Fixed point theorems for generalized Roger Hardy type $F$-contraction mappings in a metric-like space with an application to second-order differential equations

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Abstract: In this paper, we present fixed point theorems for a generalized Roger Hardy type $F$-contraction in metric-like spaces and also give some examples to illustrate the main results in this paper. Moreover the applications of second-order differential equations and fractional differential equations are shown. The existing results improve and extend the corresponding results in the literature.

Keywords: metric-like space; fixed point; Roger Hardy type $F$-contraction; second-order differential equations

1. Introduction and preliminaries
In Methews (1994) extended the concept of a metric space to a partial metric space and obtained many results in partial metric spaces. Indeed, the motivation for introducing the concept of a partial metric was to obtain appropriate mathematical models in the theory of computation and, in particular, to give the improvement of Banach’s contraction principle. Afterwards, many authors have studied the existence and uniqueness of a fixed point for nonlinear mappings satisfying various contractive conditions in the setting of partial metric spaces.

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PUBLIC INTEREST STATEMENT
Fixed point theory plays an important role in determining a solution to the nonlinear equation of the form $Tx = x$, where $T$ is a self-mapping defined on a suitable subset $K$ of metric spaces, normed linear spaces, topological vector spaces and some generalized spaces of metric spaces. Then a solution of this equation is called a fixed point of the mapping $T$ on $K$. In this work, we focus the fixed point theorem of generalized mappings in metric-like spaces. Further, an example and an application are shown.
Recently, Amini-Harandi (2012) presented generalization concept partial metric space which is metric-like space. The author also proved some fixed point theorems and gave \( \sigma \)-completeness in such space. Subsequently Shukla and Radenović (2013) have proved common fixed point theorems and introduced \( 0-\sigma \)-completeness in metric-like space which generalized Amini-Harandi’s results. Later, many results of Amini-Harandi have been generalized (see, for example Aydia & Felhi, 2016; Chen, Huang, & Li, 2016; Felhi, Aydii, & Zhang, 2016; Jaradat et al., 2017; Karapinar, Felhi, Aydii, & Sahmim, 2016; Kumari, Kumar, & Rambhadrasarma, 2012; Kumari & Panthi, 2015; Kumari, Ramana, Zoto, & Panthi, 2015; Sarma & Kumari, 2012; Shobkolaei, Rezaei Roshan, Sedghi, & Hussain, 2013).

Now, we recall some elementary results and basic definitions for our main results. In this paper, we denote \( \mathbb{N} \), \( \mathbb{R} \), and \( \mathbb{R}^+ \) by the set of positive integers, the set of real numbers and the set of non negative real numbers, respectively.

**Definition 1.1** (Matthews, 1994) Let \( X \) be nonempty set and function \( p:X \times X \rightarrow \mathbb{R}^+ \) be a function satisfying the following conditions: for all \( x, y, z \in X \),

\[
\begin{align*}
(1) & \quad p(x, x) = p(y, y) = p(x, y) \text{ if and only if } x = y, \\
(2) & \quad p(x, x) \leq p(x, y), \\
(3) & \quad p(x, y) = p(y, x), \\
(4) & \quad p(x, y) = p(x, z) + p(z, y) - p(z, z).
\end{align*}
\]

Then \( p \) is said to be a partial metric on \( X \) and a pair \((X, p)\) is called a partial metric space.

A basic example of a partial metric space is the pair \((X, p)\), where \( X = \{0, \infty\} \) and \( p(x, y) = \max\{x, y\} \) for all \( x, y \in X \). For more examples and of partial metric space see Matthews (1994), Karapinar and Erhan (2011), Shukla, Radenović, and Kadelburg (2014), Shukla and Radenović (2013) and references therein.

**Definition 1.2** (Amini Harandi, 2012) A metric-like on nonempty set \( X \) is a function \( \sigma:X \times X \rightarrow \mathbb{R}^+ \). If for all \( x, y, z \in X \), the following conditions hold:

\[
\begin{align*}
(1) & \quad \sigma(x, y) = 0 \Rightarrow x = y; \\
(2) & \quad \sigma(x, y) = \sigma(y, x); \\
(3) & \quad \sigma(x, y) = \sigma(x, z) + \sigma(z, y).
\end{align*}
\]

Then a pair \((X, \sigma)\) is said to be a metric-like space.

Each metric-like \( \sigma \) on \( X \) generates a topology \( \tau_\sigma \) on \( X \) whose base is the family of open \( \sigma \)-balls

\[ B_\sigma(x, \varepsilon) = \{y \in X:|\sigma(x, y) - \sigma(x, x)| < \varepsilon\}, \text{ for all } x \in X \text{ and } \varepsilon > 0. \]

It is easy to see that a metric space is a partial metric space and each partial metric space is a metric-like space, but the converse is not true as in the following examples:

**Example 1.3** (Amini Harandi, 2012) Let \( X = \{0, 1\} \) and \( \sigma:X \times X \rightarrow \mathbb{R}^+ \) be defined by

\[
\sigma(x, y) = \begin{cases} 
2, & \text{if } x = y = 0, \\
1, & \text{otherwise}.
\end{cases}
\]

Then \((X, \sigma)\) is a metric-like space, but it is not a partial metric space, since \( \sigma(0, 0) \not\leq \sigma(0, 1) \).
Example 1.4  (Shukla & Radenović, 2013)  Let \( X = \mathbb{R}, \ k \geq 0 \) and \( \sigma \times X \rightarrow \mathbb{R}^+ \) be defined by

\[
\sigma(x, y) = \begin{cases} 
2k, & \text{if } x = y = 0, \\
k, & \text{otherwise}.
\end{cases}
\]

Then \((X, \sigma)\) is a metric-like space, but for \( k > 0 \), it is not a partial metric space, since \( \sigma(0, 0) \not\leq \sigma(0, 1) \).

**Lemma 1.5**  (Aydi & Karapinar, 2015)  Let \((X, p)\) be a partial metric space. Then

(a) \( \{x_n\} \) is a Cauchy sequence in \((X, p)\) if and only if it is a Cauchy sequence in the metric space \((X, d_p)\).

(b) \( X \) is complete if and only if the metric space \((X, d_p)\) is complete.

**Definition 1.6**  (Amini Harandi, 2012; Shukla & Radenović, 2013)  Let \((X, \sigma)\) be a metric-like space. Then:

(1) A sequence \( \{x_n\} \) in \( X \) converges to a point \( x \in X \) if \( \lim_{n \to \infty} \sigma(x_n, x) = 0 \). The sequence \( \{x_n\} \) is said to be \( \sigma \)-Cauchy if \( \lim_{n,m \to \infty} \sigma(x_n, x_m) \) exists and is finite. The space \((X, \sigma)\) is called complete if for every \( \sigma \)-Cauchy sequence in \( \{x_n\} \), there exists \( x \in X \) such that

\[
\lim_{n \to \infty} \sigma(x_n, x) = \lim_{n,m \to \infty} \sigma(x_n, x_m).
\]

(2) A sequence \( \{x_n\} \) in \((X, \sigma)\) is said to be a \( 0-\sigma \)-Cauchy sequence if \( \lim_{n,m \to \infty} \sigma(x_n, x_m) = 0 \). The space \((X, \sigma)\) is said to be \( 0-\sigma \)-complete if every \( 0-\sigma \)-Cauchy sequence in \( X \) converges (in \( \tau_\sigma \)) to a point \( x \in X \) such that \( \sigma(x, x) = 0 \).

