A new eigenvalue inclusion set for tensors with its applications

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Abstract: In this paper, we give a new eigenvalue localization set for tensors and show that the new set is tighter than those presented by Qi (2005) and Li et al. (2014). As applications, we give a new sufficient condition of the positive (semi-) definiteness for an even-order real symmetric tensor and new lower and upper bounds of the minimum eigenvalue for $\mathcal{M}$-tensors.

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1. Introduction

For a positive integer $n$, $n \geq 2$, $N$ denotes the set $\{1, 2, \ldots, n\}$. $\mathbb{C}$ (respectively, $\mathbb{R}$) denotes the set of all complex (respectively, real) numbers. We call $\mathcal{A} = (a_{i_1 \cdots i_m})$ a complex (real) tensor of order $m$ dimension $n$, denoted by $\mathbb{C}^{[m,n]}$ ($\mathbb{R}^{[m,n]}$), if $a_{i_1 \cdots i_m} \in \mathbb{C}$ ($\mathbb{R}$) for $i_j \in N$ for $j = 1, 2, \ldots, m$. An $m$-order $n$-dimensional tensor $\mathcal{A}$ is called nonnegative, if each entry is nonnegative. A tensor of order $m$ dimension $n$ is called the unit tensor, denoted by $\mathcal{I}$, if its entries are $\delta_{i_1 \cdots i_m}$ for $i_1, \ldots, i_m \in N$, where

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PUBLIC INTEREST STATEMENT

One of many practical applications of eigenvalues of tensors is that one can identify the positive (semi-)definiteness of an even-order real symmetric tensor by using the smallest $H$-eigenvalue of a tensor; consequently, one can identify the positive (semi-)definiteness of the multivariate homogeneous polynomial determined by this tensor. However, it is not easy to compute the smallest $H$-eigenvalue of tensors when the order and dimension are very large, we always try to give a set including all eigenvalues in the complex. In particular, if one of these sets for an even-order real symmetric tensor is in the right-half complex plane, then we can conclude that the smallest $H$-eigenvalue is positive, consequently, the corresponding tensor is positive definite. Therefore, the main aim of this paper is to give a new eigenvalue inclusion set for tensors, and using the set to obtain a weaker sufficient condition for the positive (semi-)definiteness of an even-order real symmetric tensor.
\[ \delta_{i_1 \ldots i_m} = \begin{cases} 1, & \text{if } i_1 = \ldots = i_m, \\ 0, & \text{otherwise.} \end{cases} \]

A real tensor \( \mathcal{A} = (a_{i_1 \ldots i_m}) \) is called symmetric (Qi, 2005) if
\[ a_{i_1 \ldots i_m} = a_{\sigma(i_1 \ldots i_m)}, \quad \forall \pi \in \Pi_m, \]
where \( \Pi_m \) is the permutation group of \( m \) indices.

A tensor \( \mathcal{A} = (a_{i_1 \ldots i_m}) \) is called reducible if there exists a nonempty proper index subset \( J \subset N \) such that
\[ a_{i_1 \ldots i_m} = 0, \quad \forall i_1 \in J, \forall i_2, \ldots, i_m \notin J. \]

If \( \mathcal{A} \) is not reducible, then we call \( \mathcal{A} \) is irreducible (Chang, Zhang, & Pearson, 2008). Let \( \mathcal{A} = (a_{i_1 \ldots i_m}) \) be a nonnegative tensor, \( G = (g_j) \in \mathbb{R}^{m \times n}, g_j = \sum_{i_1 \ldots i_m} a_{i_1 \ldots i_m} \). \( \mathcal{A} \) is called weakly reducible if \( G \) is a reducible matrix. If \( \mathcal{A} \) is not weakly reducible, then it is called weakly irreducible; for details, see Friedland, Gaubert, and Han (2013) and Zhang, Qi, and Zhou (2014).

For a general tensor \( \mathcal{A} = (a_{i_1 \ldots i_m}) \in \mathbb{C}^{n, n}, \) Wang and Wei (2015) proved that if \( \mathcal{A} \) is irreducible, then \( \mathcal{A} \) is weakly irreducible, and for \( m = 2, \mathcal{A} \) is reducible if and only if \( \mathcal{A} \) is weakly irreducible.

Given a tensor \( \mathcal{A} = (a_{i_1 \ldots i_m}) \in \mathbb{C}^{n, n}, \) if there are \( \lambda \in \mathbb{C} \) and \( x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{C}^n \setminus \{0\} \) such that
\[ \mathcal{A}x^{m-1} = \lambda x^{m-1}, \]
then \( \lambda \) is called an eigenvalue of \( \mathcal{A} \) and \( x \) an eigenvector of \( \mathcal{A} \) associated with \( \lambda \), where \( \mathcal{A}x^{m-1} \) is an \( n \) dimension vector whose \( i \)th component is
\( (\mathcal{A}x^{m-1})_i = \sum_{i_1 \ldots i_m \in \mathbb{N}} a_{i_1 \ldots i_m} x_{i_1} \ldots x_{i_m}, \) and
\[ x^{m-1} = (x_1^{m-1}, x_2^{m-1}, \ldots, x_n^{m-1})^T. \]

