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Notes on *-finite operators class

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Abstract: Let $\mathcal{H}$ be a separable infinite-dimensional complex Hilbert space and $\mathcal{B}(\mathcal{H})$ denotes the algebra of all bounded linear operators on $\mathcal{H}$. An $A \in \mathcal{B}(\mathcal{H})$ is said to be *-finite operator if $0 \in \overline{W}(TA - AT^*)$ for each $T \in \mathcal{B}(\mathcal{H})$. In this paper, we present some properties of *-finite operators and prove that a paranormal operator under certain scalar perturbation is *-finite operator. However, we give an example of paranormal operators which is not *-finite operators.

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1. Introduction

Let $\mathcal{H}$ be a separable infinite-dimensional complex Hilbert space and $\mathcal{B}(\mathcal{H})$ denotes the algebra of all bounded linear operators on $\mathcal{H}$. Let $\overline{W}(A)$ be the closure of numerical range of the operator $A \in \mathcal{B}(\mathcal{H})$. A is said to be *-finite operator if $0 \in \overline{W}(TA - AT^*)$ for each $T \in \mathcal{B}(\mathcal{H})$. Williams (1970) has shown that for any operator B, $0 \in \overline{W}(B)$ if and only, if $\|B - \mu I\| \geq |\mu|\ \forall \mu \in \mathcal{C}$. This implies that an operator A is *-finite *iff* $\|TA - AT^* - \mu I\| \geq |\mu|;\ \forall\, \mu \in \mathcal{C}$ and $\forall\, T \in \mathcal{B}(\mathcal{H})$. *-finite operator was firstly introduced by Hamada (2002). The definition of *-finite operator is motivated by the study of finite operators given by Williams (1970). An operator A in $\mathcal{B}(\mathcal{H})$ is called finite operator if $0 \in \overline{W}(TA - AT)$ for each $T \in \mathcal{B}(\mathcal{H})$ or equivalently, $\|I - TA - AT\| \geq I$ for all $T \in \mathcal{B}(\mathcal{H})$, where I is the identity operator. This topic deals with minimizing the distance, measured by some norms or other, between a varying commutator $TT^* - T^*T$ and some fixed operator (Maher, 2006). It was shown by Mecheri that a paranormal operator is finite. For more details read (Mecheri, 2002, 2008).

In this paper, we present some properties of *-finite operators and we prove that a paranormal operator under certain scalar perturbation is *-finite operator. However, we give an example of paranormal operators which is not *-finite operator.

ABOUT THE AUTHOR

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PUBLIC INTEREST STATEMENT

Finite operators have various interesting properties and applications. This topic deals with minimizing the distance, measured by some norms or other, between a varying commutator and some fixed operator. The importance of this topic comes from its roots in quantum theory. This motivates the author to define a new operator called *-finite operator. In this paper, the author presents many properties of *-finite operators, constructs examples which are not *-finite operators and explores the relationship between paranormal operator and *-finite operator.

2. Preliminary notes

Definition 2.1 (Hamada, 2002) The operator $A \in \mathcal{B}(\mathcal{H})$ is *-finite operator if

$$0 \in \overline{W}(TA - AT^*)\ \forall\ T \in \mathcal{B}(\mathcal{H}).$$

Williams (1970) has shown that for any operator B, $0 \in \overline{W}(B)$ if and only, if $\|B - \mu I\| \geq |\mu|\ \forall\ \mu \in \mathcal{C}$. This implies that an operator A is *-finite *iff*

$$\|TA - AT^* - \mu I\| \geq |\mu|;\ \forall\ \mu \in \mathcal{C}\ \text{ and }\ \forall\ T \in \mathcal{B}(\mathcal{H}).$$

Definition 2.2 Let $A \in \mathcal{B}(\mathcal{H})$, the approximate reduced spectrum of A, $\sigma_{ra}(A)$, is the set of scalars λ for which given $\varepsilon > 0$, there exists a unit vector x in $\mathcal{H}$ satisfying $\|Ax - \lambda x\| < \varepsilon$ and $\|A^*x - \overline{\lambda}x\| < \varepsilon$.

An operator $A \in \mathcal{B}(\mathcal{H})$ is said to be normaloid if $\|A\| = r(A)$, where $r(A)$ is the spectral radius of A, paranormal if $\|Ax\|^2 \leq \|A^2x\|\|x\|$, for all $x \in \mathcal{H}$, and p-hyponormal if $|A|^{2p} - |A^*|^{2p} \geq 0\ (0 < p \leq 1)$. We have hyponormal ⊂ p-hyponormal ⊂ paranormal ⊂ normaloid.