(3) A mapping \( T : X \to X \) is continuous, if the following limits exist (finite) and

\[
\lim_{n \to \infty} \sigma(x_n, x) = \sigma(Tx, x).
\]

**Remark 1.7**  (Amini Harandi, 2012)  Let \( X = \{0, 1\}, \sigma(x, y) = 1 \) for all \( x, y \in X \) and \( x_n = 1 \) for all \( n \in \mathbb{N} \). Then it is obvious that \( x_n \to 0 \) and \( x_n \to 1 \), and so in metric-like space the limit of a convergent sequence is not necessarily unique.

**Lemma 1.8**  (Karapinar & Salimi, 2013)  Let \((X, \sigma)\) be a metric-like space.

(a) If \( x, y \in X \) then \( \sigma(x, y) = 0 \) implies that \( \sigma(x, x) = \sigma(y, y) = 0 \).

(b) If a sequence \( \{x_n\} \) in \( X \) converges to some \( x \in X \) with \( \sigma(x, x) = 0 \) then \( \lim_{n \to \infty} \sigma(x_n, y) = \sigma(x, y) \) for all \( x, y \in X \).

**Remark 1.9**  (Shukla & Radenović, 2013)  If a metric-like space is \( \sigma \)-complete, then it is \( 0-\sigma \)-complete. The converse assertion does not hold as the example given below shows.

**Example 1.10**  (Shukla & Radenović, 2013)  Let \( X = \{0, 1\} \cap \mathbb{Q} \) and \( \sigma : X \times X \to \mathbb{R}^+ \) be defined by

\[
\sigma(x, y) = \begin{cases} 
2x, & \text{if } x = y = 0, \\
\max\{x, y\}, & \text{otherwise}.
\end{cases}
\]

for all \( x, y \in X \). Then \((X, \sigma)\) is a metric-like space. Note that \((X, \sigma)\) is not a partial metric space, as \( \sigma(1, 1) = 2 > 1 = \sigma(1, 0) \). Now, it is easy to see that \((X, \sigma)\) is a \( 0-\sigma \)-complete metric-like space, while it is not a \( \sigma \)-complete metric-like space.

One of interesting generalized Banach’s contractions is an \( F \)-contraction introduced by Wardowski (2012) as follows:
Definition 1.11 Let \( F: \mathbb{R}^+ \to \mathbb{R} \) be a mapping which is satisfying the following conditions:

1. (F1) \( F \) is strictly increasing, i.e. for all \( a, b \in \mathbb{R}^+ \) such that \( F(a) < F(b) \) whenever \( a < b \).
2. (F2) For each sequence \( \{a_n\}_{n=1}^\infty \) of positive real numbers \( \lim_{n \to \infty} a_n = 0 \) iff \( \lim_{n \to \infty} F(a_n) = -\infty \).
3. (F3) There exists \( k \in (0, 1) \) such that \( \lim_{\alpha \to 0^+} a^k F(\alpha) = 0 \).

We denote with \( \mathcal{F} \) the family of all functions \( F \) that satisfy the conditions (F1)-(F3). For examples of the function \( F \) the reader is referred to Wardowski (2012) and Secelean (2013).

Definition 1.12 Let \( (X, d) \) be a metric space. A self-mapping \( T \) on \( X \) is called an \( F \)-contraction mapping if there exist \( F \in \mathcal{F} \) and \( \tau \in \mathbb{R}^+ \) such that

\[
\forall x, y \in X, \quad [d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y))].
\]

After this \( F \)-contraction, some authors have studied some fixed point theorems for this contraction (see for examples Cosentino & Vetro, 2014; Kumari & Panthi, 2016a, 2016b; Kumari, Zoto, & Panthi, 2015; Piri & Kumam, 2014). In Piri and Kumam (2014) proved some fixed point theorems for an \( F \)-Suzuki contraction \( T \), that is, a mapping \( T:X \to X \) is called an \( F \)-Suzuki contraction if there exists \( \tau \in \mathbb{R}^+ \) such that

\[
\frac{1}{2} d(x, Tx) < d(x, y) \Longrightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)) \tag{1.2}
\]

for all \( x, y \in X \) with \( Tx \neq Ty \), where \( F \in \mathcal{F} \).

In the same year, Cosentino and Vetro (2014) introduce an \( F \)-contraction of Hardy-Rogers-type. They also give sufficient conditions for unique fixed point in order metric spaces.

Definition 1.13 (Cosentino & Vetro, 2014) Given \( (X, d) \) be a metric space and \( T:X \to X \) be an \( F \)-contraction of Hardy-Rogers-type if there exists \( \tau > 0 \) and \( F \in \mathcal{F} \) such that

\[
d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(\alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty)
\]

\[
+ \delta d(x, Ty) + Ld(y, Tx)) \tag{1.3}
\]

for all \( x, y \in X \), where \( \alpha + \beta + \gamma + 2\delta = 1, \gamma \neq 1 \) and \( L \geq 0 \).

Motivated by the significance of the problems mentioned above, in this paper, we introduce a generalized Roger Hardy type \( F \)-contraction and generalized Roger Hardy type Suzuki \( F \)-contraction in metric-like spaces. We also established and proved fixed point theorems in such spaces. Some examples for support main theorems are illustrated. Furthermore, applications of second-order differential equations and fractional differential equations are shown.

2. Main results

First, we introduce the generalized Roger Hardy type \( F \)-contraction mapping in a metric-like space.

Definition 2.1 Let \( (X, \sigma) \) be a metric-like space. A mapping \( T:X \to X \) is called a generalized Roger Hardy type \( F \)-contraction mapping, if there exist \( F \in \mathcal{F} \) and \( \tau \in \mathbb{R}^+ \) such that

\[
\sigma(Tx, Ty) > 0 \Rightarrow \tau + F(\sigma(Tx, Ty)) \leq F(\alpha \sigma(x, y) + \beta \sigma(x, Tx) + \gamma \sigma(y, Ty)
\]

\[
+ \eta \sigma(x, Ty) + \delta \sigma(y, Tx)) \tag{2.1}
\]

for all \( x, y \in X \) and \( \alpha, \beta, \gamma, \eta, \delta \geq 0 \) with \( \alpha + \beta + \gamma + 2\eta + 2\delta < 1 \).

Theorem 2.2 Let \( (X, \sigma) \) be \( 0,\infty \)-complete metric-like spaces and \( T:X \to X \) be a generalized Roger Hardy type \( F \)-contraction. Then \( T \) has a unique fixed point in \( X \), either \( T \) or \( F \) is continuous.
Proof. Let \( x_n \) be an element in \( X \). If \( Tx_0 = x_0 \) then the proof is finished. We construct \( x_n = T^n x_0 \) and \( x_{n+1} = T x_n \). If there exists \( n_0 \in \{1, 2, \ldots \} \) such that right hand of (2.1) is 0 for \( x_n = x_{n_0} \) and \( y = x_{n_0} \), then is clear that \( x_{n_0-1} = x_{n_0} = T x_{n_0-1} \), so the proof is completed. Now, we let \( x_{n+1} \neq x_n \) for all \( n \geq 1 \) and let \( \sigma_n = \sigma(x_{n+1}, x_n) \). Then \( \sigma_n > 0 \) for all \( n \geq 0 \). Since \( T \) is generalized Roger Hardy type \( F \)-contraction, we obtain

\[
\tau + F(\sigma_n) = \tau + F(\sigma(x_{n+1}, x_n)) = \tau + F(T(\sigma(x_{n+1}, x_n)))
\]

\[
\leq F(a\sigma(x_n, x_{n-1}) + \beta\sigma(x_n, x_{n+1}) + \gamma\sigma(x_{n-1}, x_{n+1}) + \eta\sigma(x_{n-1}, x_{n+1}))
\]

\[
= F(a\sigma(x_n, x_{n-1}) + \beta\sigma(x_n, x_{n+1}) + \gamma\sigma(x_{n-1}, x_{n}) + \eta\sigma(x_{n-1}, x_{n}))
\]

\[
\leq F(a\sigma(x_n, x_{n-1}) + \beta\sigma(x_n, x_{n+1}) + \gamma\sigma(x_{n-1}, x_{n}) + \delta(\sigma(x_{n-1}, x_{n})) + \sigma(x_{n+1}, x_n))).
\]