If \( \lambda \) and \( x \) are all real, then \( \lambda \) is called an H-eigenvalue of \( \mathcal{A} \) and \( x \) an H-eigenvector of \( \mathcal{A} \) associated with \( \lambda \). This definition was introduced by Qi (2005) where he assumed that \( \mathcal{A} \in \mathbb{R}^{n, n} \) is symmetric and \( m \) is even. Independently, Lim (2015) gave such a definition but restricted \( x \) to be a real vector and \( \lambda \) to be a real number. Moreover, the spectral radius \( \rho(\mathcal{A}) \) of the tensor \( \mathcal{A} \) is defined as
\[ \rho(\mathcal{A}) = \max \{ |\lambda| : \lambda \in \sigma(\mathcal{A}) \}, \]
where \( \sigma(\mathcal{A}) \) is the spectrum of \( \mathcal{A} \), i.e. \( \sigma(\mathcal{A}) = \{ \lambda : \lambda \text{ is an eigenvalue of } \mathcal{A} \} \) (see Chang et al., 2008; Yang & Yang, 2010).

Let \( \mathcal{A} = (a_{i_1 \ldots i_m}) \in \mathbb{R}^{n, n} \). \( \mathcal{A} \) is called a Z-tensor, if all of its off-diagonal entries are non-positive, which is equivalent to write \( \mathcal{A} = sI - B \), where \( s > 0 \) and \( B \) is a nonnegative tensor. A Z-tensor \( \mathcal{A} = sI - B \) is an M-tensor if \( s > \rho(B) \). Here, we denote by \( r(\mathcal{A}) \) the minimal value of the real part of all eigenvalues of an M-tensor \( \mathcal{A} \), and note that if \( \mathcal{A} \) is a weakly irreducible Z-tensor, then \( r(\mathcal{A}) > 0 \) is the unique eigenvalue with a positive eigenvector; for details, see Zhang et al. (2014) and Ding, Qi, and Wei (2013).
Given an even-order symmetric tensor $\mathcal{A} = (a_{i_1 \ldots i_n}) \in \mathbb{R}^{[m,n]}$, the positive (semi-)definiteness of $\mathcal{A}$ is determined by the sign of its smallest H-eigenvalue, that is, if the smallest H-eigenvalue is positive (nonnegative), then $\mathcal{A}$ is positive (semi-)definite. However, when $m$ and $n$ are very large, it is not easy to compute the smallest H-eigenvalue of $\mathcal{A}$. Then we can try to give a set in the complex which includes all eigenvalues of $\mathcal{A}$. If this set is in the right-half complex plane, then we can conclude that the smallest H-eigenvalue is positive, consequently, $\mathcal{A}$ is positive definite; for details, see Qi (2005), Li, Li, and Kong (2014), Li and Li (2016), Li, Jiao, and Li (2016), Li, Chen, and Li (2015) and Huang, Wang, Xu, and Cui (2016).

Therefore, one of the main aims of this paper is to give a new eigenvalue inclusion set for tensors, and use this set to determine positive (semi-)definiteness of tensors.

In Qi (2005) generalized Geršgorin eigenvalue inclusion theorem from matrices to real supersymmetric tensors, which can be easily extended to general tensors (Li et al., 2014; Yang & Yang, 2010).

**Theorem 1.1** (Qi, 2005, Theorem 6) Let $\mathcal{A} = (a_{i_1 \ldots i_m}) \in \mathbb{C}^{[m,n]}$. Then
\[
\sigma(\mathcal{A}) \subseteq \Gamma(\mathcal{A}) = \bigcup_{i \in \mathbb{N}} \Gamma_i(\mathcal{A}),
\]
where
\[
\Gamma_i(\mathcal{A}) = \{ z \in \mathbb{C} : |z - a_{i_1 \ldots i_m}| \leq r_i(\mathcal{A}) \}, \quad r_i(\mathcal{A}) = \sum_{\delta_{i_2 \ldots i_m} = 0} |a_{i_2 \ldots i_m}|.
\]

To get tighter eigenvalue inclusion sets than $\Gamma(\mathcal{A})$, Li et al. (2014) extended the Brauer’s eigenvalue localization set of matrices (Varga, 2004) and proposed the following Brauer-type eigenvalue localization sets for tensors.

**Theorem 1.2** (Li et al., 2014, Theorem 2.1) Let $\mathcal{A} = (a_{i_1 \ldots i_m}) \in \mathbb{C}^{[m,n]}$. Then
\[
\sigma(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}) = \bigcup_{i,j \in \mathbb{N}, j \neq i} \mathcal{K}_{ij}(\mathcal{A}),
\]
where
\[
\mathcal{K}_{ij}(\mathcal{A}) = \{ z \in \mathbb{C} : |z - a_{i_1 \ldots i_m}| \leq r_{ij}(\mathcal{A}) |z - a_{j_1 \ldots j_m}| \}, \quad r_{ij}(\mathcal{A}) = \sum_{\delta_{i_2 \ldots i_m} = 0} |a_{i_2 \ldots i_m}| = r_i(\mathcal{A}) - |a_{j_1 \ldots j_m}|.
\]

One of many applications of eigenvalue inclusion sets is to bound the minimum H-eigenvalue of $\mathcal{M}$-tensors (He & Huang, 2014; Huang et al., 2016; Wang & Wei, 2015; Zhao & Sang, 2016). In He and Huang (2014) provided some inequalities on $\tau(\mathcal{A})$ for an irreducible $\mathcal{M}$-tensor $\mathcal{A}$ as follows.