A is said to be log-hyponormal if A is invertible and satisfies the following equality

$$log(A^*A) \geq log(AA^*).$$

It is known that invertible p-hyponormal operators are log-hyponormal operators but the converse is not true (Tanahashi, 1999). The idea of log-hyponormal operator is due to Ando (1972) and the first paper in which log-hyponormality appeared is Fujii, Himeji, and Matsumoto (1994). For properties of log-hyponormal operators (see Aluthge, 1990; Jeon,Tanahashi, & Uchiyama, 2004; Mishra, Srivastava, & Sen, 2016; Tanahashi, 1999; Uchiyama, 1999).

An operator $A \in \mathcal{B}(\mathcal{H})$ belongs to the class A if $|A^2| \geq |A|^2$. Class A was introduced by Furuta, Ito, and Yamazaki (1998) as a subclass of paranormal operators which includes the classes of p-hyponormal and log-hyponormal operators. The following theorem is one of the results associated with class A.

Theorem 2.3 *(Furuta et al., 1998)*

(1) Every log-hyponormal operator is a class A operator.
(2) Every class A operator is a paranormal operator.

3. Main results

In the following theorems we will show that a paranormal operators under some scalar perturbation is *-finite operator.

Lemma 3.1 *Let $A \in \mathcal{B}(\mathcal{H})$ be paranormal operator. Then $\sigma_{ra}(A) \neq \emptyset$.*

Proof If A is paranormal operator, then A is normaloid. Hence $\|A\| = r(A)$. This implies that there exists $\lambda \in \sigma(A)$ such that $|\lambda| = \|A\|$. Since λ is in the boundary of $\sigma(A)$, then for any $\varepsilon > 0$ there exists a unit vector x such that $\|Ax - \lambda x\| < \varepsilon$. Then $\|A^*x - \overline{\lambda}x\| < \varepsilon$ because $|\lambda| = \|A\|$.

Theorem 3.2 *Let $A \in \mathcal{B}(\mathcal{H})$ be paranormal operator. Then $A - \lambda I$ is *-finite operator for each $\lambda \in \sigma_{ra}(A)$.*

Proof Since A is paranormal operator, then by Lemma 3.1, let $\lambda \in \sigma_{ra}(A)$. Definition 2.2 implies that $\forall \varepsilon > 0$ there exist a unit vector $x \in \mathcal{H}$ and a non-zero operator $T \in \mathcal{B}(\mathcal{H})$ such that

$$\|Ax - \lambda x\| < \frac{\varepsilon}{2} \cdot \frac{1}{\|T\|},$$

and

$$\|A^*x - \overline{\lambda}x\|\,\|T\| < \frac{\varepsilon}{2}.$$

Using the Schwarz inequality, one can get

$$\left|\langle Ax, T^*x\rangle - \langle \lambda x, T^*x\rangle\right| \leq \|Ax - \lambda x\|\,\|T\| < \frac{\varepsilon}{2},$$

and

$$|\langle T^*x, A^*x\rangle - \langle T^*x, \overline{\lambda}x\rangle| \leq \|A^*x - \overline{\lambda}x\|\,\|T\| < \frac{\varepsilon}{2}.$$

Then by adding the last two inequalities, we have

$$\left|\langle (T(A-\lambda I) - (A-\lambda I)T^*)x, x\rangle\right| < \varepsilon.$$

Hence, $0 \in \overline{W}(T(A-\lambda I) - (A-\lambda I)T^*)$. Consequently, $A - \lambda I$ is *-finite.

As a consequence of the previous theorem we obtain.

Corollary 3.3 The following operators under approximate reduced spectrum perturbation are *-finite operators.

(1) Hyponormal operators,
(2) p-Hyponormal operators,
(3) Class A operators,
(4) log-hyponormal operators.

The following example shows that A is paranormal operator does not imply that A is *-finite operator.

Example 3.4 Clearly I is paranormal operator, while I is not *-finite operator. Otherwise, $0 \in \overline{W}(T - T^*)$ for each $T \in \mathcal{B}(\mathcal{H})$, in particular when $T = iI$, $0 \in \overline{W}(2iI) = \{2i\}$

Let $\mathrm{F}^*(\mathcal{H})$ be the set of all *-finite operators. In the following propositions we present some properties of $\mathrm{F}^*(\mathcal{H})$.

