cogent mathematics

APPLIED & INTERDISCIPLINARY MATHEMATICS | RESEARCH ARTICLE

Generalizations on Humbert polynomials and functions

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Received: 19 October 2016
Accepted: 20 March 2017
First Published: 24 March 2017

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Reviewing editor:
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Abstract: By starting from the standard definitions of the incomplete two-variable Hermite polynomials, we propose non-trivial generalizations and we show some applications to the Bessel-type functions as the Humbert functions. We present some relevant relations linking the Bessel-type functions, the Humbert functions and the generalized Hermite polynomials. We also present a generalization of the Laguerre polynomials in the same context of the incomplete-type and we see some useful operational relations involving special functions as the Tricomi functions; we use this family of Laguerre polynomials to introduce some operational techniques for the Humbert-type functions. Finally, we note as the formalism discussed could be exploited to generalize some distribution as for example the Poisson type.

Subjects: Advanced Mathematics; Applied Mathematics; Foundations & Theorems

Keywords: Hermite-polynomials; Laguerre-polynomials; generating-functions; orthogonal polynomials and special functions

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PUBLIC INTEREST STATEMENT

The theory of multidimensional or multi-index special functions is very relevant field of investigations to simplified a wide range of operational relations. It has also been showed that Hermite polynomials play a fundamental role in the extension of the classical special functions to the multidimensional case. The present paper is to be considered in the above mentioned framework, and is devoted to the derivation of the main properties of Humbert polynomials and functions, by using the generalized Hermite polynomials as tool. Starting from the Hermite polynomials, it has already been possible to obtain some extensions of some classical special sets of functions, including: the Bessel functions, Tricomi functions and Laguerre polynomials, and some new families of polynomials, called hybrid, since they verify properties which are typical both of the above mentioned polynomials and functions. Moreover, the formalism presented could be exploited to generalize some example of Poisson distribution.

1. Introduction

In a previous paper, Cesarano, Fornaro, and Vazquez (2015) and Dattoli, Srivastava, and Zhukovsky (2004) we have introduced a generalization of the Hermite polynomials which are a vectorial extension of the ordinary Kampé de Feriét one-variable Hermite polynomials. We have indicated this class of the Hermite polynomials, of two-index and two-variable, by the symbol $He_{m,n}(x, y)$, and we stated their definition through the following generating function:

$$e^{\underline{h}^t \hat{M} \underline{z} - \frac{1}{2} \underline{h}^t \hat{M} \underline{h}} = \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \frac{t^m}{m!} \frac{u^n}{n!} He_{m,n}(x, y), \tag{1}$$

where

$$\underline{z} = \begin{pmatrix} x \\ y \end{pmatrix} \text{and } \underline{h} = \begin{pmatrix} t \\ u \end{pmatrix}$$

are two vectors of the space $\mathbb{R}^2$ such that: $t \neq u$, $(|t|, |u|) < +\infty$, and the superscript "t" denotes transpose.

A different generalization of the Hermite polynomials could be obtained by using the slight similar procedure onto the two-variable generalized Hermite polynomials (Appell & Kampé de Fériet, 1926; Cesarano, Fornaro, & Vazquez, 2016).

$$H_n(x, y) = n! \sum_{r=0}^{\left[n/2\right]} \frac{y^r x^{n-2r}}{r!(n-2r)!}, \tag{2}$$

defined by the generating function of the form:

$$\exp(xt + yt^2) = \sum_{n=0}^{+\infty} \frac{t^n}{n!} H_n(x, y). \tag{3}$$

Let u and v continuous variables, such that $u \neq v$ and $(|u|, |v|) < +\infty$, $\tau \in \mathbb{R}$, we will say incomplete two-dimensional Hermite polynomials, the polynomials defined by following generating function:

$$\exp(xu + yv + \tau uv) = \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \frac{u^m}{m!} \frac{v^n}{n!} h_{m,n}(x, y|\tau). \tag{4}$$

This class of Hermite polynomials has been deeply studied for its importance in applications (Dattoli, Srivastava, & Khan, 2005; Lin, Tu, & Srivastava, 2001), as quantum mechanical problems, harmonic oscillator functions and also to investigate the statistical properties of chaotic light (Dodonov & Man'ko, 1994).