**Theorem 1.3** (He & Huang, 2014, Theorem 2.1) Let $\mathcal{A} = (a_{i_1 \ldots i_m}) \in \mathbb{R}^{[m,n]}$ be an irreducible $\mathcal{M}$-tensor. Then
\[
\tau(\mathcal{A}) \leq \min_{i \in \mathbb{N}} a_{i_1 \ldots i_m}, \quad \text{and} \quad \min_{i \in \mathbb{N}} R_i(\mathcal{A}) \leq \tau(\mathcal{A}) \leq \max_{i \in \mathbb{N}} R_i(\mathcal{A}),
\]
where $R_i(\mathcal{A}) = \sum_{j_1 \ldots j_m \in \mathbb{N}} a_{j_1 \ldots j_m}$.

For the weakly irreducible $\mathcal{M}$-tensor, Wang and Wei (2015) obtained the following results on $\tau(\mathcal{A})$.

**Theorem 1.4** (Wang and Wei, 2015, Lemma 4.4) Let $\mathcal{A}$ be a weakly irreducible $\mathcal{M}$-tensor. Then
\[
\tau(\mathcal{A}) \leq \min_{i \in \mathbb{N}} |a_{i_1 \ldots i_m}|.
\]
In this paper, we continue this research on the eigenvalue inclusion sets for tensors and its applications. We obtain a new eigenvalue inclusion set for tensors and prove that the new set is tighter than Theorems 1.1 and 1.2. As applications, we establish a sufficient condition for the positive (semi-)definiteness of tensors and give new lower and upper bounds of the minimum H-eigenvalue for $\mathcal{M}$-tensors, which are the correction of Theorem 4.5 in Wang and Wei (2015).

2. A new eigenvalue inclusion set for tensors

In this section, we propose a new eigenvalue inclusion set for tensors and establish the comparisons between this new set with those in Theorems 1.1 and 1.2.

**Theorem 2.1** Let $A = (a_{i\ldots j}) \in \mathbb{C}^{m \times n}$. Then

$$\sigma(A) \subseteq \Omega(A) = \bigcup_{i,j} \Omega_{ij}(A),$$

where

$$\Omega_{ij}(A) = \left\{ z \in \mathbb{C} : |z - a_{\ldots i\ldots j} - \bar{r}_i(A)| |z - a_{\ldots i\ldots j}| \leq \bar{r}_j(A) \right\},$$

$$\bar{r}_i(A) = \sum_{k \neq i} |a_{k\ldots i\ldots j}|, \quad \bar{r}_j(A) = r_j(A) - \bar{r}_i(A).$$

**Proof** For any $\lambda \in \sigma(A)$, let $x = (x_1, \ldots, x_p)^T \in \mathbb{C}^n \setminus \{0\}$ be an eigenvector corresponding to $\lambda$, i.e.

$$A x^{m-1} = \lambda x^{m-1}. \quad (1)$$

Let

$$|x_p| \geq |x_q| \geq \max \{|x_k| : k \in N, \ 1 \leq k \leq p, q \}$$

(where the last term above is defined to be zero if $n = 2$). Then, $|x_p| > 0$. From (1), we have

$$(\lambda - a_{p\ldots p}) x_p^{m-1} = \sum_{j \neq p} a_{pj\ldots j} x_j \ldots x_p + \sum_{j > p} a_{pj\ldots j} x_j^{m-1}.$$

Taking modulus in the above equation and using the triangle inequality give

$$|\lambda - a_{p\ldots p}| |x_p|^{m-1} \leq \sum_{j \neq p} |a_{pj\ldots j}| |x_j| \ldots |x_p| + \sum_{j > p} |a_{pj\ldots j}| |x_j|^{m-1}$$

$$\leq \sum_{j \neq p} |a_{pj\ldots j}| |x_p|^{m-1} + \sum_{j > p} |a_{pj\ldots j}| |x_q|^{m-1}$$

$$= \bar{r}_p(A)|x_p|^{m-1} + \bar{r}_q(A)|x_q|^{m-1},$$

equivalently,

$$(|\lambda - a_{p\ldots p}| - \bar{r}_i(A)) |x_p|^{m-1} \leq \bar{r}_p(A)|x_q|^{m-1}. \quad (2)$$

If $|x_q| = 0$, by $|x_p| > 0$, we have $|\lambda - a_{p\ldots p}| - \bar{r}_i(A) \leq 0$. Then for any $j \neq p$,

$$(|\lambda - a_{p\ldots p}| - \bar{r}_p(A)) |\lambda - a_{\ldots i\ldots j}| \leq 0 \leq \bar{r}_i(A) \bar{r}_j(A),$$

which implies that $\lambda \in \Omega_{p\ldots p}(A) \subseteq \Omega(A)$. Otherwise, $|x_q| > 0$. Similarly, from (1), we can obtain

$$|\lambda - a_{q\ldots q}| |x_q|^{m-1} \leq \bar{r}_q(A)|x_p|^{m-1}. \quad (3)$$
Multiplying (2) with (3) and noting that $|x_q|^{m-1}|x_q|^{m-1} > 0$, we have

$$((\lambda - a_{p...p} - \tilde{r}_p(A))|\lambda - a_{q...q}| \leq \tilde{r}_q(A)r_q(A),$$

and $\lambda \in \Omega_{p,q}(A) \subseteq \Omega(A)$. Hence, $\sigma(A) \subseteq \Omega(A)$.

