Payrovi et al., *Cogent Mathematics* (2017), 4: 1327683
https://doi.org/10.1080/23311835.2017.1327683

cogent mathematics

PURE MATHEMATICS | RESEARCH ARTICLE

Finiteness properties of generalized local cohomology modules for minimax modules

Sh. Payrovi[1*], I. Khalili-Gorji[1] and Z. Rahimi-Molaei[1]

Received: 25 December 2016
Accepted: 27 April 2017
First Published: 15 May 2017

*Corresponding author: Sh. Payrovi, Department of Mathematics, Imam Khomeini International University, 34149-1-6818, Qazvin, Iran
E-mail: shpayrovi@sci.ikiu.ac.ir

Reviewing editor:
Lishan Liu, Qufu Normal University, China

Additional information is available at the end of the article

Abstract: Let R be a commutative Noetherian ring, I an ideal of R, M be a finitely generated R-module and t be a non-negative integer. In this paper, we introduce the concept of I, M-minimax R-modules. We show that $\operatorname{Hom}_R(R/I, H^t_I(M, N)/K)$ is I,M-minimax, for all I,M-minimax submodules K of $H^t_I(M, N)$, whenever N and $H^0_I(M)$, $H^1_I(M), \cdots, H^{t-1}_I(M)$ are I, M-minimax R-modules. As consequence, it is shown that $\operatorname{Ass}_R H^t_I(M, N)/K$ is a finite set.

Subjects: Science; Mathematics & Statistics; Advanced Mathematics; Algebra; Fields & Rings

Keywords: generalized local cohomology; minimax module

2000 Mathematics subject classifications: Primary 13D45; Secondary 14B15; 13E05

1. Introduction

Let R be a commutative Noetherian ring, I an ideal of R, and M a finitely generated R-module. An important problem in commutative algebra is determining when the set of associated primes of the i-th local cohomology module is finite. In Huneke, (1992) raised the following question: If M is a finitely generated R-module, then the set of associated primes of $H^i_I(M)$ is finite for all ideals I of R and all $i \geq 0$. Singh (2000) and Katzman (2002) have given counterexamples to this conjecture. However, it is known that this conjecture is true in many situations; see Brodmann and Lashgari Faghani (2000), Brodmann, Rotthous, and Sharp (2000), Hellus (2001), Marley (2001). In particular, Brodmann and Lashgari Faghani (2000) have shown that, $\operatorname{Ass}_R(H^t_I(M)/K)$ is a finite set for any finitely generated submodule K of $H^t_I(M)$, whenever the local cohomology modules $H^0_I(M), H^1_I(M), \cdots, H^{t-1}_I(M)$ are finitely generated. Next, Bahmanpour and Naghipour (2008) showed that, $\operatorname{Hom}_R(R/I, H^t_I(M)/K)$ is finitely generated for any minimax submodule K of $H^t_I(M)$, whenever the local cohomology modules $H^0_I(M), H^1_I(M), \cdots, H^{t-1}_I(M)$ are minimax. After this Azami, Naghipour, and Vakili (2008) proved that, $\operatorname{Hom}_R(R/I, H^t_I(N)/K)$ is I-minimax for any I-minimax submodule K of $H^t_I(N)$, whenever N is an I-minimax R-module and the local cohomology modules $H^0_I(N), H^1_I(N), \cdots, H^{t-1}_I(N)$ are I-minimax. The main result of this note is a generalization of above theorems for generalized local cohomology modules.

Recall that an R-module N is said to have finite Goldie dimension if N dose not contain an infinite direct sum of non-zero submodules, or equivalently the injective hall $E(N)$ of N decomposes as a finite direct sum of indecomposable submodules. Also, an R-module N is said to have finite I-relative

ABOUT THE AUTHOR

Sh. Payrovi PhD, works as an associated professor in Department of Mathematics, Imam Khomeini International University, Qazvin, Iran. He published more than 30 research papers in the field of Commutative Algebra and Homological Algebra.

