Finiteness properties of generalized local cohomology modules for minimax modules

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Abstract: Let $R$ be a commutative Noetherian ring, $I$ an ideal of $R$, $M$ be a finitely generated $R$-module and $t$ be a non-negative integer. In this paper, we introduce the concept of $I, M$-minimax $R$-modules. We show that $	ext{Hom}_R(R/I, H^i_t(M, N)/K)$ is $I, M$-minimax, for all $I, M$-minimax submodules $K$ of $H^i_t(M, N)$, whenever $N$ and $H^i_t(M)$, $H^i_{t-1}(M)$, $\cdots$, $H^i_{t-r}(M)$ are $I, M$-minimax $R$-modules. As consequence, it is shown that $\text{Ass}_R H^i_t(M, N)/K$ is a finite set.

Keywords: generalized local cohomology; minimax module

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1. Introduction

Let $R$ be a commutative Noetherian ring, $I$ an ideal of $R$, and $M$ a finitely generated $R$-module. An important problem in commutative algebra is determining when the set of associated primes of the $i$-th local cohomology module is finite. In Huneke, (1992) raised the following question: If $M$ is a finitely generated $R$-module, then the set of associated primes of $H^i_t(M)$ is finite for all ideals $I$ of $R$ and all $i \geq 0$. Singh (2000) and Katzman (2002) have given counterexamples to this conjecture. However, it is known that this conjecture is true in many situations; see Brodmann and Lashgari Faghi (2000), Brodmann, Rotthous, and Sharp (2000), Hellus (2001), Marley (2001). In particular, Brodmann and Lashgari Faghi (2000) have shown that, $\text{Ass}_R H^i_t(M)/K$ is a finite set for any finitely generated submodule $K$ of $H^i_t(M)$, whenever the local cohomology modules $H^0_t(M)$, $H^i_t(M)$, $\cdots$, $H^i_{t-r}(M)$ are finitely generated. Next, Bahmanpour and Naghipour (2008) showed that, $\text{Hom}_R(R/I, H^i_t(M))/K$ is finitely generated for any minimax submodule $K$ of $H^i_t(M)$, whenever the local cohomology modules $H^0_t(M)$, $H^i_t(M)$, $\cdots$, $H^i_{t-r}(M)$ are minimax. After this Azami, Naghipour, and Vakili (2008) proved that, $\text{Hom}_R(R/I, H^i_t(N))/K$ is $1$-minimax for any $1$-minimax submodule $K$ of $H^i_t(N)$, whenever $N$ is an $1$-minimax $R$-module and the local cohomology modules $H^0_t(N)$, $H^i_t(N)$, $\cdots$, $H^i_{t-r}(N)$ are $1$-minimax. The main result of this note is a generalization of above theorems for generalized local cohomology modules.

Recall that an $R$-module $N$ is said to have finite Goldie dimension if $N$ dose not contain an infinite direct sum of non-zero submodules, or equivalently the injective hall $E(N)$ of $N$ decomposes as a finite direct sum of indecomposable submodules. Also, an $R$-module $N$ is said to have finite $I$-relative...
Goldie dimension if the Goldie dimension of the $I$-torsion submodule $\Gamma_i(N) = \bigcup_{n \geq 1} (0:_{R}^{\infty}N)_{n}$ of $N$ is finite. We say that an $R$-module $N$ has finite $I$-$M$-relative Goldie dimension if the Goldie dimension of the $R$-module $H_{i}^{N}(M, N)$ is finite. An $R$-module $N$ is called $I$-minimax if $I$-relative Goldie dimension of any quotient module of $N$ is finite. We say that an $R$-module $N$ is $I, M$-minimax if $I,M$-relative Goldie dimension of any quotient module of $N$ is finite.