By using the techniques of the generating function method (Gould & Hopper, 1962; Srivastava & Manocha, 1984), it is easy to obtain the explicit form of the above polynomials.

$$h_{m,n}(x, y|\tau) = m!n! \sum_{r=0}^{[m,n]} \frac{\tau^r x^{m-r} y^{n-r}}{r!(m-r)!(n-r)!}, \tag{5}$$

where $[m, n] = \min(m, n)$.

An interesting particular case of this class of Hermite polynomials is presented when $x = y = 1$ and $\tau = x$:

$$g_{m,n}(x) = h_{m,n}(1, 1|x). \tag{6}$$

It is significant to study the polynomials $g_{m,n}(x)$ since they can be used to define other forms of the incomplete two-dimensional Hermite polynomials of the type $h_{m,n}(.,.|.)$ themselves and since they often appear in the description of the applications in quantum optics (Arnoldus, 1999). From the relation (6) and by using definitions (4) and (5), we can immediately write the following general relation:

$$h_{m,n}(x,y|\tau) = x^m y^n g_{m,n}\left(\frac{\tau}{xy}\right). \tag{7}$$

The incomplete two-dimensional Hermite polynomials can be used to obtain different forms of the multi-index Bessel functions, in particular for the case of the Humbert functions. We remind that the ordinary cylindrical Bessel functions (Watson, 1958) are specified by the generating function:

$$\exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)\right] = \sum_{n=-\infty}^{+\infty} t^n J_n(x), \tag{8}$$

and a generalization of them, it is represented by the case of two-index, one-variable type (Cesarano & Assante, 2013; Dattoli & Torre, 1996).

$$J_{m,n}(x) = \sum_{s=-\infty}^{+\infty} J_{m-s}(x) J_{n-s}(x) J_s(x), \tag{9}$$

with the following generating function:

$$\exp\left[\frac{x}{2}\left(u-\frac{1}{u}\right)+\left(v-\frac{1}{v}\right)\left(uv-\frac{1}{uv}\right)\right] = \sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} u^m v^n J_{m,n}(x), \tag{10}$$

where $x \in \mathbb{R}$ and $u, v \in \mathbb{R}$, such that $0 < |u| \neq |v| < +\infty$.

This class of Bessel functions satisfied analogous interesting relations as the ordinary Bessel functions. For instance, by differentiating in the Equation (10) with respect to x, we have:

$$\frac{1}{2}\left(u-\frac{1}{u}\right)+\left(v-\frac{1}{v}\right)\left(uv-\frac{1}{uv}\right)\sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} u^m v^n J_{n,m}(x) = \sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} u^m v^n \frac{d}{dx} J_{m,n}(x), \tag{11}$$

which allows us to state the following recurrence relation:

$$\frac{d}{dx} J_{m,n}(x) = \frac{1}{2}\left\{\left[J_{m-1,n}(x) - J_{m+1,n}(x)\right] + \left[J_{m,n-1}(x) - J_{m,n+1}(x)\right] + \left[J_{m-1,n-1}(x) - J_{m+1,n+1}(x)\right]\right\}. \tag{12}$$

By using the same procedure, it is easy to obtain the other two recurrence relations for this class of Bessel functions:

$$\frac{2m}{x} J_{m,n}(x) = \left[J_{m,n-1}(x) - J_{m,n+1}(x)\right] + \left[J_{m-1,n-1}(x) - J_{m+1,n+1}(x)\right], \tag{13}$$

and

$$\frac{2n}{x} J_{m,n}(x) = \left[J_{m,n-1}(x) - J_{m,n+1}(x)\right] + \left[J_{m-1,n-1}(x) - J_{m+1,n+1}(x)\right]. \tag{14}$$

It is interesting to note that, for $x = 0$, from the explicit form of the generalized two-index Bessel function (Equation (9)), we get:

$$J_{m,n}(0) = \sum_{s=-\infty}^{+\infty} J_{m-s}(0) J_{n-s}(0) J_s(0), \tag{15}$$

and, since

$$J_s(0) \neq 0, \text{when } s = 0 \tag{16}$$

we, finally obtain

$$J_{m,n}(0) = \delta_{m,0}\delta_{n,0}. \tag{17}$$

As a particular case of the two-index, one-variable Bessel functions, we can introduce the Humbert functions (Aktaş, Şahin, & Altın, 2011), by setting

$$b_{m,n}(x,y|\tau) = \sum_{r=0}^{+\infty} \frac{\tau^r x^{m+r} y^{n+r}}{r!(m+r)!(n+r)!}, \tag{18}$$

defined through the following generating function:

$$\exp(xu + yv + \tau uv) = \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty} u^m v^n b_{m,n}(x,y|\tau). \tag{19}$$