PUBLIC INTEREST STATEMENT

Commutative Algebra and Homological Algebra are two fields in Pure Mathematics and local cohomology theory play an important role in those fields. The present study is about the generalized local cohomology modules and we define a new concept in this area.

Goldie dimension if the Goldie dimension of the I-torsion submodule $\Gamma_I(N) := \cup_{n\geq 1}(0:_N I^n)$ of N is finite. We say that an R-module N has finite I, M-relative Goldie dimension if the Goldie dimension of the R-module $H_I^0(M, N)$ is finite. An R-module N is called I-minimax if I-relative Goldie dimension of any quotient module of N is finite. We say that an R-module N is I, M-minimax if I,M-relative Goldie dimension of any quotient module of N is finite.

Precisely we show that, $\mathrm{Hom}_R(R/I, H_I^t(M, N)/K)$ is I, M-minimax for any I, M-minimax submodule K of $H_I^t(M, N)$, whenever the R-module N and the local cohomology modules $H_I^0(N)$, $H_I^1(N), \cdots, H_I^{t-1}(N)$ are I, M-minimax.

Throughout this paper, R will always be a commutative Noetherian ring with non-zero identity, I an ideal of R, M will be a finitely generated R-module and N an R-module. The i-th generalized local cohomology module with respect to I is defined by

$$H_I^i(M, N) = \lim_{n\in\mathbb{N}} \mathrm{Ext}_R^i(M/I^nM, N).$$

We refer the reader to Brodmann and Sharp (1998), Herzog (1974), Suzuki (1978), Yassemi, Khatami, and Sharif (2002), Payrovi, Babaei, and Khalili-Gorji (2015), Saremi and Mafi (2013) for the basic properties of local cohomology and generalized local cohomology.

2. I, M-minimax modules

For an R-module N the *Goldie dimension* is defined as the cardinal of the set of indecomposable submodule of $E(N)$ which appear in a decomposition of $E(N)$ in to a direct sum of indecomposable submodules. We shall use $\mathrm{G\,dim}N$ to denote the Goldie dimension of N. Let $\mu^0(\mathfrak{p}, N)$ denote the 0-th Bass number of N with respect to prime ideal $\mathfrak{p}$ of R. It is well known that $\mu^0(\mathfrak{p}, N) > 0$ if and only if $\mathfrak{p} \in \mathrm{Ass}_R N$ and it is clear that

$$\mathrm{G\,dim}N = \sum_{\mathfrak{p}\in \mathrm{Ass}_R N} \mu^0(\mathfrak{p}, N).$$

Also, the I-relative Goldie dimension of N is defined as

$$\mathrm{G\,dim}_I N = \sum_{\mathfrak{p}\in \mathrm{V}(I)\cap \mathrm{Ass}_R N} \mu^0(\mathfrak{p}, N).$$

The I-relative Goldie dimension of an R-module has been studied in Divaani-Aazar and Esmkhani (2005) and in Lemma 2.6 it is shown that $\mathrm{G\,dim}_I(N) = \mathrm{G\,dim}H_I^0(N)$. Having this in mind, we introduce the following generalization of the notion of I-relative Goldie dimension.

Definition 2.1 Let I be an ideal of R and M be a finitely generated R-module. We denote by $\mathrm{G\,dim}_{I,M}N$ the I, M-relative Goldie dimension of N and we define I, M-relative Goldie dimension of N as

$$\mathrm{G\,dim}_{I,M}N = \mathrm{G\,dim}H_I^0(M, N).$$

The class of I-minimax modules is defined in Azami et al. (2008) and an R-module N is said to be *minimax with respect to I* or *I-minimax* if I-relative Goldie dimension of any quotient module of N is finite. This motivates the following definition.

Definition 2.2 Let I be an ideal of R and M be a finitely generated R-module. An R-module N is said to be *I, M-minimax* if the I, M-relative Goldie dimension of any quotient module of N is finite; i.e. for any submodule K of N, $\mathrm{G\,dim}_{I,M}N/K < \infty$.

Proposition 2.3 Let N be an R-module. Then N is I,M-minimax if and only if N is $\mathrm{Ann}(M/IM)$-minimax.