Precisely we show that, $\text{Hom}_{R}(R/I, H_{i}^{N}(M, N)/K)$ is $I, M$-minimax for any $I, M$-minimax submodule $K$ of $H_{i}^{N}(M, N)$, whenever the $R$-module $N$ and the local cohomology modules $H_{i}^{N}(N), H_{i+1}^{N}(N), \cdots , H_{i-1}^{N}(N)$ are $I, M$-minimax.

Throughout this paper, $R$ will always be a commutative Noetherian ring with non-zero identity, $I$ an ideal of $R$, $M$ will be a finitely generated $R$-module and $N$ an $R$-module. The $i$-th generalized local cohomology module with respect to $I$ is defined by

$$H_{i}^{N}(M, N) = \lim_{n \in \mathbb{N}} \text{ Ext}_{R}^{i}(M/I^{n}M, N).$$


2. I, M-minimax modules

For an $R$-module $N$ the Goldie dimension is defined as the cardinal of the set of indecomposable submodule of $E(N)$ which appear in a decomposition of $E(N)$ in to a direct sum of indecomposable submodules. We shall use $G \text{dim} N$ to denote the Goldie dimension of $N$. Let $\mu^{0}(p, N)$ denote the 0-th Bass number of $N$ with respect to prime ideal $p$ of $R$. It is well known that $\mu^{0}(p, N) > 0$ if and only if $p \in \text{Ass}_{R}N$ and it is clear that

$$G \text{dim} N = \sum_{p \in \text{Ass}_{R}N} \mu^{0}(p, N).$$

Also, the $I$-relative Goldie dimension of $N$ is defined as

$$G \text{dim}_{I} N = \sum_{p \in \text{VU} \cap \text{Ass}_{R}N} \mu^{0}(p, N).$$

The $I$-relative Goldie dimension of an $R$-module has been studied in Divaani-Aazar and Esmkhani (2005) and in Lemma 2.6 it is shown that $G \text{dim}_{I} N = G \text{dim} H^{0}_{I}(N)$. Having this in mind, we introduce the following generalization of the notion of $I$-relative Goldie dimension.

**Definition 2.1** Let $I$ be an ideal of $R$ and $M$ be a finitely generated $R$-module. We denote by $G \text{dim}_{I,M} N$ the $I, M$-relative Goldie dimension of $N$ and we define $I, M$-relative Goldie dimension of $N$ as

$$G \text{dim}_{I,M} N = G \text{dim} H^{0}_{I}(M, N).$$

The class of $I$-minimax modules is defined in Azami et al. (2008) and an $R$-module $N$ is said to be $\text{minimax with respect to } I$ or $I$-$\text{minimax}$ if $I$-relative Goldie dimension of any quotient module of $N$ is finite. This motivates the following definition.

**Definition 2.2** Let $I$ be an ideal of $R$ and $M$ be a finitely generated $R$-module. An $R$-module $N$ is said to be $I, M$-$\text{minimax}$ if the $I, M$-relative Goldie dimension of any quotient module of $N$ is finite; i.e. for any submodule $K$ of $N$, $G \text{dim}_{I,M} N/K < \infty$.

**Proposition 2.3** Let $N$ be an $R$-module. Then $N$ is $I,M$-minimax if and only if $N$ is $\text{Ann}(M/IM)$-$\text{minimax}$.

**Proof** It is sufficient to show that for each $p \in \text{Ann}(M/IM)$, there is an integer $n_{p}$ such that
G \dim_{I,N} = \sum_{p \in \text{Ass}(I,N) / \text{Ass}(M/I)} n_p \mu^0(p, N).

