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## COMPUTATIONAL SCIENCE | RESEARCH ARTICLE

# On the approximate evaluation of oscillatory-singular integrals

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**Abstract:** In this paper an efficient numerical scheme is proposed for the numerical computation of the Cauchy type oscillatory integral  $\int_{-1}^1 \frac{\cos wx}{x} f(x) dx$ ; where  $f(x)$  is a well-behaved function without having any kind of singularity in the range of integration  $[-1; 1]$ . The scheme is devised with the help of quadrature rule meant for the approximate evaluation of Cauchy principal value of integrals of the type  $\int_{-1}^1 \frac{f(x)}{dx}$ , and a quasi exact quadrature meant for the numerical integration of Filon-type integrals. The error bounds are determined and the scheme numerically verified by some standard test integrals.

**Subjects:** Science; Mathematics & Statistics; Advanced Mathematics; Analysis - Mathematics; Mathematical Numerical Analysis

**Keywords:** analytic function; Cauchy principal value; Filon's integral; Quasi-exact method; holder's condition; error bound

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### PUBLIC INTEREST STATEMENT

Integrals frequently appeared in sciences and engineering. In practice, we are confronted with different kinds of difficulties in evaluating integrals analytically. Thus, an alternating technique becomes absolutely necessary in order to evaluate, which has given birth to the technique of numerical integration or mechanical quadrature.

In adopting this technique the exact value of the integral needs to be sacrificed and we have to be content with its approximate value. For this reason numerical integration is widely known as "Approximate integration".

## 1. Introduction

Singular integral of these two kinds:

$$I(f) = P \int_{-1}^1 \frac{f(x)}{x} \quad (1.1)$$

and

$$I(f, w) = P \int_{-1}^1 \frac{f(x)}{x} \cos wx dx \quad (1.2)$$

where,  $f(x)$  is analytic on  $[-1, 1]$  though look as similar but, the second differs from the former by its high oscillatory characteristics for large  $w \in \mathbb{R} - \{0\}$ . Further, this difference is very much noticed when both the integrals are numerically approximated. It is observed that classical quadrature rules meant for the numerical approximation of the CPV integral (1.1) diverge significantly and finally lead to uncontrolled instabilities when these rules are employed for the numerical integration of the second integral (1.2) with increasing values of  $|w|$ . It is well-known that these two improper integrals exist, if  $f$  satisfies the Holder's condition in  $[-1, 1]$ . Again, since these two integrals frequently appear in physical and applied sciences: image analysis, quantum physics, fluid dynamics, aerodynamics and engineering, etc. Thus they have attracted the attention of many researchers to devise quadrature rules for their numerical approximations. Some notable work Capobianco and Criscuolo (2003), Chung, Evans, and Webster (2000), Huybrechs and Vandewalle (2006), Keller (2012), Milovanović (1998), Okecha (1987), Wang and Xiang (2010) i.e. interpolating the function  $f$  by Lagrange's interpolation with nodes as zeros of Legendre polynomial and/(or not) the singular point; or using a special case of Hermite interpolation; or method based on interpolation at the zeros of orthogonal polynomial with Jacobi weight function; or interpolating at Clenshaw-Curtis points and expressing in terms of Chebyshev polynomial of first and second kind; or transforming the integral to an indefinite integral of oscillatory and singular function, etc. are recently seen in this field. However, to apply over an unknown integral, all these methods are not so simple and straight like standard quadrature rules (trapezoidal rule, Simpson's (1 = 3)rd rule or Gauss-Legendre  $n$ -point rules etc.) meant for the numerical integration of real definite integrals without having any kind of singularities. Thus, in this paper we have proposed a rapidly convergent simple numerical scheme consisting of a non-classical rule to approximate Filon-type integrals (Davis & Rabinowitz, 1984) which later incorporated with the standard classical quadrature rules meant for the numerical integration of integral (1.1) to approximate the integral (1.2). Further, we have compared the results of our numerical computation with the approximate results obtain by integrating the same integral by the scheme already proposed by Okecha (2006). The detailed comparison has been given in Table 1 with their absolute error for different values of  $w$ . Again, we have also determined the error bound of our proposed scheme which may be an extra advantage over (Okecha, 2006).