Proof It is sufficient to show that for each $\mathfrak{p} \in Ann(M/IM)$, there is an integer $n_{\mathfrak{p}}$ such that

$$\mathrm{G\,dim}_{I,M}N = \sum_{\mathfrak{p}\in V(\mathrm{Ann}(M/IM))} n_{\mathfrak{p}}\mu^{0}(\mathfrak{p}, N).$$

We have $H_I^0(M, N) \cong \mathrm{Hom}_R(M, \Gamma_I(N)) \cong \Gamma_I(\mathrm{Hom}_R(M, N))$ so that it follows

$$\begin{aligned}\mathrm{G\,dim}_{I,M}(N) &= \mathrm{G\,dim}H_I^0(M, N) = \sum_{\mathfrak{p}\in \mathrm{Ass}H_I^0(M,N)} \mu^{0}(\mathfrak{p}, H_I^0(M, N))\\ &= \sum_{\mathfrak{p}\in \mathrm{Ass\,Hom}_R(M,N)\cap V(I)} \mu^{0}(\mathfrak{p}, \mathrm{Hom}_R(M, N)).\end{aligned}$$

On the other hand, $\mathrm{Ass\,Hom}_R(M, N) = \mathrm{Ass}N \cap \mathrm{Supp}M$. Hence,

$$\mathrm{G\,dim}_{I,M}(N) = \sum_{\mathfrak{p}\in \mathrm{Ass}N\cap V(\mathrm{Ann}(M/IM))} \mu^{0}(\mathfrak{p}, \mathrm{Hom}_R(M, N)).$$

For $\mathfrak{p} \in \mathrm{Ass}(N) \cap V(\mathrm{Ann}(M/IM))$ we have

$$\begin{aligned}\mu^{0}(\mathfrak{p}, \mathrm{Hom}_R(M, N)) &= \dim_{k(\mathfrak{p})} \mathrm{Hom}_{R_{\mathfrak{p}}}(K(\mathfrak{p}), \mathrm{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}))\\ &= \dim_{k(\mathfrak{p})} \mathrm{Hom}_{R_{\mathfrak{p}}}(k(\mathfrak{p}) \otimes_{R_{\mathfrak{p}}} M_{\mathfrak{p}}, N_{\mathfrak{p}}),\end{aligned}$$

where $k(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ and $k(\mathfrak{p}) \otimes_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$ is a finite dimensional $k(\mathfrak{p})$-vector space with dimension $n_{\mathfrak{p}}$. Hence, $k(\mathfrak{p}) \otimes_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \cong \oplus_{n_{\mathfrak{p}}} k(\mathfrak{p})$ which implies that

$$\mu^{0}(\mathfrak{p}, \mathrm{Hom}_R(M, N)) = \dim_{k(\mathfrak{p})} \mathrm{Hom}_{R_{\mathfrak{p}}}(\oplus_{n_{\mathfrak{p}}} k(\mathfrak{p}), N_{\mathfrak{p}}) = n_{\mathfrak{p}}\mu^{0}(\mathfrak{p}, N).$$

It is clear that the above argument is true for each quotient of N. □

Remark 2.4 The following statements are true for any R-module N.

(i) The I, R-minimax modules are precisely I-minimax.
(ii) The I, M-minimax modules are I-minimax.
(iii) If N is Noetherian or Artinian R-module, then N is I, M-minimax.
(iv) If J is a second ideal of R such that $I \subseteq J$ and N is J, M-minimax, then N is I,M-minimax.
(v) Let N is $\mathrm{Ann}_R(M)$-torsion, i.e. $\Gamma_{\mathrm{Ann}_R(M)}(N) = N$. Then N is I, M-minimax if and only if N is I-minimax.

Proposition 2.5 Let $0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$ be an exact sequence of R-modules. Then N is I, M-minimax if and only if N' and N'' are both I, M-minimax.