It follows that (2.2) and \( F \) is strictly increasing, then

\[
\sigma_n < a\sigma_{n-1} + \beta\sigma_n + \gamma\sigma_{n-1} + 2\eta\sigma_{n-1} + \delta(\sigma_{n-1} + \sigma_n)
\]

implies that

\[
(1 - \beta - \delta)\sigma_n < (a + \gamma + 2\eta + \delta)\sigma_{n-1}, \quad \text{for all } n \in \mathbb{N}.
\]

Since \( a + \beta + \gamma + 2\eta + 2\delta < 1 \), we deduce that \( 1 - \beta - \delta > 0 \) and so

\[
\sigma_n < \frac{a + 2\eta + \gamma + \delta}{1 - \beta - \delta} \sigma_{n-1} < \sigma_{n-1}, \quad \text{for all } n \in \mathbb{N}.
\]

Consequently

\[
\tau + F(\sigma_n) \leq F(\sigma_{n-1}).
\]

Similarly method shows that

\[
F(\sigma_n) \leq F(\sigma_{n-1}) - \tau \leq \ldots \leq F(\sigma_0) - n\tau, \quad \text{for all } n \in \mathbb{N}
\]

and so \( \lim_{n \to \infty} F(\sigma_n) = -\infty \). Thus from (F2), we obtain

\[
\lim_{n \to \infty} \sigma_n = 0.
\]

Also from (F3), there exists \( k \in (0, 1) \) such that

\[
\lim_{n \to \infty} \sigma_n F(\sigma_n) = 0.
\]

Now, it follows that

\[
\sigma_n F(\sigma_n) - \sigma_n F(\sigma_0) \leq \sigma_n F(\sigma_0) - n\tau - \sigma_n F(\sigma_0)
\]

\[
\leq \sigma_n F(\sigma_0) - \sigma_n n\tau - \sigma_n F(\sigma_0)
\]

\[
= -\sigma_n n\tau
\]

\[
\leq 0, \quad \text{for all } n \in \mathbb{N}.
\]
Letting $n \to \infty$ in (2.3), we get

$$\lim_{n \to \infty} n \sigma_n = 0.$$  \hfill (2.4)

From above (2.4), there exists $n_1 \in \mathbb{N}$ such that $n \sigma_n \leq 1$ for all $n \geq n_1$.

Therefore

$$\sigma_n \leq \frac{1}{n^2}, \quad \text{for all } n_1 \geq n. \hfill (2.5)$$

Next, we will show that $(x_n)$ is an $\omega$-Cauchy sequence, let $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. From (2.5) and property (3), we obtain

$$\sigma(x_n, x_m) \leq \sigma(x_n, x_{n+1}) + \sigma(x_{n+1}, x_{n+2}) + \cdots + \sigma(x_{m-1}, x_m)$$

$$= \sigma_{n+1} + \cdots + \sigma_{m-1}$$

$$= \sum_{i=n}^{m-1} \sigma_i$$

$$\leq \sum_{i=n}^{\infty} \sigma_i$$

$$\leq \sum_{i=n}^{\infty} \frac{1}{i^2}.$$  \hfill (2.6)

By the convergence of $P$ series $\sum_{i=n}^{\infty} \frac{1}{i^2}$, and so as $n \to \infty$, then $\sigma(x_n, x_m) \to 0$. Hence $(x_n)$ is an $\omega$-Cauchy sequence in $(X, \sigma)$. From $X$ is an $\omega$-complete metric-like space, there exists $x^* \in X$ such that

$$\lim_{n \to \infty} x_n = x^*$$

equivalently,

$$\lim_{n \to \infty} \sigma(x_n, x^*) = \lim_{n \to \infty} \sigma(x_n, x^*) = \sigma(x^*, x^*) = 0. \hfill (2.7)$$

Now, if $T$ is continuous, we obtain from (2.7) that

$$\lim_{n \to \infty} \sigma(Tx_n, Tx^*) = \lim_{n \to \infty} \sigma(x_{n+1}, Tx^*) = \sigma(x^*, Tx^*) = 0. \hfill (2.8)$$

Therefore, $x^*$ is a fixed point of $T$.

Now, suppose $F$ is continuous. In this case, we claim that $x^* = Tx^*$. Assume the contrary, that is, $x^* \neq Tx^*$. In this case, there exist an $n_0 \in \mathbb{N}$ and a subsequence $(x_{n_k})$ of $(x_n)$ such that $\sigma(x_{n_k}, Tx^*) > 0$ for all $n_k \geq n_0$. (Otherwise, there exists $n_1 \in \mathbb{N}$ such that $x_n = Tx^*$ for all $n \geq n_1$, which implies that $x_n \to x^*$. That is a contradiction, with $x^* \neq Tx^*$). Since $\sigma(x_{n_k}, Tx^*) > 0$ for all $n_k \geq n_0$, then from (2.1), we get

$$\tau + F(\sigma(x_{n_k}, Tx^*)) = \tau + F(\sigma(x_{n_k}^*, Tx^*))$$

$$\leq F(\alpha(\sigma(x_{n_k}, Tx^*)) + \beta(\sigma(x_{n_k}, Tx^*)) + \gamma(\sigma(x^*, Tx^*))$$

$$+ \delta(\sigma(x^*, x_{n_k})))$$

$$= F(\alpha(\sigma(x_{n_k}^*, Tx^*)) + \beta(\sigma(x_{n_k}, Tx^*)) + \gamma(\sigma(x^*, Tx^*))$$

$$+ \delta(\sigma(x^*, x_{n_k}^*))). \hfill (2.9)$$

Passing $n \to \infty$ in (2.9) and using $F$ is continuous we obtain

$$\tau + F(\sigma(x^*, Tx^*)) \leq F(\gamma + \eta(\sigma(x^*, Tx^*)))$$

$$< F(\sigma(x^*, Tx^*)). \hfill (2.10)$$
From (2.10), the assumption is a contradiction. Therefore we claim is true, that is \( x^* = Tx^* \). For the uniqueness fixed point of \( T \), if \( x^*, y^* \in X \) be two distinct fixed point of \( T \), this is \( Tx^* = x^* \neq y^* = Ty^* \). Therefore \( \sigma(Tx^*, Ty^*) = \sigma(x^*, y^*) > 0 \), which implies that

\[
F(\sigma(x^*, y^*)) = F(\sigma(Tx^*, Ty^*)) \leq \tau + F(\sigma(Tx^*, Ty^*)) \\
\leq F(\alpha\sigma(x^*, y^*) + \beta\sigma(x^*, Tx^*) + \gamma\sigma(y^*, Ty^*)) \\
+ \eta\sigma(x^*, Ty^*) + \delta\sigma(y^*, Ty^*) \\
= F(\alpha\sigma(x^*, y^*) + \beta\sigma(x^*, x^*) + \gamma\sigma(y^*, y^*)) \\
+ \eta\sigma(x^*, y^*) + \delta\sigma(y^*, x^*)) \\
= F((\alpha + \eta + \delta)\sigma(x^*, y^*)) \\
< F(\sigma(x^*, y^*)).
\]

Since \( F(\sigma(x^*, y^*)) = F(\sigma(Tx^*, Ty^*)) < \tau + F(\sigma(Tx^*, Ty^*)) \leq F(\sigma(x^*, y^*)) \), which is a contradiction. Therefore, the fixed point is unique.