Next, a comparison theorem is given for Theorems 1.1, 1.2 and 2.1.

**Theorem 2.2** Let $A = (a_{i_{1}...i_{n}}) \in C^{[m,n]}$. Then

$$\Omega(A) \subseteq \mathcal{K}(A) \subseteq \Gamma(A).$$

**Proof** According to Theorem 2.3 in Li et al. (2014), $\mathcal{K}(A) \subseteq \Gamma(A)$. Hence it suffices to show that $\Omega(A) \subseteq \mathcal{K}(A)$. Let $z \in \Omega(A)$, then there exist $p, q \in N$, $p \neq q$ such that $z \in \Omega_{p,q}(A)$, i.e.

$$|z - a_{p...p} - \tilde{r}_p(A)|z - a_{q...q}| \leq \tilde{r}_p(A)r_q(A).$$

The following proof will be divided into two cases according to a certain rule.

(i) Suppose that $\tilde{r}_p(A)r_q(A) = 0$. Then $\tilde{r}_p(A) = 0$ or $r_q(A) = 0$.

(ii) If $r_q(A) = 0$, then $|a_{pq...q}| = 0$, $r_q(A) = r_q(A)$, and

$$|(z - a_{p...p} - r_q(A))|z - a_{q...q}| = (|z - a_{p...p} - \tilde{r}_p(A))|z - a_{q...q}| \leq \tilde{r}_p(A)r_q(A) = 0 \leq |a_{pq...q}|r_q(A),$$

which implies that $z \in \mathcal{K}_{p,q}(A) \subseteq \mathcal{K}(A)$, consequently, $\Omega(A) \subseteq \mathcal{K}(A)$.

We can also have $z \in \mathcal{K}_{p,q}(A) \subseteq \mathcal{K}(A)$ and $\Omega(A) \subseteq \mathcal{K}(A)$.

(ii) Suppose that $\tilde{r}_p(A)r_q(A) > 0$. Dividing (5) by $\tilde{r}_p(A)r_q(A)$, we have

$$\frac{|z - a_{p...p} - \tilde{r}_p(A)|}{\tilde{r}_p(A)} \frac{|z - a_{q...q}|}{r_q(A)} \leq 1,$$

which implies

$$\frac{|z - a_{p...p} - \tilde{r}_p(A)|}{\tilde{r}_p(A)} \leq 1$$

or

$$\frac{|z - a_{q...q}|}{r_q(A)} \leq 1.$$
Furthermore, we have
\((|z - a_{p...p}| - r^z_p(A))z - a_{q...q} | \leq |a_{pq...q}||r_q(A)|,\)
which implies \(\Omega(A) \subseteq \mathcal{K}(A)\). When (7) holds and \(|a_{pq...q}| = 0\), then
\(|z - a_{p...p}| \leq f_p(A) + \hat{f}_p(A) = r^z_p(A) + |a_{pq...q}|,\)
i.e.
\(|z - a_{p...p}| - r^z_p(A) \leq 0 = |a_{pq...q}|.\)
Obviously,
\((|z - a_{p...p}| - r^z_p(A))|z - a_{q...q}| \leq 0 = |a_{pq...q}||r_q(A)|,\)
which also implies \(\Omega(A) \subseteq \mathcal{K}(A)\).

(ii) When (8) holds, we only need to prove \(\Omega(A) \subseteq \mathcal{K}(A)\) under the condition of
\(\frac{|z - a_{p...p}|}{r_p(A)} > 1,\) i.e. \(\frac{|z - a_{p...p}|}{f_p(A)} > 1.\)

If \(|a_{qp...p}| > 0\), then from Lemmas 2.2 and 2.3 in Li and Li (2016) and (6), we have
\(\frac{|z - a_{p...p}|}{r_p(A)} \leq \frac{|z - a_{p...p}|}{f_p(A)} \frac{|z - a_{q...q}|}{r_q(A)} \leq 1,\)
which leads to
\(|z - a_{q...q}| - r^z_q(A))|z - a_{p...p}| \leq |a_{qp...p}||r_p(A).\)

This implies \(z \in \Omega_{a_{r...p}}(A) \subseteq \Omega(A).\) Furthermore, \(\Omega(A) \subseteq \mathcal{K}(A)\). If \(|a_{qp...p}| = 0\), by (8), we can have
\(|z - a_{q...q}| - r^z_q(A) \leq 0 = |a_{qp...p}|.\)
Then
\(|z - a_{q...q}| - r^z_q(A))|z - a_{p...p}| \leq 0 = |a_{qp...p}||r_p(A),\)
which also implies \(\Omega(A) \subseteq \mathcal{K}(A)\). This proof is completed.