Proposition 3.5 Let $A \in \mathcal{B}(\mathcal{H})$. Then $A \in \mathrm{F}^*(\mathcal{H}) \Longleftrightarrow A^* \in \mathrm{F}^*(\mathcal{H})$.

Proof Since $(A^*) = A$, it suffices to prove one implication. Let $A \in \mathrm{F}^*(\mathcal{H})$. Then

$$\|TA - AT^* - \mu I\| \geq |\mu|;\ \forall \mu \in \mathcal{C} \text{ and } \forall T \in \mathcal{B}(\mathcal{H}).$$

Since the map $\mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H}), T \rightarrow T^*$ is surjective,

$$\|T^*A - AT - \overline{\mu}I\| \geq |\mu|;\ \forall \mu \in \mathcal{C} \text{ and } \forall T \in \mathcal{B}(\mathcal{H}).$$

Hence

$$\|A^*T^* - TA^* - \overline{\mu}I\| = \|(TA - AT^* - \mu I)^*\| \geq |\mu|;\ \forall \mu \in \mathcal{C} \text{ and } \forall T \in \mathcal{B}(\mathcal{H}).$$

which proves that $A^* \in \mathrm{F}^*(\mathcal{H})$.

Proposition 3.6 $\mathrm{F}^*(\mathcal{H})$ is invariant by unitary equivalence.

Proof Let $A \in F^*(\mathcal{H})$ and let U unitary in $\mathcal{B}(\mathcal{H})$. We have

$$\|TA - AT^* - \mu I\| \geq |\mu|,$$

for all $\mu \in \mathcal{C}$ and for all $T \in \mathcal{B}(\mathcal{H})$. Since the map: $T \to U^*TU$ is surjective, $\forall T \in \mathcal{B}(\mathcal{H})$, there exists $S \in \mathcal{B}(\mathcal{H})$ such that $T = U^*SU$.

Hence

$$\begin{aligned}\|T(U^*AU) - (U^*AU)T^* - \mu I\| &= \|(U^*SU)(U^*AU) - (U^*AU)(U^*S^*U) - \mu U^*U\| \\ &= \|U^*(SA - AS^* - \mu I)U\| = \|SA - AS^* - \mu I\| \geq |\mu|,\end{aligned}$$

where $U^*AU \in F^*(\mathcal{H})$, we deduce that $\mathcal{U}(F^*(\mathcal{H})) \subset F^*(\mathcal{H})$, where $\mathcal{U}(F^*(\mathcal{H}))$ is the unitary orbit of similarity of $F^*(\mathcal{H})$.

We conclude this section by studying the density of $F^*(\mathcal{H})$ in the operator norm topology.

Proposition 3.7 The set $F^*(\mathcal{H})$ is not dense in $\mathcal{B}(\mathcal{H})$ in the operator norm topology.

Proof To prove first that set $F^*(\mathcal{H})$ is closed in $\mathcal{B}(\mathcal{H})$, let $\{A_n\}$ be a sequence of $*$-finite operators such that $A_n \to A$, let $\mu \in \mathcal{C}$ then

$$\|TA_n - A_nT^* - \mu I\| \longrightarrow \|TA - AT^* - \mu I\|.$$

Thus

$$\inf_T \|TA_n - A_nT^* - \mu I\| \longrightarrow \inf_T \|TA - AT^* - \mu I\|,$$

i.e. $|\mu| \longrightarrow \inf_T \|TA - AT^* - \mu I\|$. This implies that $\inf_T \|TA - AT^* - \mu I\| = |\mu|$ for each $T \in \mathcal{B}(\mathcal{H})$. which means that $\|TA - AT^* - \mu I\| \geq |\mu|$ for each $T \in \mathcal{B}(\mathcal{H})$. Consequently, A is $*$-finite and $F^*(\mathcal{H})$ is closed in $\mathcal{B}(\mathcal{H})$. It suffices to prove that $F^*(\mathcal{H}) \neq \mathcal{B}(\mathcal{H})$. But this is clear since I is not *-finite operator. See Example 3.4.

Remark 3.8 It was shown in Example 3.4 that identity operator, I, is not *-finite operator, while I is trivially finite operator. This implies that we have two different sets of operators finite operators and *-finite operators. The question arises how large is the difference between finite and *-finite operators.

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