Next we give example for support Theorem 2.2.

**Example 2.3** Let \( X = [0, 1] \cap \mathbb{Q} \) and \( \sigma : X \times X \to \mathbb{R}^+ \) be defined by

\[
\sigma(x, y) = \begin{cases} 
2x^2 & \text{if } x = y, \\
\max\{2x^2, 2y^2\} & \text{otherwise}
\end{cases}
\]

for all \( x, y \in X \). Then \( (X, \sigma) \) is a 0-\( \sigma \)-complete metric-like space which is not \( \sigma \)-complete. Let \( TX \to X \) be mapping given by \( Tx = \frac{x}{2} \). Take \( F(t) = \ln(t) + t \) and \( \tau = \ln 2 \), where \( t > 0 \). We will show that \( T \) is a generalized Roger Hardy type \( F \)-contraction with \( \alpha = 0.25, \beta = 0.25, \gamma = \eta = \delta = 0 \).

Thus \( \alpha + \beta + \gamma + 2\eta + 2\delta = 0.5 < 1 \).

Then for \( x > y \)

\[
\sigma(Tx, Ty) = \sigma\left(\frac{x}{2}, \frac{y}{2}\right) = \frac{2x^2}{4} = \frac{x^2}{2} > 0
\]

and

\[
\alpha\sigma(x, y) + \beta\sigma(x, Tx) + \gamma\sigma(y, Ty) + \eta\sigma(x, Ty) + \delta\sigma(y, Tx) = 0.25\sigma(x, y) + 0.25\sigma\left(x, \frac{x}{2}\right)
\]

\[
= 0.25(2x^2) + 0.25(2x^2)
\]

\[
= x^2.
\]

Hence, we get

\[
\tau + F(\sigma(Tx, Ty)) = \ln 2 + \frac{x^2}{2} + \ln\left(\frac{x^2}{2}\right) \leq x^2 + \ln x^2 \\
= F(\alpha\sigma(x, y) + \beta\sigma(x, Tx) + \gamma\sigma(y, Ty) \\
+ \eta\sigma(x, Ty) + \delta\sigma(y, Tx)).
\]

Similarly, for \( x = y \neq 0 \) (otherwise \( \sigma(Tx, Ty) = 0 \)) one gets that

\[
\tau + F(\sigma(Tx, Ty)) = \ln 2 + \frac{x^2}{2} + \ln\left(\frac{x^2}{2}\right) \leq x^2 + \ln x^2 \\
= F(\alpha\sigma(x, y) + \beta\sigma(x, Tx) + \gamma\sigma(y, Ty) \\
+ \eta\sigma(x, Ty) + \delta\sigma(y, Tx)).
\]
Therefore, all the conditions of Theorem 2.2 are satisfied. Then $T$ has a unique fixed point which is 0.

Following Figures 1–3 to show that R.H.S. expression dominates the L.H.S. expression in $[0, 1] \cap \mathbb{Q}$, which validates our inequalities in Example 2.3.

Next, we introduce the notion of generalized Roger Hardy type $F$-Suzuki contraction in metric-like space and prove a corresponding fixed point theorem.

**Definition 2.4** Let $(X, \sigma)$ be a metric-like space. A mapping $T:X \to X$ is called a generalized Roger Hardy type $F$-Suzuki contraction if there exist $F \in \mathcal{F}$ and $\tau \in \mathbb{R}^+$ such that

![Figure 1. Plot of inequality, 2D view.](image)

![Figure 2. Plot of inequality, 3D view.](image)
\begin{align}
\frac{1}{2} \sigma(x, T x) < \sigma(x, y) \Rightarrow \tau + F(\sigma(T x, T y)) \leq F(\Delta(x, y)) \tag{2.11}
\end{align}

for all \( x, y \in X \) and \( \Delta(x, y) = a \sigma(x, y) + \beta \sigma(x, T x) + \gamma \sigma(y, T y) + \eta \sigma(x, T y) + \delta \sigma(y, T x) \) where \( a, \beta, \gamma, \eta, \delta \geq 0 \) with \( a + \beta + \gamma + 2\eta + 2\delta < 1 \) and satisfying \( \sigma(T x, T y) > 0 \).

**Theorem 2.5** Let \( (X, \sigma) \) be a 0-\( \sigma \) complete metric-like space and \( T : X \to X \) be a generalized Roger Hardy type \( F \)-Suzuki contraction. Then \( T \) has a unique fixed point in \( X \).

**Proof** Let \( x_0 \in X \) and defined a sequence \( \{x_n\}_{n=1}^{\infty} \) by
\[
x_1 = T x_0, \quad x_2 = T^2 x_0, \ldots, \quad x_{n+1} = T^{n+1} x_0, \quad \forall n \in \mathbb{N}.
\]

If there exists \( n \in \mathbb{N} \) such that \( \sigma(x_n, T x_n) = 0 \), so the proof is finished. Now, we assume that \( 0 < \sigma(x_n, T x_n) = \sigma(x_n, x_{n+1}) = \sigma_{n}, \) for all \( n \in \mathbb{N} \). Therefore
\[
\frac{1}{2} \sigma(x_n, T x_n) < \sigma(x_n, T x_n), \quad \forall n \in \mathbb{N}
\]
which implies that
\[
\tau + F(\sigma(T x_n, T^2 x_n)) \leq F(\Delta(x_n, T x_n))
\]
\[
= F(a \sigma(x_n, T x_n) + \beta \sigma(x_n, T x_n) + \gamma \sigma(T x_n, T^2 x_n) + \eta \sigma(T x_n, T x_n) + \delta \sigma(T x_n, T x_n))
\]
\[
\leq F(a \sigma(x_n, T x_n) + \beta \sigma(x_n, T x_n) + \gamma \sigma(T x_n, T^2 x_n) + \eta \sigma(T x_n, T x_n) + \delta \sigma(T x_n, T x_n))
\]
\[
(2.12)
\]
Following (2.12) and \( F \) is strictly increasing, then
\[
\sigma(T x_n, T^2 x_n) < (a + \beta + \eta + 2\delta) \sigma(x_n, T x_n) + (\gamma + \eta) \sigma(T x_n, T^2 x_n)
\]
\[
(2.13)
\]
implies that
\[
(1 - \gamma - \eta) \sigma(T x_n, T^2 x_n) < (a + \beta + \eta + 2\delta) \sigma(x_n, T x_n), \quad \forall n \in \mathbb{N}.
\]

From \( a + \beta + \gamma + 2\eta + 2\delta < 1 \), we deduce that \( 1 - \gamma - \eta > 0 \) and so
\begin{align*}
\sigma(Tx_n, T^2x_n) &< \frac{\alpha + \beta + \eta + 2\delta}{1 - \eta} \sigma(x_n, Tx_n) < \sigma(x_n, Tx_n), \text{ for all } n \in \mathbb{N}.
\end{align*}

Consequently
\begin{align*}
\tau + F(\sigma(Tx_n, T^2x_n)) &\leq F(\sigma(x_n, Tx_n))
\end{align*}
i.e.
\begin{align*}
F(\sigma(Tx_n, T^2x_n)) &\leq F(\sigma(x_n, Tx_n)) - \tau.
\end{align*}