In the following, a numerical example is given to verify Theorem 2.2.

**Example 2.1** Let \(A \in \mathbb{R}^{14\times 2}\) with entries be defined as follows:
\(a_{1111} = 14, a_{1222} = 6, a_{1333} = 9, a_{2111} = 2, a_{2222} = 15, a_{2333} = 8, a_{3111} = 3, a_{3222} = 5, a_{3333} = 17,\)
and other $a_{ijkl} = 0$. The eigenvalue inclusion sets $\Gamma(A)$, $K'(A)$, $\Omega(A)$ and the exact eigenvalues are drawn in Figure 1, where $\Gamma(A)$, $K'(A)$ and $\Omega(A)$ are represented by green boundary, blue boundary, and red boundary, respectively. The exact eigenvalues are plotted by black “+”. It is easy to see $\sigma(A) \subseteq \Omega(A) \subset K'(A), \Omega(A)$ can capture all eigenvalues of $A$ more precisely than $K'(A)$ and $\Gamma(A)$.

3. Determining the positive definiteness for an even-order real symmetric tensor

As shown in Qi (2005), Li et al. (2014), Li and Li 2016, Li et al. (2016), Li et al. (2015) and Huang et al. (2016), an eigenvalue localization set can provide a sufficient condition for the positive definiteness and positive semi-definiteness of tensors. As applications of the results in Section 2, we in this section provide some sufficient conditions for the positive definiteness and positive semi-definiteness of tensors, respectively.

**Theorem 3.1** Let $A = (a_{ijkm}) \in \mathbb{R}^{[m,n]}$ be an even-order symmetric tensor with $a_{ijkm} > 0$ for all $k \in N$. If for any $i, j \in N, i \neq j$,

$$(a_{ijkm} - \bar{r}(A))a_{ijkm} > \bar{r}(A)\bar{r}(A).$$

then $A$ is positive definite.

**Proof** Let $\lambda$ be an $H$-eigenvalue of $A$. Suppose that $\lambda \leq 0$. By Theorem 2.1, we have $\lambda \in \Omega(A)$, that is, there are some $i, j \in N, i \neq j$ such that

$$(|\lambda - a_{ijkm} - \bar{r}(A)|) |a_{ijkm}| \leq \bar{r}(A)\bar{r}(A).$$

From $a_{ijkm} > 0, k \in N$, we have

$$(|\lambda - a_{ijkm} - \bar{r}(A)|) |a_{ijkm}| \geq (a_{ijkm} - \bar{r}(A))a_{ijkm} > \bar{r}(A)\bar{r}(A).$$

This is a contradiction. Hence, $\lambda > 0$, and $A$ is positive definite. The conclusion follows. $\square$

Similar to the proof of Theorem 3.1, we can easily obtain the following conclusion:

Let $A = (a_{ijkm}) \in \mathbb{R}^{[m,n]}$ be an even-order symmetric tensor with $a_{ijkm} \geq 0$ for all $k \in N$. If for any $i, j \in N, i \neq j$, then $A$ is positive semi-definite. The conclusion follows.
\[(a_{i...j} - \tilde{r}_i(A))q_{i...j} \geq \tilde{r}_i(A)r_j(A),\]

then \(A\) is positive semi-definite.

4. **New bounds for the minimum eigenvalue of \(M\)-tensors**

In this section, new lower and upper bounds for the minimum H-eigenvalue of \(M\)-tensors are given, which are the correction and generalization of Theorem 4.5 in Wang and Wei (2015).

**THEOREM 4.1** Let \(A \in \mathbb{R}^{(m,n)}\) be a weakly irreducible \(M\)-tensor. Then

\[
\min_{i \neq j} L_y(A) \leq r(A) \leq \max_{i \neq j} L_y(A),
\]

where

\[
L_y(A) = \frac{1}{2} \left\{ a_{i...j} + a_{j...i} - \tilde{r}_i(A) - \left[ (a_{i...j} - a_{j...i} - \tilde{r}_i(A))^2 + 4\tilde{r}_i(A)r_j(A) \right]^{1/2} \right\}.
\]

**Proof** Because \(r(A)\) is an eigenvalue of \(A\), from Theorem 2.1, there are \(i, j \in \mathbb{N}, j \neq i\), such that

\[
((r(A) - a_{i...j})(r(A) - a_{j...i}) \leq \tilde{r}_i(A)r_j(A).
\]

From Theorem 2.1, we can get

\[
(a_{i...j} - r(A) - \tilde{r}_i(A))(a_{j...i} - r(A)) \leq \tilde{r}_i(A)r_j(A),
\]
equivalently,

\[
r(A)^2 - (a_{i...j} + a_{j...i} - \tilde{r}_i(A))r(A) + a_{j...i}(a_{i...j} - \tilde{r}_i(A)) - \tilde{r}_i(A)r_j(A) \leq 0.
\]