It is evident, the similar structure between these functions and the incomplete two-dimensional Hermite polynomials presented previously (see Equations (4) and (5)). For this reason (Dattoli, Srivastava, & Sacchetti, 2003), the Humbert functions are usual exploited in connection with the Hermite polynomials of the type $h_{m,n}(x, y|\tau)$.

We can immediately note, for instance, that the Humbert functions could be expressed in terms of the incomplete Hermite polynomials. By rewriting, in fact, the expression in Equation (19), we have:

$$\exp\left[xu + yv + \tau uv - \tau\left(uv - \frac{1}{uv}\right) - \frac{\tau}{uv}\right] = \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty} u^m v^n b_{m,n}(x,y|\tau), \tag{20}$$

and, from the generating function of the ordinary Bessel function (Equation (8)), we find

$$b_{m,n}(x,y|\tau) = \sum_{r=0}^{[m,n]} \frac{(-1)^r h_{m-r,n-r}(x,y|\tau) J_r(2\tau)}{r!(m-r)!(n-r)!}. \tag{21}$$

In the following, we will indicate with:

$g_{m,n}(x)$ and $b_{m,n}(x)$,

the Humbert polynomials and the Humbert functions, respectively. In the next sections, we will study the properties of these particular polynomials and functions and we will see some their non-trivial generalizations along with the analysis of the related applications to facilitate some operational computation.

2. Relevant properties for Humbert polynomials and functions

In Section 1, we have introduced the incomplete two-dimensional Hermite polynomials through the relations (4) and (5). By using the equivalences stated in Equations (6) and (7), we can now state the expression of the generating function for the Humbert polynomials $g_{m,n}(x)$. We have (Srivastava & Djordjević 2011).

$$\exp(u + v + xuv) = \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty} \frac{u^m}{m!}\frac{v^n}{n!} g_{m,n}(x) \tag{22}$$

where, again, $u \neq v$ and $(|u|, |v|) < +\infty$, $\tau \in \mathbb{R}$.

By following the same procedure used to derive the recurrence relations related to the two-index, one-variable Bessel function in the previous section, we can find similar expressions for this class of Humbert polynomials. In fact, by deriving, respectively, with respect to x, u and v, we obtain:

$$\begin{aligned}&\tfrac{d}{dx}g_{m,n}(x)=mng_{m-1,n-1}(x),\\&g_{m+1,n}(x)=g_{m,n}(x)+mxg_{m-1,n}(x),\\&g_{m,n+1}(x)=g_{m,n}(x)+nxg_{m,n-1}(x).\end{aligned}\tag{23}$$

By using the above relations, it is possible to state the differential equation satisfied by the polynomials $g_{m,n}(x)$. After an easy manipulation of the equations in the (23), we get:

$$mg_{m-1,n}(x)=\left(m-x\frac{d}{dx}\right)g_{m,n}(x),\tag{24}$$

$$ng_{m,n-1}(x)=\left(n-x\frac{d}{dx}\right)g_{m,n}(x)\tag{25}$$

and also

$$mng_{m-1,n-1}(x)=\left(m-x\frac{d}{dx}\right)\left(n-x\frac{d}{dx}\right)g_{m,n}(x).\tag{26}$$

After equating Equation (24) with the first of the relations obtained in (23), we can state the following differential equation solved by the Humbert polynomials:

$$x^2y''-\left[(m+n-1)x+1\right]y'+mny=0.\tag{27}$$

We note that, from the Equation (25), it also follows:

$$\begin{aligned}&g_{m+1,n}(x)=g_{m,n}(x)+nxg_{m,n}(x)-x^2\tfrac{d}{dx}g_{m,n}(x),\\&g_{m,n+1}(x)=g_{m,n}(x)+mxg_{m,n}(x)-x^2\tfrac{d}{dx}g_{m,n}(x)\end{aligned}\tag{28}$$