Similarly method show that
\begin{align*}
F(\sigma(x_n, Tx_n)) &\leq F(\sigma(x_{n-1}, Tx_{n-1})) - \tau \\
&\leq F(\sigma(x_{n-2}, Tx_{n-2})) - 2\tau
&\vdots \\
&\leq F(\sigma(x_0, Tx_0)) - n\tau, \quad \forall n \in \mathbb{N}.
\end{align*}
From (2.14), then \(\lim_{n \to \infty} F(\sigma_n) = -\infty\). Thus from (F2) we obtain
\begin{align*}
\lim_{n \to \infty} \sigma_n = 0.
\end{align*}

Also from (F3), there exists \(k \in (0, 1)\) such that
\begin{align*}
\lim_{n \to \infty} \sigma_n^k F(\sigma_n) = 0.
\end{align*}

Now, it follows that
\begin{align*}
\sigma_n^k F(\sigma_n) - \sigma_n^k F(\sigma_0) &\leq \sigma_n^k (F(\sigma_0) - n\tau) - \sigma_n^k F(\sigma_0) \\
&\leq \sigma_n^k F(\sigma_0) - \sigma_n^k n\tau - \sigma_n^k F(\sigma_0) \\
&= -\sigma_n^k n\tau \\
&\leq 0, \quad \text{for all } n \in \mathbb{N}.
\end{align*}
Letting \(n \to \infty\) in (2.15), we get
\begin{align*}
\lim_{n \to \infty} n\sigma_n^k = 0. \quad (2.16)
\end{align*}
From above (2.16), there exists \(n_1 \in \mathbb{N}\) such that \(n\sigma_n^k \leq 1\) for all \(n \geq n_1\).

Therefore
\begin{align*}
\sigma_n &\leq \frac{1}{n^k}, \quad \text{for all } n_1 \geq n. \quad (2.17)
\end{align*}
Next, we will show that \(\{x_n\}\) is a 0-\(\sigma\)-Cauchy sequence, let \(m, n \in \mathbb{N}\) such that \(m > n \geq n_1\). From (2.17) and property (F3), we obtain
\[\sigma(x_n, x_m) \leq \sigma(x_n, x_{n+1}) + \sigma(x_{n+1}, x_{n+2}) + \cdots + \sigma(x_{m-1}, x_m)\]
\[= \sigma_n + \sigma_{n+1} + \cdots + \sigma_{m-1}\]
\[= \sum_{j=n}^{m-1} \sigma_j\]
\[\leq \sum_{i=n}^{\infty} \frac{1}{i^2}.\]

(2.18)

By the convergence of \(P\) series \(\sum_{i=n}^{\infty} \frac{1}{i^2}\), and so as \(n \to \infty\), then \(\sigma(x_n, x_m) \to 0\). Hence \(\{x_n\}\) is a \(0-\sigma\)-Cauchy sequence in \((X, \sigma)\). Since \(X\) is \(0-\sigma\)-complete metric-like space, there exists a \(x^* \in X\) such that \(\lim_{n \to \infty} x_n = x^*\); equivalently,

\[\lim_{n,m \to \infty} \sigma(x_n, x_m) = \lim_{n \to \infty} \sigma(x_n, x^*) = \sigma(x^*, x^*) = 0.\]

(2.19)

Next, we claim that

\[\frac{1}{2} \sigma(x_n, Tx_n) < \sigma(x_n, x^*) \text{ or } \frac{1}{2} \sigma(Tx_n, T^2x_n) < \sigma(Tx_n, x^*), \quad \forall n \in \mathbb{N}.\]

(2.20)

Suppose contrary that there exists \(m \in \mathbb{N}\) such that

\[\frac{1}{2} \sigma(x_m, Tx_m) \geq \sigma(x_m, x^*) \text{ or } \frac{1}{2} \sigma(Tx_m, T^2x_m) \geq \sigma(Tx_m, x^*).\]

(2.21)

Hence,

\[2\sigma(x_m, x^*) \leq \sigma(x_m, Tx_m) \leq \sigma(x_m, x^*) + \sigma(x^*, Tx_m)\]

which implies that

\[\sigma(x_m, x^*) \leq \sigma(x^*, Tx_m).\]

(2.22)

From (2.21) and (2.22), we obtain

\[\sigma(x_m, x^*) \leq \sigma(x^*, Tx_m) \leq \frac{1}{2} \sigma(Tx_m, T^2x_m).\]

In fact, \(\frac{1}{2} \sigma(x_m, Tx_m) < \sigma(x_m, Tx_m)\), then from (2.11), we get

\[\tau + F(\sigma(Tx_m, T^2x_m)) \leq F(\alpha \sigma(x_m, Tx_m) + \beta \sigma(x_m, T^2x_m) + \gamma \sigma(Tx_m, T^2x_m) + \delta \sigma(Tx_m, Tx_m))\]
\[+ \eta \sigma(x_m, T^2x_m) + \delta \sigma(Tx_m, Tx_m))\]
\[\leq F((\alpha + \beta)\sigma(x_m, Tx_m) + \gamma \sigma(Tx_m, T^2x_m) + \delta \sigma(Tx_m, T^2x_m)) + \eta \sigma(x_m, Tx_m) + \delta \sigma(Tx_m, T^2x_m))\]
\[= F((\alpha + \beta + \eta + 2\delta)\sigma(x_m, Tx_m) + (\gamma + \eta)\sigma(Tx_m, T^2x_m)).\]

(2.23)

It follows that (2.33) and property (F1), then

\[\sigma(Tx_m, T^2x_m) < (\alpha + \beta + \eta + 2\delta)\sigma(x_m, Tx_m) + (\gamma + \eta)\sigma(Tx_m, T^2x_m)\]

which implies that

\[(1 - \gamma - \eta)\sigma(Tx_m, T^2x_m) < (\alpha + \beta + \eta + 2\delta)\sigma(x_m, Tx_m).\]

Since \(1 - \gamma - \eta > 0\), we obtain
\[ \sigma(Tx_m, T^2x_m) < \frac{\alpha + \beta + \eta + 2\delta}{1 - \gamma - \eta} \sigma(x_m, Tx_m) < \sigma(x_m, Tx_m). \]  \hspace{1cm} (2.24)

It follows from (2.21), (2.22), (2.24) that

\[
\sigma(Tx_m, T^2x_m) < \sigma(x_m, Tx_m) \\
\leq \sigma(x_m, x') + \sigma(x', v Tx_m) \\
\leq \sigma(x', Tx_m) + \sigma(x', Tx_m) \\
\leq \frac{1}{2} \sigma(Tx_m, T^2x_m) + \frac{1}{2} \sigma(Tx_m, T^2x_m) \\
= \sigma(Tx_m, T^2x_m).
\]

This is a contradiction. Thus (2.20) true for all \( n \in \mathbb{N} \), which implies that

\[
\tau + F(\sigma(Tx_m, Tx')) \leq F(\alpha \sigma(x_m, x') + \beta \sigma(x_m, Tx_m) \\
+ \gamma \sigma(x', Tx') + \eta \sigma(x', Tx') + \delta \sigma(x', Tx'))
\]

is true for all \( n \in \mathbb{N} \) or

\[
\tau + F(\sigma(T^2x_m, Tx')) \leq F(\alpha \sigma(x_m, x') + \beta \sigma(Tx_m, T^2x_m) \\
+ \gamma \sigma(x', Tx') + \eta \sigma(Tx_m, Tx') + \delta \sigma(x', vT^2x_m))
\]

is true for all \( n \in \mathbb{N} \).