Solving for \(r(A)\) gives

\[
r(A) \geq \frac{1}{2} \left\{ a_{i...j} + a_{j...i} - \tilde{r}_i(A) - \left[ (a_{i...j} + a_{j...i} - \tilde{r}_i(A))^2 - 4(a_{j...i}(a_{i...j} - \tilde{r}_i(A)) - \tilde{r}_i(A)r_j(A)) \right]^{1/2} \right\}
\]

\[
= \frac{1}{2} \left\{ a_{i...j} + a_{j...i} - \tilde{r}_i(A) - \left[ (a_{i...j} - a_{j...i} - \tilde{r}_i(A))^2 + 4\tilde{r}_i(A)r_j(A) \right]^{1/2} \right\}
\]

\[
\geq \min_{i \neq j} \frac{1}{2} \left\{ a_{i...j} + a_{j...i} - \tilde{r}_i(A) - \left[ (a_{i...j} - a_{j...i} - \tilde{r}_i(A))^2 + 4\tilde{r}_i(A)r_j(A) \right]^{1/2} \right\}.
\]

Next, we prove that the second inequality in (9) holds. Suppose that \(x = (x_1, ..., x_n)^T > 0\) is an eigenvalue of \(A\) corresponding to \(r(A)\), i.e.

\[
Ax^{m-1} = r(A)x^{m-1},
\]

and

\[
x_i \leq x_u \leq \min\{x_k : k \in \mathbb{N}, k \neq i, u\}.
\]

From (11), we have

\[
(a_{u...u} - r(A)x_u^{m-1}) = -\sum_{\delta_{i...i}u=0} a_{u...u} x_1 \cdots x_u \geq r_u(A)x_u^{m-1},
\]

and
\[
(a_{i,j} - r(A)x_i^{-1})x_j^{-1} = - \sum_{i,j \in \mathbb{N}} a_{i,j} x_i \cdots x_j - \sum_{i,j \in \mathbb{N}} a_{i,j} x_j
\geq \tilde{f}(A)x_i^{-1} + \tilde{f}(A)x_j^{-1},
\]
i.e.,
\[
(a_{i,j} - r(A) - \tilde{f}(A))x_i^{-1} \geq \tilde{f}(A)x_j^{-1}.
\] (13)

Multiplying (12) with (13) and noting that \(|x_i^{-1}|x_j^{-1} > 0\), we have
\[
(a_{u,v} - r(A))(a_{i,j} - r(A) - \tilde{f}(A)) \geq \tilde{f}(A)r_u(A).
\] (14)

Then solving for \(r(A)\) gives
\[
r(A) \leq \frac{1}{2} \left\{ a_{i,j} + a_{u,v} - \tilde{f}(A) - \left[ (a_{i,j} - a_{u,v} - \tilde{f}(A))^2 + 4\tilde{f}(A)r_u(A) \right]^{1/2} \right\}
\leq \max_{i,j} \frac{1}{2} \left\{ a_{i,j} + a_{j,j} - \tilde{f}(A) - \left[ (a_{i,j} - a_{j,j} - \tilde{f}(A))^2 + 4\tilde{f}(A)r_u(A) \right]^{1/2} \right\}.
\]
This proof is completed. \(\Box\)

**Remark 4.1**  Note that \(L_i(A) \neq W_i(A)\). Hence, the bounds (9) in Theorem 4.1 are slightly different from the bounds in Theorem 4.5 of Wang and Wei (2015). In fact, the bounds (9) are the correction of the bounds in Theorem 4.5 of Wang and Wei (2015). Because the left (right) inequality of (4.2) in Theorem 4.5 of Wang and Wei (2015) obtained by solving for \(r(A)\) from inequality (4.2) ((14), respectively); for details, see the proof of Theorem 4.5 in Wang and Wei (2015). However, solving for \(r(A)\) by inequalities (10) and (14) gives the bounds (9).

In the following, a counterexample is given to show that the result in Theorem 4.5 in Wang and Wei (2015) is false. Consider the tensor \(A = (a_{ijkl})\) of order 4 dimension 2 with entries defined as follows:
\[
\begin{align*}
A(:, :, 1, 1) &= (30 - 2 - 2), \quad A(:, :, 2, 1) = (-3 - 1 - 3), \\
A(:, :, 1, 2) &= (-2 - 3 - 2), \quad A(:, :, 2, 2) = (-1 - 427).
\end{align*}
\]

By Theorem 4.5 in Wang and Wei (2015), we have \(11 \leq r(A) \leq 11\). By Theorem 2.1, we have \(8.9585 \leq r(A) \leq 12.6893\). In fact, \(r(A) = 10.8851\).

Next, we can extend the results of Theorem 4.1 to a more general case.

**THEOREM 4.2** Let \(A \in \mathbb{R}^{[m,n]}\) be an \(M\)-tensor. Then
\[
\min_{i,j} L_i(A) \leq r(A) \leq \max_{i,j} L_j(A).
\] (15)

**Proof**  Because \(A\) is an \(M\)-tensor, by Theorems 2 and 3 in Ding et al. (2013), there is \(x = (x_1, \ldots, x_n)^T \geq 0\) such that \(Ax^{m-1} > 0\). Let
\[
\delta = \min_{i \in \mathbb{N}} \{Ax^{m-1} \} = \min_{i \in \mathbb{N}} \sum_{i,j \in \mathbb{N}} a_{i,j} x_i \cdots x_j = \max_{i \in \mathbb{N}} x_i.
\]

Then \(x_{\max} > 0\). Let \(A_k = A - \frac{1}{\delta} \varepsilon\), where \(k = 1, 2, \ldots\), and \(\varepsilon\) denote the tensor with every entry being 1.