Proof This is immediate from Proposition 2.3 and Azami et al. (2008, Proposition 2.5). □

Proposition 2.6 Let t be a non-negative integer. Then for all R-module N the following statements are equivalent:

(i) $\mathrm{Ext}_R^i(R/I, N)$ is I, M-minimax for all $i \leq t$.
(ii) $\mathrm{Ext}_R^i(R/J, N)$ is I, M-minimax for all ideal J of R with $I \subseteq J$ and for all $i \leq t$.
(iii) $\mathrm{Ext}_R^i(L, N)$ is I, M-minimax for all finitely generated R-module L with $\mathrm{Supp}L \subseteq \mathrm{V}(I)$ and for all $i \leq t$.
(iv) For any minimal prime ideal $\mathfrak{p}$ over I, $\mathrm{Ext}_R^i(R/\mathfrak{p}, N)$ is I,M-minimax for all $i \leq t$.

Proof The proof is similar to that of Azami et al. (2008, Corollary 2.8) □

Proposition 2.7 If N is an I, M-minimax module such that $Ass_R N \subseteq V(I)$, then $H^i_I(L, N)$ is I, M-minimax for all finitely generated R-module L and all $i \geq 0$.

Proof If $i = 0$, then $H^0_I(L, N) = \mathrm{Hom}_R(L, \Gamma_I(N))$ and so by Azami et al. (2008, Corollary 2.5), $H^0_I(L, N)$ is I, M-minimax. As $\mathrm{Ass}_R N/\Gamma_I(N) \subseteq \mathrm{Ass}_R N$, it easily follows from $\mathrm{Ass}_R N \subseteq V(I)$ that $N = \Gamma_I(N)$. Consequently, $H^i_I(L, N) = \mathrm{Ext}^i_R(L, N)$ for all $i \geq 0$, by Yassemi et al. (2002, Theorem 2.3). So that $H^i_I(L, N)$ is I, M-minimax for all $i \geq 0$, as required. □

Proposition 2.8 Let N be an R-module and let t be a non-negative integer. If $H^i_I(N)$ is I, M-minimax for all $i < t$, then $H^i_I(M, N)$ is I, M-minimax for all $i < t$.

Proof We use induction on t. When $t = 1$, the R-module $\Gamma_I(N)$ is I, M-minimax by assumption. Since $H^0_I(M, N) \cong \mathrm{Hom}(M, \Gamma_I(N))$, it follows that $H^0_I(M, N)$ is I, M-minimax, by Azami et al. (2008, Theorem 2.7). Now suppose, inductively that $t > 1$ and the result has been proved for $t - 1$. Since $H^i_I(N) \cong H^i_I(N/\Gamma_I(N))$ and $H^i_I(M, N) \cong H^i_I(M, N/\Gamma_I(N))$ for all $i > 0$, it follows that $H^i_I(N/\Gamma_I(N))$ is I, M-minimax for all $i < t$. Therefore, we may assume that N is I-torsion free. Let E be an injective envelope of N and put $N_1 = E/N$. Then $\Gamma_I(E) = 0$. Consequently, $H^i_I(N_1) \cong H^{i+1}_I(N)$. Thus $H^i_I(N_1)$ is I, M-minimax for all $i < t - 1$ and by induction hypothesis $H^i_I(M, N_1)$ is I, M-minimax for all $i < t - 1$. Also, we have $H^i_I(M, N_1) \cong H^{i+1}_I(M, N)$ so that $H^i_I(M, N)$ is I, M-minimax for all $i < t$. □

3. Finiteness of associated primes

It will be shown in this section that the subject of the previous section can be use to prove a finiteness result about generalized local cohomology modules. In fact we will generalize the main results of Brodmann and Lashgari Faghani (2000) and Azami et al. (2008). Throughout this section I is an ideal of R and M is a finitely generated R-module.