In first case we consider

\[
\tau + F(\sigma(Tx_m, Tx')) \leq F(\alpha \sigma(x_m, x') + \beta \sigma(x_m, Tx_m) + \gamma \sigma(x', Tx') \\
+ \eta \sigma(x', Tx') + \delta \sigma(x', Tx_m)) \\
\leq F(\alpha \sigma(x_m, x') + \beta \sigma(x_m, Tx_m) + \gamma (\sigma(x', x_m) \\
+ \sigma(x_m, Tx_m) + \sigma(Tx_m, T^2x_m) + \eta \sigma(x_m, Tx_m) \\
+ \sigma(x_m, T^2x_m)) + \delta (\sigma(x', x_m) + \sigma(x_m, Tx_m))) \\
= F((\alpha + \gamma + \delta)\sigma(x_m, x') + (\beta + \gamma + \eta + \delta)\sigma(x_m, Tx_m) \\
+ (\gamma + \eta)\sigma(Tx_m, T^2x_m)).
\]

From \( \tau > 0 \) and \( \sigma(x_m, Tx_m) < 2\sigma(x_m, x') \), this implies that

\[
\sigma(Tx_m, T^2x_m) < (\alpha + \gamma + \delta)\sigma(x_m, x') + (\beta + \gamma + \eta + \delta)\sigma(x_m, Tx_m) \\
+ (\gamma + \eta)\sigma(Tx_m, T^2x_m) \\
< (\alpha + \gamma + \delta)\sigma(x_m, x') + (\beta + \gamma + \eta + \delta)2\sigma(x_m, x') \\
+ (\gamma + \eta)\sigma(Tx_m, T^2x_m) \\
= (\alpha + 2\beta + 3\gamma + 2\eta + 3\delta)\sigma(x_m, x') + (\gamma + \eta)\sigma(Tx_m, T^2x_m). \hspace{1cm} (2.25)
\]

Following from (2.25), we get

\[
(1 - \gamma - \eta)\sigma(Tx_m, T^2x') < (\alpha + 2\beta + 3\gamma + 2\eta + 3\delta)\sigma(x_m, x'),
\]

consequently

\[
\tau + F((1 - \gamma - \eta)\sigma(Tx_m, T^2x')) \leq F((\alpha + 2\beta + 3\gamma + 2\eta + 3\delta)\sigma(x_m, x')) = F((\alpha + 2\beta + 3\gamma + 2\eta + 3\delta)\sigma(x_{m+1}, x')).
\]

From above letting \( n \to \infty \) and using (2.19) and condition (F2), we obtain

\[
\lim_{n \to \infty} \sigma(Tx_m, Tx^m) = 0.
\]
Therefore $\sigma(x^*, Tx^*) = \lim_{n \to \infty} \sigma(x_{n+1}, Tx^*) = \lim_{n \to \infty} \sigma(Tx_n, Tx^*) = 0.$

For second case we consider

$$F(\sigma(T^2x_n, Tx^*)) \leq F(\alpha \sigma(Tx_n, x^*) + \beta \sigma(Tx_n, T^2x_n) + \gamma \sigma(x^*, Tx^*) + \eta \sigma(Tx_n, Tx^*) + \delta \sigma(x^*, T^2x_n))$$

$$\leq F(\alpha \sigma(Tx_n, x^*) + \beta \sigma(x_n, T^2x_n) + \gamma \sigma(x^*, Tx_n) + \sigma(Tx_n, T^2x_n) + \eta \sigma(Tx_n, T^2x_n) + \delta \sigma(x^*, Tx_n) + \sigma(Tx_n, T^2x_n))$$

$$= F((a + \gamma + \beta)\sigma(Tx_n, x^*) + (\beta + \gamma + \eta + \delta)\sigma(Tx_n, T^2x_n) + (\gamma + \eta)\sigma(T^2x_n, Tx^*).$$

From $\tau > 0$ and $\sigma(Tx_n, T^2x_n) < 2\sigma(Tx_n, x^*)$, this implies that

$$\sigma(T^2x_n, Tx^*) < (a + \gamma + \beta)\sigma(Tx_n, x^*) + (\beta + \gamma + \eta + \delta)\sigma(Tx_n, T^2x_n) + (\gamma + \eta)\sigma(T^2x_n, Tx^*)$$

$$< (a + \gamma + \beta)\sigma(Tx_n, x^*) + (\beta + \gamma + \eta + \delta)2\sigma(Tx_n, x^*) + (\gamma + \eta)\sigma(T^2x_n, Tx^*)$$

$$= (a + 2\beta + 3y + 2\eta + 3\delta)\sigma(Tx_n, x^*) + (\gamma + \eta)\sigma(T^2x_n, Tx^*).$$

Following from (2.26), we get

$$(1 - \gamma - \eta)\sigma(T^2x_n, Tx^*) < (a + 2\beta + 3y + 2\eta + 3\delta)(T^2x_n, x^*),$$

consequently

$$\tau + F((1 - \gamma - \eta)\sigma(T^2x_n, Tx^*)) \leq F((a + 2\beta + 3y + 2\eta + 3\delta)\sigma(Tx_n, x^*))$$

$$= F((a + 2\beta + 3y + 2\eta + 3\delta)\sigma(x_{n+1}, x^*).$$

From above letting $n \to \infty$ and using (2.19) and condition (F2), we obtain

$$\lim_{n \to \infty} \sigma(T^2x_n, Tx^*) = 0.$$

Therefore $\sigma(x^*, Tx^*) = \lim_{n \to \infty} \sigma(x_{n+2}, Tx^*) = \lim_{n \to \infty} \sigma(T^2x_n, Tx^*) = 0.$

Hence $x^*$ is a fixed point of $T$. For the proof of the uniqueness fixed point of $T$ is similarity Theorem 2.2.

**Corollary 2.6** Let $(X, d)$ be a $0$-$\sigma$ complete metric-like space and $T: X \to X$ be a mapping. If there exist $0 \leq k < \frac{1}{2}$ such that

$$\frac{1}{2} \sigma(x, Tx) < \sigma(x, y) \Rightarrow \tau + F(k(\sigma(x, Tx) + \sigma(y, Ty))).$$

Then $T$ has unique fixed point in $X$.

**Proof** Letting $\alpha = \eta = \delta = 0$ and $\beta = \gamma = k$ in Theorem (2.5) then we have same results.

**Corollary 2.7** Let $(X, d)$ be a $0$-$\sigma$ complete metric-like space and $T: X \to X$ be a mapping. If there exist $0 \leq k < \frac{1}{2}$ such that

$$\frac{1}{2} \sigma(x, Tx) < \sigma(x, y) \Rightarrow \tau + F(k(\sigma(x, Ty) + \sigma(y, Tx))).$$

Then $T$ has unique fixed point in $X$. 
Proof Letting $a = b = \gamma = 0$ and $\eta = \delta = k$ in Theorem 2.5 then we have same results.

3. Some applications

3.1. An application to second-order differential equations

Now, we consider the boundary value problem for second-order differential equation

$$
\begin{align*}
\begin{cases}
    x'(t) = -f(t, x(t)), & t \in I, \\
    x(0) = x(1) = 0,
\end{cases}
\end{align*}
$$

(3.1)

where $I = [0, 1]$ and $f: I \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

In this section we are going to apply Theorem 2.2 to study existence and uniqueness of solutions for type of second-order differential equations. Our approach is inspired by Section 5 of Aydi and Karapinar (2015).

It is known, and easy to check, that the problem (3.1) is equivalent to the integral equation

$$
x(t) = \int_0^1 G(t, s)f(s, x(s)) \, ds, \quad \text{for } t \in I, \sigma_i
$$

(3.2)

where $G$ is the Green’s function defined by

$$
G(t, s) = \begin{cases}
    t(1 - s) & \text{if } 0 \leq t \leq s \leq 1 \\
    s(1 - t) & \text{if } 0 \leq s \leq t \leq 1.
\end{cases}
$$

i.e. if $x \in C^2(I, \mathbb{R})$, then $x$ is a solution of problem (3.1) if and only if $x$ is a solution of the integral Equation (3.2).