We have \( H^i_I(M, N) \cong \text{Hom}_R(M, \Gamma_i(N)) \cong \Gamma_i(\text{Hom}_R(M, N)) \) so that it follows

\[
G \dim_{I,N}(N) = G \dim_{I,N}(M, N) = \sum_{p \in \text{Ass}(I, M, N)} \mu^0(p, H^i_I(M, N))
= \sum_{p \in \text{Ass}(M, N) / \text{Ass}(M/I)} \mu^0(p, \text{Hom}_R(M, N)).
\]

On the other hand, \( \text{Ass} \text{Hom}_R(M, N) = \text{Ass} \cap \text{Supp} M \). Hence,

\[
G \dim_{I,N}(N) = \sum_{p \in \text{Ass}(N) \cap V(\text{Ann}(M/I))} \mu^0(p, \text{Hom}_R(M, N)).
\]

For \( p \in \text{Ass}(N) \cap V(\text{Ann}(M/I)) \) we have

\[
\mu^0(p, \text{Hom}_R(M, N)) = \dim_{k_p} \text{Hom}_{k_p}(k(p), \text{Hom}_{k_p}(M_{p'}, N_{p'})) = \dim_{k_p} \text{Hom}_{k_p}(k(p) \otimes_{k_p} M_{p'}, N_{p'}),
\]

where \( k(p) = R_\wp / \wp R_\wp \) and \( k(p) \otimes_{k_p} M_{p'} \) is a finite dimensional \( k(p) \)-vector space with dimension \( n_p \).

Hence, \( k(p) \otimes_{k_p} M_{p'} \cong \oplus_{n_p} k(p) \) which implies that

\[
\mu^0(p, \text{Hom}_R(M, N)) = \dim_{k_p} \text{Hom}_{k_p}(\oplus_{n_p} k(p), N_{p'}) = n_p \mu^0(p, N).
\]

It is clear that the above argument is true for each quotient of \( N \).

\[\square\]

**Remark 2.4** The following statements are true for any \( R \)-module \( N \).

(i) The \( I, R \)-minimax modules are precisely \( I \)-minimax.

(ii) The \( I, M \)-minimax modules are \( I \)-minimax.

(iii) If \( N \) is Noetherian or Artinian \( R \)-module, then \( N \) is \( I, M \)-minimax.

(iv) If \( J \) is a second ideal of \( R \) such that \( I \subseteq J \) and \( N \) is \( J, M \)-minimax, then \( N \) is \( I, M \)-minimax.

(v) Let \( N \) be \( \text{Ann}_R(M) \)-torsion, i.e. \( \Gamma_{\text{Ann}_R(M)}(N) = 0 \). Then \( N \) is \( I, M \)-minimax if and only if \( N \) is \( I \)-minimax.

**Proposition 2.5** Let \( 0 \to N' \to N \to N'' \to 0 \) be an exact sequence of \( R \)-modules. Then \( N \) is \( I, M \)-minimax if and only if \( N' \) and \( N'' \) are both \( I, M \)-minimax.

**Proof** This is immediate from Proposition 2.3 and Azami et al. (2008, Proposition 2.5).

**Proposition 2.6** Let \( t \) be a non-negative integer. Then for all \( R \)-module \( N \) the following statements are equivalent:

(i) \( \text{Ext}^i_I(R/I, N) \) is \( I, M \)-minimax for all \( i \leq t \).

(ii) \( \text{Ext}^i_J(R/J, N) \) is \( I, M \)-minimax for all ideal \( J \) of \( R \) with \( I \subseteq J \) and for all \( i \leq t \).

(iii) \( \text{Ext}^i_J(L, N) \) is \( I, M \)-minimax for all finitely generated \( R \)-module \( L \) with \( \text{Supp} L \subseteq \text{V}(I) \) and for all \( i \leq t \).

(iv) For any minimal prime ideal \( \wp \) over \( I \), \( \text{Ext}^i_I(R/\wp, N) \) is \( I, M \)-minimax for all \( i \leq t \).

**Proof** The proof is similar to that of Azami et al. (2008, Corollary 2.8).
Proposition 2.7  If $N$ is an $I$, $M$-minimax module such that $\text{Ass}_N(N) \subseteq V(I)$, then $H^i_I(L, N)$ is $I, M$-minimax for all finitely generated $R$-module $L$ and all $i \geq 0$.