Which suggest the introduction of the following operators:

$$\begin{aligned}&\hat{S}_m^{+}=1+\hat{n}x-x^2\tfrac{d}{dx},\\&\hat{S}_n^{+}=1+\hat{m}x-x^2\tfrac{d}{dx},\end{aligned}\tag{29}$$

where we have denoted with the symbols:

$\hat{m}$ and $\hat{n}$

a kind of number operators, in the sense that their action read as following:

$$\hat{m}\hat{n}g_{s,r}(x)=srg_{s,r}(x).$$

It is now evident, by using the relations stated in the Equations (23)–(28) and by the definition of the operators expressed in Equation (29), that the following expressions hold:

$$\begin{aligned}&\hat{S}_m^{+}g_{m,n}(x)=g_{m+1,n}(x),\\&\hat{S}_n^{+}g_{m,n}(x)=g_{m,n+1}(x).\end{aligned}\tag{30}$$

The above relations, combined with the first in the Equation (23), allow us to state the following relevant differential equation:

$$\tfrac{d}{dx}\left[1+(n+1)x-x^2\tfrac{d}{dx}\right]\left[1+mx-x^2\tfrac{d}{dx}\right]g_{m,n}(x)=(m+1)(n+1)g_{m,n}(x).\tag{31}$$

It is possible to derive similar relations regarding the Humbert functions. Before to proceed, we remind that, the function defined by the following generating function:

$$\exp\left(t - \frac{x}{t}\right) = \sum_{n=-\infty}^{+\infty} t^n C_n(x) \tag{32}$$

is known as the Tricomi function (Cesarano, Cennamo, & Placidi, 2014b), which its explicit form is

$$C_n(x) = \sum_{r=0}^{+\infty} \frac{(-1)^r x^r}{r!(n+r)!}. \tag{33}$$

It is possible to introduce a generalization of the above function in the sense of the Humbert functions. In fact, from the Equation (18) it is immediately recognized that we can call generalized Tricomi function, the function expressed by the following relation:

$$C_{m,n}(x) = b_{m,n}(1, 1|x). \tag{34}$$

By using the same procedure outlined above, we can derive, for the Humbert functions the analogous recurrence relations stated for the polynomials $g_{m,n}(x)$. In fact, by considering the relation (34), we have:

$$\begin{aligned} &\tfrac{d}{dx}C_{m,n}(x) = C_{m+1,n+1}(x), \\ &mC_{m,n}(x) = C_{m-1,n}(x) - xC_{m+1,n+1}(x), \\ &nC_{m,n}(x) = C_{m,n-1}(x) - xC_{m+1,n+1}(x). \end{aligned} \tag{35}$$

We can combine the above relations, to get

$$\begin{aligned} &C_{m-1,n}(x) = \left(m + x\tfrac{d}{dx}\right)C_{m,n}(x), \\ &C_{m,n-1}(x) = \left(n + x\tfrac{d}{dx}\right)C_{m,n}(x). \end{aligned} \tag{36}$$

These last relations suggest to introduce similar operators acting on these generalized Tricomi function as well as we have done for the Humbert polynomials. We have indeed

$$\begin{aligned} &\hat{E}_m^- = \hat{m} + x\tfrac{d}{dx}, \\ &\hat{E}_n^- = \hat{n} + x\tfrac{d}{dx}, \\ &\hat{S}_{m,n}^{+,+} = \tfrac{d}{dx}. \end{aligned} \tag{37}$$

We have used, again, the same notation as expressed for the operators in Equation (29). By following the same procedure used for the Humbert polynomials, we can easily to state the following differential equation (Srivastava, Özarslan, & Yılmaz, 2014).

$$x^2 y''' - (m+n+3)xy'' + (mn+m+n+1)y' = y. \tag{38}$$

3. Further generalizations for Humbert polynomials and functions and incomplete Laguerre polynomials

In the paper (Cesarano et al., 2014b), we have showed some relations linked the cylindrical Bessel function and the Tricomi function; in particular, we have seen that:

$$x^{-\frac{n}{2}} J_0(2\sqrt{x}) = C_n(x) = \sum_{r=0}^{+\infty} \frac{(-1)^r x^r}{r!(n+r)!} \tag{39}$$

The relations stated in the previous sections and the structure of the 0*th* order Tricomi function, allow us to introduce a generalization of the Laguerre polynomials.