Let $X = C(I)$ be the space of all continuous functions defined on $I$. Consider the metric-like $\sigma$ on $X$ defined by

$$
\sigma(x, y) = \|x - y\|_\infty + \|x\|_\infty + \|y\|_\infty \quad \text{for all } x, y \in X,
$$

where $\|u\|_\infty = \max_{t \in [0, 1]} |u(t)|$ for all $u \in X$.

Note that $\sigma$ is also a partial metric on $X$ and since

$$
d_\sigma(x, y) = 2\sigma(x, y) - \sigma(x, x) - \sigma(y, y) = 2\|x - y\|_\infty.
$$

By Lemma 1.5, hence $(X, \sigma)$ is complete since the metric space $(X, \| \cdot \|)$ is complete.

THEOREM 3.1 Suppose the following conditions:

1. there exist continuous functions $p: I \to \mathbb{R}^+$ such that

$$
|f(s, a) - f(s, b)| \leq 8p(s)|a - b|
$$

for all $s \in I$ and $a, b \in \mathbb{R}$;

2. there exist continuous functions $q: I \to \mathbb{R}^+$ such that

$$
|f(s, a)| \leq 8q(s)|a|
$$

for all $s \in I$ and $a \in \mathbb{R}$;

3. $\max_{s \in I} p(s) = a_1 \lambda_1 < \frac{1}{\gamma}$, which is $0 \leq a < \frac{1}{\gamma}$;

4. $\max_{s \in I} q(s) = a_2 \lambda_2 < \frac{1}{\epsilon}$, which is $0 \leq a < \frac{1}{\epsilon}$.
Then problem (3.1) has a unique solution \( u \in X = C(I, \mathbb{R}) \).

**Proof** Define the mapping \( T : X \to X \) by

\[
T_x(t) = \int_0^1 G(t, s)f(s, x(s)) \, ds,
\]

for all \( x \in X \) and \( t \in T \). Then the problem (3.1) is equivalent to finding a fixed point \( u \) of \( T \) in \( X \). Let \( x, y \in X \), we obtain

\[
|T_x(t) - T_y(t)| = \left| \int_0^1 G(t, s)f(s, x(s)) \, ds - \int_0^1 G(t, s)f(s, y(s)) \, ds \right|
\]

\[
\leq \int_0^1 G(t, s)|f(s, x(s)) - f(s, y(s))| \, ds
\]

\[
\leq 8 \int_0^1 G(t, s)p(s)|x(s) - y(s)| \, ds
\]

\[
\leq 8\lambda_1\|x - y\|_{\infty}
\]

\[
= \alpha \lambda_1\|x - y\|_{\infty}.
\]

In the above equality, we used that for each \( t \in I \), we have \( \int_0^1 G(t, s) \, ds = \frac{1}{t}(1 - t) \) and so \( \sup_{t \in I} \int_0^1 G(t, s) \, ds = \frac{1}{8} \). Therefore,

\[
\|T_x - T_y\|_{\infty} \leq \alpha \lambda_1\|x - y\|_{\infty}.
\]

(3.3)

Moreover, we have

\[
T_x(t) = \int_0^1 G(t, s)f(s, x(s)) \, ds
\]

\[
\leq 8 \int_0^1 G(t, s)q(s)|x(s)| \, ds
\]

\[
\leq 8\lambda_2\|x\|_{\infty}.
\]

Hence,

\[
\|T_x\|_{\infty} \leq \alpha \lambda_2\|x\|_{\infty}.
\]

(3.4)

Similarly method, we obtain

\[
\|T_y\|_{\infty} \leq \alpha \lambda_2\|y\|_{\infty}.
\]

(3.5)

Let \( e^{-\tau} = \lambda_1 + 2\lambda_2 < 1 \) where \( \tau > 0 \). Form (3.3), (3.4) and (3.5), we obtain

\[
s(x, y) = \|T_x - T_y\|_{\infty} + \|T_x\|_{\infty} + \|T_y\|_{\infty}
\]

\[
\leq \alpha \lambda_1\|x - y\|_{\infty} + \alpha \lambda_2\|x\|_{\infty} + \alpha \lambda_2\|y\|_{\infty}
\]

\[
\leq (\lambda_1 + 2\lambda_2)(\alpha)(\|T_x - T_y\|_{\infty} + \|T_x\|_{\infty} + \|T_y\|_{\infty})
\]

\[
= (e^{-\tau})\alpha s(x, y).
\]

(3.6)

Let \( \beta, \gamma, \eta, \delta > 0 \) where \( \beta < \frac{1}{7}, \gamma < \frac{1}{7}, \eta < \frac{1}{7}, \delta < \frac{1}{7} \).
Following (3.6), we get
\[ \sigma(Tx, Ty) \leq (e^{-r})[\alpha\sigma(x, y) + \beta \sigma(x, Tx) + \gamma \sigma(y, Ty) + \eta \sigma(x, Ty) + \delta \sigma(y, Tx)], \]  
(3.7)

where \( \alpha + \beta + \gamma + 2\eta + 2\delta < 1 \). Taking the function \( F: \mathbb{R}^+ \to \mathbb{R} \) in (3.7), where \( F(t) = \ln(t) \), which is \( F \in \mathcal{P} \), we get
\[ \tau + F(\sigma(Tx, Ty)) \leq F(\alpha \sigma(x, y) + \beta \sigma(x, Tx) + \gamma \sigma(y, Ty) + \eta \sigma(x, Ty) + \delta \sigma(y, Tx)). \]

Therefore all hypothesis of Theorem 2.1 are satisfied, and so \( T \) has a unique fixed point \( u \in X \), that is, the problem (3.1) has a unique solution \( u \in C^2(I) \).

### 3.2. An application to fractional differential equations

Next, we apply Theorem 2.2 to establish the existence of solution of fractional order functional differential equation.

Consider the following initial valued problem (IVP for short) of the form
\[ D^\alpha y(t) = f(t, y_t), \quad \text{for all } t \in J = [0, b], \quad 0 < \alpha < 1, \]
(3.8)
\[ y(t) = \phi(t), \quad t \in (-\infty, 0] \]
(3.9)

where \( D^\alpha \) is the standard Riemann–Liouville fractional derivative \( f: J \times B \to \mathbb{R} \), \( \phi \in B \), \( \phi(0) = 0 \) and \( B \) is called a phase space or state space satisfying some fundamental axioms (H-1, H-2, H-3) given below which were introduced by Hale and Kato (1978).

For any function \( y \) defined on \((-\infty, b]\) and any \( t \in J \), we denote by \( y_t \) the element of \( B \) defined by \( y_t(\theta) = y(t + \theta), \quad \theta \in (-\infty, 0] \).

Here \( y_t(\cdot) \) represents the history of the state from \(-\infty\) up to present time \( t \).

By \( C(J, \mathbb{R}) \) we denote the Banach space of all continuous functions from \( J \) into \( \mathbb{R} \) with the norm \( \|y\|_\infty = \sup \{|y(t)|: t \in J\} \),

where \( | \cdot | \) denotes a suitable complete norm on \( \mathbb{R} \).

Now consider the metric-like space \( \sigma \) on \( X \) given by
\[ \sigma(x, y) = 2d(x, y) \quad \text{for all } x, y \in X. \]

Then \( (X, \sigma) \) is complete as the metric space \( (X, d) \) is complete.