Proof  If $i = 0$, then $H^0_I(L, N) = \text{Hom}_R(L, \Gamma_0(N))$ and so by Azami et al. (2008, Corollary 2.5), $H^0_I(L, N)$ is $I, M$-minimax. As $\text{Ass}_N(N) \subseteq \text{Ass}_N(N)$, it easily follows from $\text{Ass}_N(N) \subseteq V(I)$ that $N = \Gamma_0(N)$. Consequently, $H^0_I(L, N) = \text{Ext}_R^0(L, N)$ for all $i \geq 0$, by Yassemi et al. (2002, Theorem 2.3). So that $H^i_I(L, N)$ is $I, M$-minimax for all $i \geq 0$, as required.

Proposition 2.8  Let $N$ be an $R$-module and let $t$ be a non-negative integer. If $H^t_I(N)$ is $I, M$-minimax for all $i < t$, then $H^t_I(M, N)$ is $I, M$-minimax for all $i < t$.

Proof  We use induction on $t$. When $t = 1$, the $R$-module $\Gamma_1(N)$ is $I, M$-minimax by assumption. Since $H^1_I(M, N) \cong \text{Hom}_R(M, \Gamma_1(N))$, it follows that $H^1_I(M, N)$ is $I, M$-minimax, by Azami et al. (2008, Theorem 2.7). Now suppose, inductively, that $t > 1$ and the result has been proved for $t - 1$. Since $H_t^0(N) \cong H_t^0(N/\Gamma_1(N))$ and $H_t^0(M, N) \cong H_t^0(M, N/\Gamma_1(N))$ for all $i > 0$, it follows that $H_t^0(N/\Gamma_1(N))$ is $I, M$-minimax for all $i < t - 1$. Therefore, we may assume that $N$ is $I$-torsion free. Let $E$ be an injective envelope of $N$ and put $N_t = E/N$. Then $\Gamma_1(E) = 0$. Consequently, $H_t^0(N_t) \cong H_t^0(N)$. Thus $H_t^0(N_t)$ is $I, M$-minimax for all $i < t - 1$ and by induction hypothesis $H_t^0(M, N_t)$ is $I, M$-minimax for all $i < t$. Also, we have $H_t^0(M, N_t) \cong H_t^0(M, N)$ so that $H_t^0(M, N)$ is $I, M$-minimax for all $i < t$.

3. Finiteness of associated primes

It will be shown in this section that the subject of the previous section can be used to prove a finiteness result about generalized local cohomology modules. In fact we will generalize the main results of Brodmann and Lashgari Faghani (2000) and Azami et al. (2008). Throughout this section $I$ is an ideal of $R$ and $M$ is a finitely generated $R$-module.

Theorem 3.1  Let $N$ be an $R$-module and let $t$ be a non-negative integer. If $H^t_I(N)$ is $I, M$-minimax for all $i < t$, and $\text{Ext}^t_I(R/I, N)$ is $I, M$-minimax, then for any $I, M$-minimax submodule $K$ of $H_I^t(M, N)$ and for any finitely generated $R$-module $L$ with $\text{Supp} L \subseteq V(I)$ the $R$-module $\text{Hom}_R(L, H_I^t(M, N)/K)$ is $I, M$-minimax.

Proof  The exact sequence

$$0 \longrightarrow K \longrightarrow H^t_I(M, N) \longrightarrow H^t_I(M, N)/K \longrightarrow 0$$

provides the following exact sequence:

$$\cdots \longrightarrow \text{Hom}_R(L, H^t_I(M, N)) \longrightarrow \text{Hom}_R(L, H^t_I(M, N)/K) \longrightarrow \text{Ext}^t_I(L, K) \longrightarrow \cdots .$$

