 &= ([0.2861, 0.5920], [0.3194, 0.6142]) \\
\alpha_2 &= ([0.2980, 0.5842], [0.3653, 0.5734]) \\
\alpha_3 &= ([0.2892, 0.6956], [0.2999, 0.5852]) \\
\alpha_4 &= ([0.3727, 0.6200], [0.3796, 0.6910])
\end{align*}
\]

Step 3: In this step, we can find the scores of $\alpha_j$.

\[
\begin{align*}
S(\alpha_1) &= -0.0234, S(\alpha_2) = -0.0168 \\
S(\alpha_3) &= 0.0675, S(\alpha_4) = -0.0491
\end{align*}
\]

Step 4: Arrange the scores of the all alternatives in the form of descending order and select that alternative, which has the highest score function. Since $\alpha_3 > \alpha_2 > \alpha_1 > \alpha_4$. Hence $A_3 > A_2 > A_1 > A_4$. Thus $A_3$ is the best option for the customer.

<table>
<thead>
<tr>
<th>Table 2. Interval-valued Pythagorean ordered fuzzy decision matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_1$</td>
</tr>
<tr>
<td>-------</td>
</tr>
<tr>
<td>$F_1$</td>
</tr>
<tr>
<td>$F_2$</td>
</tr>
<tr>
<td>$F_3$</td>
</tr>
<tr>
<td>$F_4$</td>
</tr>
</tbody>
</table>
6. Conclusions

In this paper, we have introduced the notion of interval-valued Pythagorean fuzzy weighted geometric (IVPFWG) operator, interval-valued Pythagorean fuzzy ordered weighted geometric (IVPFOWG) operator, and interval-valued Pythagorean fuzzy hybrid geometric (IVPFHG) operator. We have also discussed some properties and given some examples to developed these operators. At the last, we have applied the IPFWG operator and IPFOWG operator to MADM problem under interval-valued Pythagorean fuzzy information.

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Citation information


References