**Table 1. Computation and comparison with Okecha method of  $I = \int_{-1}^1 \frac{\cos wx}{x} e^x dx$  for different values of  $w$  and  $n$**

$w$	$n$	Approximate value of $I$	Absolute error	Approx. value of $I$ by Okecha	Absolute error
10	5	0.13343797422572656330	$9.0 \times 10^{-5}$	0.133437974225952854362	$9.3 \times 10^{-3}$
	10	0.13352779149519644619	$8.1 \times 10^{-9}$	0.13352779149536281456	$8.4 \times 10^{-7}$
	15	-0.13352779958744365185	$7.8 \times 10^{-16}$	-0.13352779958762387461	$8.1 \times 10^{-12}$
20	5	0.10797671522500618124	$3.8 \times 10^{-5}$	0.10797671522512056026	$3.9 \times 10^{-3}$
	10	0.10793877833910459074	$2.3 \times 10^{-9}$	0.1079389200162337660	$3.1 \times 10^{-7}$
	15	0.10793877604092738664	$4.0 \times 10^{-16}$	0.10793877604203667112	$5.2 \times 10^{-12}$
40	5	0.04351704158587210713	$5.1 \times 10^{-5}$	0.04362883452011123952	$6.0 \times 10^{-3}$
	10	0.04346609542204626919	$3.9 \times 10^{-9}$	0.04346609560023672920	$4.3 \times 10^{-7}$
	15	0.04346609152928015973	$1.2 \times 10^{-16}$	0.043466091529400216455	$1.2 \times 10^{-12}$
80	5	-0.029149138120060147372	$6.2 \times 10^{-5}$	-0.02927334519002385099	$6.2 \times 10^{-3}$
	10	-0.02921145635443256338	$5.2 \times 10^{-9}$	-0.02921146900112638226	$6.0 \times 10^{-7}$
	15	-0.02921146156050591358	$3.2 \times 10^{-16}$	-0.02921146156063002779	$4.0 \times 10^{-12}$
160	5	0.00325212124320817252	$5.7 \times 10^{-5}$	0.00338001174399272820	$6.6 \times 10^{-3}$
	10	0.00319527828384336132	$4.5 \times 10^{-9}$	0.003195279144282654403	$6.0 \times 10^{-7}$
	15	0.00319527375383055201	$2.7 \times 10^{-16}$	0.00319527375395883162	$3.3 \times 10^{-12}$
320	5	-0.00308047698190483387	$5.8 \times 10^{-5}$	-0.00314043556333108275	$7.0 \times 10^{-3}$
	10	-0.00313829547324173674	$3.6 \times 10^{-9}$	-0.00313840100023456700	$4.8 \times 10^{-7}$
	15	-0.00313830011593685398	$1.6 \times 10^{-16}$	-0.00313830011599923456	$1.6 \times 10^{-12}$

**2. Generalized classical rule for the numerical integration of  $I(f) = P \int_{-1}^1 \frac{f(x)}{x} dx$**

Price (1960) has given a four-point degree eight Gauss-type quadrature rule.

$$R_p(f) = \sum_{k=1}^2 w_k [f(w_k) - f(-x_k)] \tag{2.1}$$

with nodes and weights as depicted in Table 2 for  $n = 2$  to approximate the CPV integral (1.1) numerically. Latter, Chawla and Jayarajan (1975), Elliot and Paget (1979), Hunter (1972), Lebedev and Baburin (1965), Monegato (1982), Piessens (1970) have framed 2n-point rule.

$$Q_n(x) = \sum_{k=1}^n w_k [f(w_k) - f(-x_k)] \tag{2.2}$$

**Table 2. Nodes and weights of 2n-point Gauss-type quadrature rules for CPV integrals**

$n$	ADP	Nodes	Weights
1	1	0.577350269189626	1.732050807568877
2	8	0.861136311594053	0.403948643732766
		0.339981043584856	1.918180931460600
3	12	0.932469514203153	0.183732003855993
		0.661209386466264	0.545608668649091
		0.238619186083197	1.960923353453849
4	16	0.960289856497541	0.105414563743876
		0.796666477413669	0.279139440102061
		0.525532409916391	0.596931112065570
		0.183434642495682	1.977182599993103

$$\text{where, } w_k = \int_{-1}^1 \frac{P_n(x)}{P_n'(x_k)(x-x_k)} dx$$

for  $i = 1(1)n$ ; of precision  $\leq 4n$  by interpolating the function  $f(x)$  by Lagrange's interpolation at the zeros  $0 < x_k < 1$  of orthogonal polynomial  $P_n(x)$  as the nodes of interpolant. Though, analytically it looks simple and straight to determine the weights with the evaluated nodes but, numerically the computational complexity increases with increasing  $n$ . Therefore, in this section we suggest an algebraic method to determine the nodes and weights of the generalized quadrature rule  $Q_n(x)$  meant for the numerical approximation of the CPV integral (1.1). Here, we assume that the proposed 2n-point rule  $Q_n(f)$  meant for the numerical integration of the CPV integral (1.1) is based on the roots  $\pm x_i$ ;  $x_i \in (0, 1)$ ;  $\forall i = 1(1)n$  of the monic polynomial

$$P_{2n}(x) = \prod_{i=1}^n (x^2 - x_i^2) = \sum_{k=0}^n c_{2k} x^{2k}; c_{2n} = 1$$