Then \(A_k\) is an irreducible \(\varepsilon\)-tensor, and \(|A_k|\) is a monotonically increasing sequence.

Taking \(k > \left\lfloor \frac{m^{m-1} - 1}{\delta} \right\rfloor + 1\), then for any \(i \in \mathbb{N},\)
\[
\sum_{i_1, \ldots, i_m \in \mathbb{N}} \left(a_{i_1 \ldots i_m} - \frac{1}{k}\right) x_{i_1} \cdots x_{i_m} \geq \min_{\mathbb{N}} \sum_{i_1, \ldots, i_m \in \mathbb{N}} a_{i_1 \ldots i_m} x_{i_1} \cdots x_{i_m} - \frac{n^{m-1}}{k} y^{m-1}
\]

which implies that \( A_y x^{m-1} > 0 \). Then, by Theorems 2 and 3 in Ding et al. (2013), we can conclude that \( A_y \) is an irreducible \( M \)-tensor. By Theorem 4.1 in He and Huang (2014), \( \{\tau(A_y)\} \) is a monotonically increasing sequence with upper bound \( \tau(A) \), so \( \tau(A_y) \) has a limit, and let

\[
\lim_{k \to +\infty} \tau(A_y) = \lambda \leq \tau(A).
\]

By Theorem 2.6 in Wang and Wei (2015), we see that \( \tau(A_y) \) is the eigenvalue of \( A_y \) with a positive eigenvector \( y^{m} \), i.e. \( A_y y^{m} = \tau(A_y) y^{m} \). As homogeneous multivariable polynomials, we can restrict \( y^{m} \) on the unit ball \( \|y^{m}\| = 1 \). Then \( (y^{m}) \) is a bounded sequence, so it has a convergent subsequence. Without loss of generality, suppose that it is the sequence itself. Let \( y^{m} \to y \) as \( k \to +\infty \), we get \( y \geq 0 \) and \( \|y\| = 1 \). Letting \( k \to +\infty \), we have \( A_y y = \lambda y^{m-1} \) from \( A_y y^{m} = \tau(A_y) y^{m} \). So \( \lambda \) is an eigenvalue of \( A_y \), furthermore, \( \lambda \geq \tau(A) \). Together with (16) results in \( \lambda = \tau(A) \), which means that

\[
\lim_{k \to +\infty} \tau(A_y) = \tau(A).
\]

Using Theorem 4.1 for \( A_y \), we have

\[
\min_{i,j} L_i(A_y) \leq \tau(A_y) \leq \max_{i,j} L_i(A_y),
\]

where

\[
L_i(A_y) = \frac{1}{2} \left\{ a_{i \ldots i} + a_{j \ldots j} - \frac{2}{k} - \tilde{r}_i(A_y) - \left[ a_{i \ldots j} - a_{j \ldots i} - \tilde{r}_i(A_y) \right]^2 + 4 \tilde{r}_i(A_y) \tilde{r}_j(A_y) \right\}^{\frac{1}{2}},
\]

\[
\tilde{r}_i(A_y) = r_i(A_y) + \frac{n^{m-1}}{k}, \quad \tilde{r}_j(A_y) = r_j(A_y) + \frac{n^{m-1} - (n-1)}{k}.
\]

Letting \( k \to +\infty \) in (17), we have that (15) holds. \( \square \)

Similar to the proof of Theorem 4.2, we can extend the results of Theorem 1.3 to a more general case.

**Theorem 4.3** Let \( A \) be an \( M \)-tensor. Then

\[
\tau(A) \leq \min_{i \in \mathbb{N}} a_{i \ldots i}, \quad \text{and} \quad \min_{i \in \mathbb{N}} R_i(A) \leq \tau(A) \leq \max_{i \in \mathbb{N}} R_i(A).
\]

Next, we compare the bounds in Theorem 4.2 with those in Theorem 4.3.

**Theorem 4.4** Let \( A = (a_{i_1 \ldots i_m}) \in \mathbb{R}^{m,n} \) be an \( M \)-tensor. Then

\[
\min_{i \in \mathbb{N}} R_i(A) \leq \min_{i,j} L_i(A) \leq \max_{i,j} L_i(A) \leq \max_{i \in \mathbb{N}} R_i(A).
\]

**Proof** Similar to the proof of Theorem 5 in Zhao and Sang (2016), we can obtain \( \min_{i \in \mathbb{N}} R_i(A) \leq \min_{i,j} L_i(A) \) easily. Next, we only prove that the last inequality in (18) holds.