Theorem 3.1 *Let N be an R-module and let t be a non-negative integer. If $H^i_I(N)$ is I, M-minimax for all $i < t$, and $\mathrm{Ext}^t_R(R/I, N)$ is I, M-minimax, then for any I, M-minimax submodule K of $H^t_I(M, N)$ and for any finitely generated R-module L with $\mathrm{Supp} L \subseteq V(I)$ the R-module $\mathrm{Hom}_R(L, H^t_I(M, N)/K)$ is I, M-minimax.*

Proof The exact sequence

$$0 \longrightarrow K \longrightarrow H^t_I(M, N) \longrightarrow H^t_I(M, N)/K \longrightarrow 0$$

provides the following exact sequence:

$$\cdots \longrightarrow \mathrm{Hom}_R(L, H^t_I(M, N)) \longrightarrow \mathrm{Hom}_R(L, H^t_I(M, N)/K) \longrightarrow \mathrm{Ext}^1_R(L, K) \longrightarrow \cdots.$$

By Azami et al. (2008, Corollary 2.5), $\mathrm{Ext}^1_R(L, K)$ is I, M-minimax, so in view of Azami et al. (2008, Proposition 2.3), it is thus sufficient for us to show that the R-module $\mathrm{Hom}_R(L, H^t_I(M, N))$ is I, M-minimax. To this end, it is enough to show that $\mathrm{Hom}_R(R/I, H^t_I(M, N))$ is I, M-minimax by Proposition 2.6. We use induction on t. When $t = 0$, the R-module $\mathrm{Hom}_R(R/I, N))$ is I, M-minimax, by assumption. On the other hand,

$$\begin{aligned}\mathrm{Hom}_R(R/I, H^0_I(M, N)) &\cong \mathrm{Hom}_R(R/I, \ \mathrm{Hom}_R(M, \Gamma_I(N)) \\ &\cong \mathrm{Hom}_R(M/IM, \Gamma_I(N)) \cong \mathrm{Hom}_R(M/IM, N)\end{aligned}$$

and $\mathrm{Supp}(M/IM) \subseteq V(I)$, it follows that $\mathrm{Hom}_R(M/IM, N)$ is I, M-minimax, by Proposition 2.6. Hence $\mathrm{Hom}_R(R/I, H^0_I(M, N))$ is I, M-minimax. Now suppose, inductively, that $t > 0$ and that the result has been proved for $t - 1$. Since $\Gamma_I(N)$ is I, M-minimax, it follows that $\mathrm{Ext}^i_R(R/I, \Gamma_I(N))$ is I, M-minimax for all $i \geq 0$. The exact sequence

$$0 \longrightarrow \Gamma_I(N) \longrightarrow N \longrightarrow N/\Gamma_I(N) \longrightarrow 0$$

induces the exact sequence

$$\mathrm{Ext}_R^t(R/I,\, N)) \longrightarrow\ \mathrm{Ext}_R^t(R/I,\, N/\Gamma_I(N)) \longrightarrow\ \mathrm{Ext}_R^{t+1}(R/I,\, \Gamma_I(N)).$$

Now, the R-module $\mathrm{Ext}_R^t(R/I,\, N/\Gamma_I(N))$ is I, M-minimax, by Azami et al. (2008, Proposition 2.3) and the assumption. Also, $H_I^0(N/\Gamma_I(N)) = 0$ and $H_I^i(N/\Gamma_I(N)) \cong H_I^i(N)$ for all $i > 0$, so that $H_I^i(N/\Gamma_I(N))$ is I, M-minimax for all $i < t$. Therefore, we may assume that N is I-torsion free. Let E be an injective envelope of N and put $T = E/N$. Then $H_I^0(E) = 0$, $H_I^0(M,\, E) = 0$ and $\mathrm{Hom}_R(R/I,\, E) = 0$. Consequently, $\mathrm{Ext}_R^i(R/I,\, T) \cong\ \mathrm{Ext}_R^{i+1}(R/I,\, N)$, $H_I^i(T) \cong H_I^{i+1}(N)$ and $H_I^i(M,\, T) \cong H_I^{i+1}(M,\, N)$ for all $i \geq 0$. The induction hypothesis applied to T yields that $\mathrm{Hom}_R(M/IM,\, H_I^{t-1}(M,\, T))$ is I, M-minimax. Hence $\mathrm{Hom}_R(M/IM,\, H_I^t(M,\, N))$ is I, M-minimax. □