Since, the rule  $Q_n(f)$  is a fully symmetric quadrature rule and exactly integrates all monomials of even degree thus, in order to determine the weights  $w_i$  and the nodes  $x_i$ ; for  $i = 1(1)n$ ; it is assumed here that

$$E(x^j) = I(x^j) - Q_n(x^j) = 0; \forall j = 1(2)(2n - 1)$$

Now with this assumption the following system of linear equations

$$AW = M \tag{2.3}$$

$$\text{where } \begin{cases} A = (\eta_{ij}); W = (w_i)^T; M = (m_i)^T \\ \eta_{ij} = x_j^{(2i-1)}; m_i = \frac{1}{2i-1}; \forall i, j = 1(1)n \end{cases}$$

with  $2n$  unknowns is obtained. Since for all  $j = 1(1)n$ ;

$$\sum_{k=0}^n c_{2k} m_{k+j} = \sum_{i=1}^n w_i x_i^{(2j-1)} P_{2n}(x_i) = 0 \Rightarrow \sum_{k=0}^{n-1} c_{2k} m_{k+j} = -m_n + j_i \tag{2.4}$$

thus, solving the above system of Equation (2.4) for the  $n$ -unknowns  $c_{2i-2}$ ;  $\forall i = 1(1)n$ , we get

$$[c_{2i-2}]^T = -[\theta_{ij}]^{-1} [m_{n+i}]^T$$

as the coefficients of the monic polynomial  $P_{2n}(x)$ ; where  $\theta_{ij} = m_{i+j-1}$ ;  $\forall i, j = 1(1)n$ : It is to be noted here that the system  $[\theta_{ij}]$  is non-singular since its determinant is the Gram determinant, i.e.

$$D = |[\theta_{ij}]| = |m_{i+j-1}| = \left| \int_{-1}^1 x^{i+j-1} dx \right|$$

of the linearly independent functions  $1; x^2; \dots; x^{2n-2}$  (Davis, 1963). As a result, the system may be solved uniquely for the coefficients  $c_{2i-2}$ ;  $\forall i = 1(1)n$  and hence the nodes  $x_i \in (0, 1)$ ;  $\forall i = 1(1)n$ ; of the quadrature rule  $Q_n(f)$  may be obtained from the equation  $P_{2n}(x) = 0$ ; all of which will be real and simple. Now having these nodes, the weights  $w_i$ ;  $\forall i = 1(1)n$ ; of the quadrature rule  $Q_n(f)$  for different  $n \geq 1$  may be obtained from the system (2.3). The nodes and weights of the family of rules  $Q_n(f)$ ; for different  $n = 1, 2, 3, \dots, 12$  are given in Tables 2-9 with their respective Algebraic Degree of Precision (ADP). It is to be noted here that the rule due to Price (1960) is obtained from the rule  $Q_n(x)$  for  $n = 2$ :

**Table 3. Nodes and weights of  $2n$ -point Gauss-type quadrature rules for CPV integrals**

$n$	ADP	Nodes	Weights
5	20	0.973906528517160	0.068457641833702
		0.865063366688886	0.172763470175274
		0.679409568298880	0.322465818467250
		0.433395394129139	0.621295756617324
		0.148874338981603	1.985058182197562
6	24	0.981560634246462	0.048061561090791
		0.904117256369162	0.118280372643593
		0.769902674190968	0.207920213699644
		0.587317954281241	0.345924086337482
		0.367831498993049	0.634781244063258
		0.125233408509444	1.989461508513519

**Table 4. Nodes and weights of  $2n$ -point Gauss-type quadrature rules for CPV integrals**

$n$	ADP	Nodes	Weights
7	28	0.986283808687120	0.035607864639996
		0.928434883613854	0.086336789611788
		0.827201314953169	0.146903261112928
		0.687292904618987	0.228728052003020
		0.515248636116577	0.360094883304720
		0.319112368715717	0.643028863051947
		0.108054948621674	1.992170243434278