1. For any \( i, j \in \mathbb{N}, j \neq i \), if \( R_i(A) \leq R_j(A), \) i.e. \( a_{i \ldots i} - \tilde{r}_i(A) - \tilde{r}_j(A) \leq a_{j \ldots j} - r_j(A) \), then

\[
\tilde{r}_i(A) \geq a_{i \ldots i} - a_{j \ldots j} - \tilde{r}_j(A) + r_j(A).
\]

Hence,
\[
(a_{i,j} - a_{j,i} - \bar{r}_i(A))^2 + 4\bar{r}_j(A)r_j(A)
\geq [a_{i,j} - a_{j,i} - \bar{r}_i(A)]^2 + 4(a_{i,j} - a_{j,i} - \bar{r}_i(A) + \bar{r}_j(A))(a_{i,j} - a_{j,i} - \bar{r}_i(A) + \bar{r}_j(A))
\]
\[
= [a_{i,j} - a_{j,i} - \bar{r}_i(A)]^2 + 4(a_{i,j} - a_{j,i} - \bar{r}_i(A))r_j(A) + 4r_j(A)^2
\]
\[
= [a_{i,j} - a_{j,i} - \bar{r}_i(A) + 2r_j(A)]^2.
\]

When
\[a_{i,j} - a_{j,i} - \bar{r}_i(A) + 2r_j(A) > 0,
\]
we have
\[
a_{i,j} + a_{j,i} - \bar{r}_i(A) - [(a_{i,j} - a_{j,i} - \bar{r}_i(A))^2 + 4\bar{r}_j(A)r_j(A)]^{\frac{1}{2}}
\leq a_{i,j} + a_{j,i} - \bar{r}_i(A) - [(a_{i,j} - a_{j,i} - \bar{r}_i(A))^2 + 4\bar{r}_j(A)r_j(A)]^{\frac{1}{2}}
\]
\[
= a_{i,j} + a_{j,i} - \bar{r}_i(A) - [a_{i,j} - a_{j,i} - \bar{r}_i(A)]
\]
\[
= a_{i,j} + a_{j,i} - \bar{r}_i(A) + [a_{i,j} - a_{j,i} - \bar{r}_i(A)]
\]
\[
= 2a_{i,j} - 2r_j(A)
\]
\[
\leq 2a_{i,j} - 2r_j(A)
\]
\[
= 2R_j(A).
\]

And when
\[a_{i,j} - a_{j,i} - \bar{r}_i(A) + 2r_j(A) \leq 0,
\]
i.e. \[a_{i,j} - \bar{r}_i(A) \leq a_{j,i} - 2r_j(A),\]
we have
\[
a_{i,j} + a_{j,i} - \bar{r}_i(A) - [(a_{i,j} - a_{j,i} - \bar{r}_i(A))^2 + 4\bar{r}_j(A)r_j(A)]^{\frac{1}{2}}
\leq a_{i,j} + a_{j,i} - \bar{r}_i(A) - [(a_{i,j} - a_{j,i} - \bar{r}_i(A))^2 + 4\bar{r}_j(A)r_j(A)]^{\frac{1}{2}}
\]
\[
= a_{i,j} + a_{j,i} - \bar{r}_i(A) - [a_{i,j} - a_{j,i} - \bar{r}_i(A)]
\]
\[
= a_{i,j} + a_{j,i} - \bar{r}_i(A) + [a_{i,j} - a_{j,i} - \bar{r}_i(A)]
\]
\[
= 2a_{i,j} - 2r_j(A)
\]
\[
\leq 2a_{i,j} - 2r_j(A)
\]
\[
\leq 2a_{i,j} - 2r_j(A)
\]
\[
= 2R_j(A).
\]

Therefore,
\[
L_j(A) = \frac{1}{2} \left\{ a_{i,j} + a_{j,i} - \bar{r}_j(A) - \left[ (a_{i,j} - a_{j,i} - \bar{r}_j(A))^2 + 4\bar{r}_j(A)r_j(A) \right]^{\frac{1}{2}} \right\} \leq R_j(A),
\]
which implies
\[
\max_{i,j \neq k} L_{i,j}(A) \leq \max_{i,j \neq k} K_{i,j}(A).
\]

(II) For any \( i, j \in N, j \neq i, \) if \( R_j(A) \leq R_i(A), \) i.e.
\[a_{j,i} - \bar{r}_j(A) \leq a_{i,j} - \bar{r}_i(A) - \bar{r}_j(A),\]

then
\[r_j(A) \geq a_{j,i} - a_{i,j} + \bar{r}_i(A) + \bar{r}_j(A).\]
Similarly, we can obtain
\[
L_j(A) = \frac{1}{2} \left( a_{i_1j} + a_{j_1j} - r_j(A) - \left[ a_{i_1j} + a_{j_1j} - r_j(A) \right]^2 + 4r_j(A) r_j(A) \right)^{\frac{1}{2}} \leq R_j(A),
\]
which implies
\[
\max_{\lambda \in \sigma} L_j(A) \leq \max_{\lambda \in \sigma} R_j(A).
\]
The conclusion follows from I and II. 

5. Conclusion
In this paper, a new eigenvalue localization set for tensors is given. It is proved that the new set is tighter than those in Qi (2005) and Li et al. (2014). As applications of the obtained results, a new sufficient condition of the positive (semi-)definiteness for an even-order real symmetric tensor, and new lower and upper bounds of the minimum eigenvalue for M-tensors, which are the correction of the bounds in Wang and Wei (2015), are obtained. Finally, we extend Lemma 4.4 in Wang and Wei (2015) and Theorem 2.1 in He and Huang (2014) to a more general case.

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