1. **(H-1)** If \( y: (-\infty, b] \to \mathbb{R} \), and \( y_0 \in B \) then for every \( t \in [0, b] \) the following conditions hold:
   (1) \( y_t \) is in \( B \),
   (2) \( \|y_t\|_B \leq K(t) \sup \{|y(s)|: 0 \leq s \leq t\} + M(t)\|y_0\|_B \),
   (3) \( |y(t)| \leq H\|y_t\|_B \), where \( H \geq 0 \) is a constant, \( K: [0, b] \to [0, \infty] \) is continuous,
   \( M: [0, \infty] \to [0, \infty] \) is locally bounded and \( H, K, M \) are independent of \( y(\cdot) \).

2. **(H-2)** For the function \( y(\cdot) \) in (H-1), \( y_t \) is a \( B \)-valued continuous function on \([0, b]\).
The space $B$ is complete.

By a solution of problem (3.8)--(3.9), we mean a space $\Omega = \{y:(-\infty, b] \to \mathbb{R} | y|_{]-\infty,0]} \in B \text{ and } y|_{[0,b)} \text{ is continuous} \}$. Therefore a function $y \in \Omega$ is called a solution of (3.8)--(3.9) if $y$ satisfies the equation $D^\alpha y(t) = f(t, y_t)$ on $J$, and the condition $y(t) = \phi(t)$ on $(-\infty, 0]$.

The following lemma is crucial to prove our existence theorem for the problem (3.8)--(3.9).

**Lemma 3.2** (see Delbosio & Rodino, 1996) Let $0 < \alpha < 1$ and $h: (0, b] \to \mathbb{R}$ be continuous and $\lim_{t \to 0^+} h(t) = h(0^+) \in \mathbb{R}$. Then $y$ is a solution of the fractional integral equation

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) \, ds,$$

if and only if $y$ is a solution of the initial value problem for the fractional differential equation

$$D^\alpha y(t) = h(t), \quad t \in (0, b), \quad y(0) = 0.$$

Now we are ready to prove following existence theorem.

**Theorem 3.3** Let $f: J \times B \to \mathbb{R}$. Assume

(H) there exists $q > 0$ such that

$$|f(t, u) - f(t, v)| \leq q||u - v||_B, \quad \text{for } t \in J \text{ and for all } u, v \in B.$$

If $\frac{\partial h}{\partial x} < k_\alpha < 1$ where $0 \leq k_\alpha < \frac{1}{\alpha}$ and

$$K_\alpha = \sup\{|K(t); t \in [0, b)]\},$$

then there exists a unique solution for the IVP (3.8)--(3.9) on the interval $(-\infty, b]$.

**Proof** To prove the existence of solution for the IVP (3.8)--(3.9), we transform it into a fixed point problem. For this, consider the operator $N: \Omega \to \Omega$ defined by

$$N(y)(t) = \begin{cases} 
\phi(t) & \text{if } t \in (-\infty, 0], \\
\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_s) \, ds & \text{if } t \in [0, b]. 
\end{cases}$$

Let $x:(-\infty, b] \to \mathbb{R}$ be the function defined by

$$x(t) = \begin{cases} 
\phi(t) & \text{if } t \in (-\infty, 0], \\
0 & \text{if } t \in [0, b]. 
\end{cases}$$

Then $x_0 = \phi$. For each $z \in C([0, b], \mathbb{R})$ with $z(0) = 0$, we denote by $z$ the function defined by

$$z(t) = \begin{cases} 
0 & \text{if } t \in (-\infty, 0], \\
z(t) & \text{if } t \in [0, b]. 
\end{cases}$$

If $y(\cdot)$ satisfies the integral equation

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_s) \, ds,$$

then there exists a unique solution for the IVP (3.8)--(3.9) on the interval $(-\infty, b]$.\]
we can decompose $y(\cdot)$ as $y(t) = \overline{z}(t) + x(t)$, $0 \leq t \leq b$, which implies $y_t = \overline{z}_t + x_t$, for every $0 \leq t \leq b$, and the function $z(\cdot)$ satisfies

$$z(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \overline{z}_s + x_s) \, ds.$$  

Set

$$C_0 = \{ z \in C([0, b], \mathbb{R}); z_0 = 0 \},$$  

and let $\| \cdot \|_b$ be the semi-norm in $C_0$ defined by

$$\| z \|_b = \| z_0 \|_b + \sup \{ |z(t)|; 0 \leq t \leq b \} = \sup \{ |z(t)|; 0 \leq t \leq b \}, \quad z \in C_0.$$

$C_0$ is a Banach space with norm $\| \cdot \|_b$. Let the operator $P : C_0 \to C_0$ be defined by

$$(Pz)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \overline{z}_s + x_s) \, ds, \quad t \in [0, b]. \quad (3.10)$$

That the operator $N$ has a fixed point is equivalent to $P$ has a fixed point, and so we turn to proving $P$ has a fixed point. Indeed, consider $z, z^* \in C_0$. Then we have for all $t \in [0, b]$.

$$|P(z)(t) - P(z^*)(t)| = \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \overline{z}_s + x_s) \, ds \right|$$

$$- \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \overline{z}_s + x_s) \, ds \right|$$

\leq \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \overline{z}_s + x_s) - f(s, \overline{z}_s + x_s) \, ds \right|$$

\leq \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} q \| \overline{z}_s - \overline{z}_s \|_b \, ds \right|$$

\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} q K_b \sup_{s \in [0, t]} \| z(s) - z^*(s) \| \, ds$$

\leq \frac{K_b}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} q \| z - z^* \|_b.$$  

Therefore

$$\|P(z) - P(z^*)\|_b \leq \frac{q b^* K_b}{\Gamma(\alpha + 1)} \| z - z^* \|_b,$$

i.e.

$$\sigma(P(z), P(z^*)) \leq 4 k_2 \sigma(z, z^*). \quad (3.11)$$

Suppose $F \in \mathcal{F}$ where $F(t) = \ln(t)$ for all $t \in \mathbb{R}^+$. Taking $F$ to (3.11) we obtain

$$\ln(\sigma(P(z), P(z^*))) \leq \ln(e^{\lambda t} k_2 \sigma(z, z^*))$$

where $\lambda = e^{-t}$. Suppose $k_2, k_3, k_4, k_5$ is nonnegative constant with $k_2, k_3, k_4, k_5 < \frac{1}{\lambda}$. From above inequality we get
\[
\ln(\sigma(P(z), P(z^*))) \leq \ln(e^{-r}(k_1 \sigma(z, z^*) + k_2 \sigma(z, P(z)) + k_4 \sigma(z^*, P(z^*)) + k_5 \sigma(z, P(z^*)) + k_6 \sigma(z^*, P(z))))
\]

where \(k_1 + k_2 + k_3 + k_4 + k_5 < 1\). Hence, we obtain

\[
\tau + F(\sigma(P(z), P(z^*))) \leq F(k_1 \sigma(z, z^*) + k_2 \sigma(z, P(z)) + k_3 \sigma(z^*, P(z^*)) + k_4 \sigma(z, P(z^*)) + k_5 \sigma(z^*, P(z))).
\]

Thus, we deduce that the operator \(P\) satisfies all the hypothesis of Theorem 2.2. Therefore \(P\) has a unique fixed point.

4. Conclusions
In this paper, we introduced generalized Roger Hardy type \(F\)-contraction and generalized Roger Hardy type Suzuki \(F\)-contractions in metric-like spaces. We also established and obtained fixed point theorems in such spaces. Moreover, we give some examples for support main theorems. Finally, we also applied our main results to second-order differential equations and fractional differential equations.

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References