By Azami et al. (2008, Corollary 2.5), $\text{Ext}^t_I(L, K)$ is $I, M$-minimax, so in view of Azami et al. (2008, Proposition 2.3), it is thus sufficient for us to show that the $R$-module $\text{Hom}_R(L, H^t_I(M, N/K))$ is $I, M$-minimax. To this end, it is enough to show that $\text{Hom}_R(L/R/I, H^t_I(M, N))$ is $I, M$-minimax by Proposition 2.6. We use induction on $t$. When $t = 0$, the $R$-module $\text{Hom}_R(L/R/I, N)$ is $I, M$-minimax, by assumption. On the other hand,

$$\text{Hom}_R(L/R/I, H^t_I(M, N)) \cong \text{Hom}_R(L/R/I, \text{Hom}_R(L/R/I, \Gamma_0(N)))$$

$$\cong \text{Hom}_R(M/IM, \Gamma_0(N)) \cong \text{Hom}_R(M/IM, N)$$

and $\text{Supp}(M/IM) \subseteq V(I)$, it follows that $\text{Hom}_R(M/IM, N)$ is $I, M$-minimax, by Proposition 2.6. Hence $\text{Hom}_R(L/R/I, H^t_I(M, N))$ is $I, M$-minimax. Now suppose, inductively, that $t > 0$ and that the result has been proved for $t - 1$. Since $\Gamma_1(N)$ is $I, M$-minimax, it follows that $\text{Ext}^t_I(L/R/I, \Gamma_0(N))$ is $I, M$-minimax for all $i > 0$. The exact sequence

$$0 \longrightarrow \Gamma_0(N) \longrightarrow N \longrightarrow N/\Gamma_0(N) \longrightarrow 0$$

induces the exact sequence

$$0 \longrightarrow \Gamma_0(N) \longrightarrow N \longrightarrow N/\Gamma_0(N) \longrightarrow 0$$
\[ \text{Ext}^i(R/I, N) \to \text{Ext}^i(R/I, N/\Gamma_i(N)) \to \text{Ext}^{i+1}(R/I, \Gamma_i(N)). \]

Now, the R-module \( \text{Ext}^i(R/I, N/\Gamma_i(N)) \) is \( I, M \)-minimax, by Azami et al. (2008, Proposition 2.3) and the assumption. Also, \( H^0_i(N/\Gamma_i(N)) = 0 \) and \( H^j_i(N/\Gamma_i(N)) \cong H^j_i(N) \) for all \( i > 0 \), so that \( H^j_i(N/\Gamma_i(N)) \) is \( I, M \)-minimax for all \( i < t \). Therefore, we may assume that \( N \) is \( I \)-torsion free. Let \( E \) be an injective envelope of \( N \) and put \( T = E/N \). Then \( H^0_i(T) = 0 \), \( H^j_i(T) = 0 \) and \( \text{Hom}_R(R/I, E) = 0 \). Consequently, \( \text{Ext}^i(R/I, T) \cong \text{Ext}^{i+1}_R(R/I, N) \cong H^{i+1}_i(N) \) and \( H^j_i(M, T) \cong H^{j+1}_i(M, N) \) for all \( i > 0 \). The induction hypothesis applied to \( T \) yields that \( \text{Hom}_R(M/IM, H^{j+1}_i(M, T)) \) is \( I, M \)-minimax. Hence \( \text{Hom}_R(M/IM, H^j_i(M, N)) \) is \( I, M \)-minimax.

**Theorem 3.2** Let \( N \) be an \( I, M \)-minimax \( R \)-module and let \( t \) be a non-negative integer such that \( H^j_i(N) \) is \( I, M \)-minimax for all \( i < t \). Then for any \( I, M \)-minimax submodule \( K \) of \( H^j_i(M, N) \) and for any finitely generated \( R \)-module \( L \) with \( \text{Supp} \subset \text{V}(I) \) the \( R \)-module \( \text{Hom}_R(L, H^j_i(M, N)/K) \) is \( I, M \)-minimax. In particular, the set of associated prime ideals of \( H^j_i(M, N)/K \) is finite.

**Proof** Apply the last theorem and Azami et al. (2008, Corollary 2.5).

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