**Table 5. Nodes and weights of  $2n$ -point Gauss-type quadrature rules for CPV integrals**

$n$	ADP	Nodes	Weights
8	32	0.989400935145512	0.027443332680608
		0.944575023870436	0.065906382749733
		0.865631204262509	0.109929621256614
		0.755404411571257	0.164983111002224
		0.617876248834496	0.242113187153308
		0.458016782509084	0.369323845930055
		0.281603554633516	0.648441448307212
		0.095012511310392	1.993954382308664

**Table 6. Nodes and weights of  $2n$ -point Gauss-type quadrature rules for CPV integrals**

$n$	ADP	Nodes	Weights
9	36	0.991565172982806	0.021799880115863
		0.955823972910600	0.052012216510590
		0.892602520591791	0.085621195799762
		0.803705049935446	0.125595835175727
		0.691687167759437	0.177182958035649
		0.559770974370531	0.251250804747739
		0.411751296855497	0.375675073248891
		0.251886322537489	0.652185217457345
		0.084775048085327	1.995191456441141

**Table 7. Nodes and weights of 2n-point Gauss-type quadrature rules for CPV integrals**

<b>n</b>	<b>ADP</b>	<b>Nodes</b>	<b>Weights</b>
10	40	0.993128481353814	0.017736182480252
		0.963971320096234	0.042119613103321
		0.912232997438596	0.068702852775516
		0.839114486021899	0.099244936468853
		0.746328296717370	0.136576900639120
		0.636049118107663	0.185827311429501
		0.510861976453832	0.257777536876273
		0.373701411831024	0.380237469492232
		0.227782493373115	0.654884411744119
		0.076525295117100	1.996084968526942

**Table 8. Nodes and weights of 2n-point Gauss-type quadrature rules for CPV integrals**

<b>n</b>	<b>ADP</b>	<b>Nodes</b>	<b>Weights</b>
11	44	0.994295819358128	0.014708731388521
		0.970066984508076	0.034809525059847
		0.926972617496419	0.056400795869904
		0.865841586966162	0.080593973311720
		0.787861936709951	0.109060394059333
		0.694549513754975	0.144550649801581
		0.587717132944936	0.192134598899053
		0.469439178192063	0.262550708654742
		0.342012625060904	0.383575946546346
		0.207915122419338	0.656859018272333
0.069759137829373	1.996738422555679		

**Table 9. Nodes and weights of 2n-point Gauss-type quadrature rules for CPV integrals**

<b>n</b>	<b>ADP</b>	<b>Nodes</b>	<b>Weights</b>
12	48	0.995084870605378	0.012665633140288
		0.974193852151391	0.029902794784635
		0.936981745338937	0.048226832903149
		0.884077129137604	0.068397534219232
		0.816396910841891	0.091489925899187
		0.735138994928865	0.119130261830653
		0.641782131070862	0.154013398383860
		0.538085903572963	0.201102672048141
		0.426078216999278	0.270913128281256
		0.307988642888426	0.390825501277473
		0.186055967207425	0.662007852138094
		0.062200292463922	1.998634975417466

### 3. Scheme for the approximate evaluation of Cauchy type oscillatory integral

$$J(f; w) = P \int_{-1}^1 \frac{\cos wx}{x} f(x) dx$$

For the construction of the scheme, the integral  $J(f; w)$  is rewritten as follows:

$$J(f; w) = \int_{-1}^1 \frac{\cos wx - 1}{x} f(x) dx + P \int_{-1}^1 \frac{f(x)}{x} dx = J_0(f) + I(f) \tag{3.1}$$

where

$$J_0(f) = \int_{-1}^1 \frac{\cos wx - 1}{x} f(x) dx \tag{3.2}$$

is a Filon-type oscillatory, but truly non-singular integral and

$$I(f) = P \int_{-1}^1 \frac{f(x)}{x} dx$$

is the singular integral as given in Equation (1.1) which can be numerically approximated by the proposed quadrature rule  $Q_n(f)$  for suitable  $n$ . Further, since large value of  $|w|$  causes high oscillations of the integrand function of the first integral  $J_0(f)$  and due to its very fast oscillation characteristic the result of its numerical approximation very often adversely affects to the result of approximation of the second singular integral  $I(f)$  thus, a desirable accuracy for the integral  $J(f, w)$  may not be achieved. Therefore, in order to circumvent this difficulty that appears in this section, we propose a rapidly converging quasi-exact method which almost exactly integrates the integral  $J_0(f)$ .