THEOREM 3.2 *Let N be an I, M-minimax R-module and let t be a non-negative integer such that $H_I^i(N)$ is I, M-minimax for all $i < t$. Then for any I, M-minimax submodule K of $H_I^t(M,\, N)$ and for any finitely generated R-module L with $\mathrm{Supp}L \subseteq\ V(I)$ the R-module $\mathrm{Hom}_R(L, H_I^t(M,N)/K)$ is I, M-minimax. In particular, the set of associated prime ideals of $H_I^t(M,\, N)/K$ is finite.*

Proof Apply the last theorem and Azami et al. (2008, Corollary 2.5). □

Funding
The authors received no direct funding for this research.

Author details
Sh. Payrovi[1]
E-mail: shpayrovi@sci.ikiu.ac.ir
I. Khalili-Gorji[1]
E-mail: i.khalili@edu.ikiu.ac.ir
Z. Rahimi-Molaei[1]
E-mail: z.rahimi_math@yahoo.com
[1] Department of Mathematics, Imam Khomeini International University, 34149-1-6818, Qazvin, Iran.

Citation information
Cite this article as: Finiteness properties of generalized local cohomology modules for minimax modules, Sh. Payrovi, I. Khalili-Gorji & Z. Rahimi-Molaei, *Cogent Mathematics* (2017), 4: 1327683.

References

Azami, J., Naghipour, R., & Vakili, B. (2008). Finiteness properties of local cohomology modules for α-minimax modules. *Proceedings of the American Mathematical Society, 136*, 439–448.

Bahmanpour, K., & Naghipour, R. (2008). On the cofiniteness of local cohomology modules. *Proceedings of the American Mathematical Society, 136*, 2359–2363.

Brodmann, M. P., & Lashgari Faghani, A. (2000). A finiteness result for associated primes of local cohomology modules. *Proceedings of the American Mathematical Society, 128*, 2851–2853.

Brodmann, M. P., Rotthous, C., & Sharp, R. Y. (2000). On annihilators and associated primes of local cohomology module. *Journal of Pure and Applied Algebra, 135*, 197–227.

Brodmann, M. P., & Sharp, R. Y. (1998). *Local cohomology; An algebraic introduction with geometric application*. Cambridge: Cambridge University Press.

Divaani-Aazar, K., & Esmkhani, M. A. (2005). Artinianness of local cohomology modules of ZD-modules. *Communications in Algebra, 33*, 2857–2863.

Hellus, M. (2001). On the set of associated primes of a local cohomology modules. *Journal of Algebra, 237*, 406–419.

Herzog, J. K. (1974). *Auflosungen und dualitat in der lokalen algebra*. Preprint univ Regensburg.

Huneke, C. (1992). Problems on local cohomology modules. Free resolution in commutative algebra and algebraic geometry. (Sundance, UT, 1990), *Res. Notes Math., 2* (pp. 93-108). Boston, MA: Jones and Bartlett

Katzman, M. (2002). An example of an infinite set of associated primes of local cohomology modules. *Journal of Algebra, 252*, 161–166.

Marley, T. (2001). The associated primes of local cohomology modules over rings of small dimension. *Manuscripta Mathematica, 104*, 519–525.

Payrovi, S., Babaei, S., & Khalili-Gorji, I. (2015). Bass numbers of generalized local cohomology modules. *Publications De LInstitut Mathematiqe, 97*, 233–238.

Saremi, H., & Mafi, A. (2013). Finiteness dimension and bass numbers of generalized local cohomology modules. *Journal of Algebra and its Applications*, *12*, 6 pages. 1350036.

Singh, A. K. (2000). p-Tortion element in local cohomology modules. *Mathematical Research Letters, 7*, 165–176.

Suzuki, N. (1978). On the generalized local cohomology and its duality. *Journal of Mathematics of Kyoto University, 18*, 71–85.

Yassemi, S., Khatami, L., & Sharif, T. (2002). Associated primes of generalized local cohomology modules. *Communications in Algebra, 30*, 327–330.