#### 3.1. The proposed quasi-exact method

To construct the method, we assume here that the function  $f(x)$  is continuous up to its maximum derivatives in  $[-1, 1]$ . Now with this assumption expanding  $f(x)$  by using Taylor's expansion about the singular point  $x = 0$ , we get

$$f(x) = \sum_{k=0}^{\infty} c_k x^k$$

where  $c_k = \frac{f^{(k)}(0)}{k!}$ ,  $\forall k = 0, 1, 2, \dots$  are the Taylor's coefficients. Truncating the above series after the first  $(n + 1)$  terms, the interpolating polynomial  $g_n(x)$  with the interpolating conditions

$$g_n^{(i)}(x) = f^{(i)}(x); \forall_i = 0(1)n$$

is obtained as:

$$g_n(x) = f(0) + \sum_{k=1}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

Using standard method (Atkinson, 1989) it can be proved that the truncation error  $\tilde{E}_n(f)$  associated with the polynomial  $g_n(x)$  is

$$\tilde{E}_n(f) = \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(\xi)$$

for  $\xi \in [-1, 1]$ . Now as  $f(x) \cong g_n(x)$ ; thus,

$$\begin{aligned}
 J_0(f) &= \int_{-1}^1 \frac{\cos wx - 1}{x} f(x) dx \cong \int_{-1}^1 \frac{\cos wx - 1}{x} g_n(x) dx \\
 &= \sum_{k=1}^n \frac{F^{(k)}(0)}{k!} \int_{-1}^1 x^{k-1} (\cos wx - 1) dx = \sum_{k=1}^n \frac{F^{(k)}(0)}{k!} \left[ \int_{-1}^1 x^{k-1} \left( \cos wx dx - \frac{1 - (-1)^k}{k} \right) \right]
 \end{aligned} \tag{3.3}$$

**THEOREM 3.1.** If  $a_{k-1} = \int_{-1}^1 x^{k-1} \cos wx dx$  and  $a_i = 0$ ; for  $i \in Z$ ;  $i < 0$ ; then

$$w^2 a_{k-1} + (k-1)(k-2)a_{k-3} = [1 + (-1)^{k-1}]w \sin w + (k-1) \cos w \tag{3.4}$$

the non-homogenous linear recurrence relation holds  $\forall k = 1(1)n$ .

*Proof* The proof of Theorem 3.1 is very straight and can be established by following the method of integration by parts.

As a result, the integral  $J_0(f)$  can be numerically evaluated as

$$J_0(f) \cong \sum_{k=1}^n \frac{f^{(k)}(0)}{k!} a_{k-1} \tag{3.5}$$

with

$$a_0 = \frac{2 \sin w}{w}, a_i = 0;$$

for all  $k$  as positive odd integers. However, the recurrence relation given in Equation (3.4) can be rewritten as:

$$a_k + \rho k(k-1)a_{k-2} = \rho \sigma_k; \forall k = 2, 3, \dots \tag{3.6}$$

where

$$\rho = \frac{1}{w^2}, \sigma_k = [1 + (-1)^k] \left\{ \frac{\sin w}{w} + k \frac{\cos w}{w^2} - \frac{1}{k+1} \right\}$$

On solving by following standard method of solution of recurrence relation, we obtain its particular solution as:

$$a_k = \begin{cases} k! \left[ \sum_{m=0}^{(k/2)-1} \left\{ 2_{q_0} \frac{(-1)^m}{(k-2m)! w^{2m}} + 2_{q_1} \frac{(-1)^m}{(k-2m-1)! w^{2m}} \right\} \right] & \text{for } k \text{ even} \\ 0; & \text{for } k \text{ odd} \end{cases}$$

where  $q_0 = \frac{\sin w}{w}; q_1 = \frac{\cos w}{w^2}$

Here, with the values of  $a_k$  for different  $k$ , we denote the right side of the Equation (3.5) as the quadrature rule

$$R_0(f) = \sum_{k=1}^n \frac{f^{(k)}(0)}{k} \left[ \sum_{m=0}^{(k/2)-3/2} \left\{ 2_{q_0} \frac{(-1)^m}{(k-2m-1)! w^{2m}} + 2_{q_1} \frac{(-1)^m}{(k-2m-2)! w^{2m}} \right\} + (-1)^{(k-1)/2} \frac{2q_0}{w^{k-1}} \right] \tag{3.7}$$

meant for the approximate evaluation of the integral  $J_0(f)$ : Therefore, from Equations (2.2), (3.1) and (3.7); the integral  $J(f)$  can be numerically evaluated as:

$$J(f) \cong S(f) = R_0(f) + Q_n(f) \tag{3.8}$$

**4. Error analysis**

Here we assume that the function  $f(x)$  is infinitely differentiable in the interval of integration  $[-1, 1]$ . Now with this assumption denoting  $E_J(f)$  as the error associated with the scheme  $S(f)$  meant for the numerical integration of the Cauchy type oscillatory integral  $J(f)$  (Equation (3.1)) is obtained as:

$$|E_J(f)| \leq |J_0(f) - R_0(f)| + |I(f) - Q_n(f)| = |E_0(f)| + |E_n(f)| \tag{4.1}$$

where,

$$E_0(f) = J_0(f) - R_0(f)$$

and

$$E_n(f) = I(f) - Q_n(f)$$

are the error terms associated with the quadrature rules  $R_0(f)$  and  $Q_n(f)$  meant for the approximate evaluation of the Fylon type oscillatory integral  $J_0(f)$  (Equation (3.2)) and the CPV integral  $I(f)$  (Equation (1.1)), respectively. However,

$$|E_0(f)| = \left| \frac{f^{(n+1)}(\xi)}{(n+1)!} \int_{-1}^1 x^n (\cos wx - 1) dx \right| \leq \frac{M_{n+1}}{(n+1)!} \left[ \left| \int_{-1}^1 x^n \cos wx dx \right| + \frac{2}{n+1} \right]$$

where

$$M_{n+1} = \text{Max}_{\xi \in (-1, 1)} |f^{(n+1)}(\xi)|.$$

Further,

$$\int_{-1}^1 x^n \cos wx dx = \frac{1}{w} \left[ \{x^n \sin wx\}_{-1}^1 - n \int_{-1}^1 x^{n-1} \cos wx dx \right]$$

$$\begin{aligned} \left| \int_{-1}^1 x^n \cos wx dx \right| &\leq \frac{1}{|w|} \left[ 2 + n \int_{-1}^1 |x|^{n-1} dx \right] = \frac{1}{|w|} \left[ 2 + n \left\{ \int_{-1}^0 |x|^{n-1} dx + \int_0^1 |x|^{n-1} dx \right\} \right] \\ &= \frac{1}{|w|} \left[ 2 + 2n \int_0^1 |x|^{n-1} dx \right] = \frac{4}{|w|} \end{aligned}$$

As a result,

$$|E_0(f)| \leq \frac{M_{n+1}}{(n+1)!} \left[ \frac{4}{|w|} + \frac{2}{n+1} \right] \tag{4.2}$$

Again, since the quadrature rule  $Q_n(x)$  exactly integrates all polynomial of degree  $\leq 4n$  thus, by the help of the Taylor's series it can be shown that the corresponding error

$$|E_n(f)| \leq \frac{2M_{4n+1}}{(4n+1) \times (4n+1)!} \tag{4.3}$$

where  $M_{4n+1} = \text{Max}_{\zeta \in (-1, 1)} |f^{(4n+1)}(\zeta)|$ . Now for  $M = \text{Max}_{\xi, \zeta \in (-1, 1)} \{M_{n+1}, M_{4n+1}\}$ ; by Equations (4.1), (4.2) and (4.3)

$$|E_J(f)| \leq M \left[ \frac{4}{(n+1)!|w|} + \frac{2}{(n+1)! \times (n+1)} + \frac{2}{(4n+1)! \times (4n+1)} \right] \leq \frac{4M}{(n+1)} \left[ \frac{1}{|w|} + \frac{1}{(n+1)} \right]$$

which  $\rightarrow 0$ ; for  $|w|$ ;  $n \rightarrow \infty$ . This fact is also very much observed in the numerical integration of some standard such integrals.

## 5. Numerical experiments

To test the accuracy of the scheme, the integral considered here as:

$$I = \int_{-1}^1 \frac{\cos wx}{x} e^x dx$$

for different  $w$ . The result of their numerical approximations is depicted in Table 1.

## 6. Conclusion

Cauchy-type oscillatory integral has numerically evaluated in this paper. It has been observed that a stable accuracy can be achieved with a fixed number of points for any value of  $w$ . Thus, the proposed mixed scheme (classical + non-classical) may be considered as a suitable method for the numerical approximations of this type of integrals. Further, we compared our scheme with the scheme already has been devised by Okecha (2006). It is observed that the scheme formulated in this paper provides more accurate result than the former.

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