We will say incomplete two-dimensional Laguerre polynomials, the polynomials defined by the following generating function:

$$\exp(u+v)C_0(xuv) = \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty} \frac{u^m}{m!}\frac{v^n}{n!} l_{m,n}(x), \tag{40}$$

where their explicit form reads:

$$l_{m,n}(x) = m!n! \sum_{r=0}^{[m,n]} \frac{(-1)^r x^r}{(r!)^2 (m-r)!(n-r)!}. \tag{41}$$

It is evident the similar structure with the Humbert polynomials discussed in Sections 2.

We remind that the ordinary Laguerre polynomials (Cesarano et al., 2014b) have the following operational expression:

$$L_n(x) = \left(1 - \hat{D}^{-1}\right)^n, \tag{42}$$

where $\hat{D}_x^{-1}$ denotes the inverse of the derivative operator (Dattoli & Cesarano, 2003; Dattoli, Lorenzutta, Ricci, & Cesarano, 2004; Dattoli, Srivastava, & Cesarano, 2001), being essentially an integral operator, it will be specified by the operational rule:

$$\hat{D}_x^{-1}(1) = \frac{x^n}{n!}. \tag{43}$$

From the above considerations, we can firstly write the following expression for the 0*th* Tricomi function:

$$C_0(x) = \sum_{r=0}^{+\infty} \frac{(-1)^r \hat{D}_x^{-r}}{r!} = \exp\left(-\hat{D}_x^{-1}\right), \tag{44}$$

and then we can state the important equivalence between the Humbert polynomials (Cesarano, Cennamo, & Placidi, 2014a) and the incomplete two-dimensional Laguerre polynomials, that is:

$$l_{m,n}(x) = g_{m,n}\left(-\hat{D}_x^{-1}\right). \tag{45}$$

By recalling the explicit forms of the generalized two-variable Laguerre polynomials:

$$L_n(x,y) = \left(y - \hat{D}_x^{-1}\right)^n = n! \sum_{r=0}^{n} \frac{(-1)^r y^{n-r} x^r}{(r!)^2 (n-r)!} \tag{46}$$

and from the particular expression of their generating function, in terms of the Tricomi function:

$$\sum_{n=0}^{+\infty} \frac{t^n}{n!} L_n(x,y) = \exp(yt) C_0(xt), \tag{47}$$

we can finally establish a link between the incomplete two-dimensional Laguerre polynomials and the generalized Laguerre of the form $L_n(x, y)$. We have:

$$l_{m,n}(x,y) = m!n! \sum_{r=0}^{[m,n]} \frac{g_{m-r,n-r}(-y) L_r(x,y)}{r!(m-r)!(n-r)!}. \tag{48}$$

The considerations and the following results obtained to define the incomplete two-dimensional polynomials, can be used to introduce a similar generalization for the Humbert functions.

By considering indeed the following generating function:

$$\exp(u+v)C_0\left(\frac{x}{uv}\right)=\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}u^m v^n A_{m,n}(x), \tag{49}$$

we easily obtain the explicit form of the function $A_{m,n}(x)$:

$$A_{m,n}(x)=\sum_{r=0}^{+\infty}\frac{(-1)^r x^r}{(r!)^2(m+r)!(n+r)!}. \tag{50}$$

It is easy to note the analogy between the above expression and the generalized Tricomi function presented in the previous section. We find in fact:

$$A_{m,n}(x)=C_{m,n}\left(-\hat{D}_x^{-1}\right). \tag{51}$$

In the same way, it is possible obtain an expression of the functions $A_{m,n}(x)$ involving the generalized two-variable Laguerre polynomials. From the relation stated in Equation (48) and form the (50), we have:

$$A_{m,n}(x)=\sum_{r=0}^{+\infty}\frac{g_{m+r,n+r}(-y)L_r(x,y)}{r!(m+r)!(n+r)!}. \tag{52}$$

4. Concluding remarks

Before closing the paper, we want just to mention how the concepts and the formalism discussed in the previous sections allows also the generalizations of other simple distribution functions like the Poisson distribution.

By using the Tricomi function of order m

$$C_m(-x)=\sum_{r=0}^{+\infty}\frac{x^r}{r!(m+r)!}, \tag{53}$$

we can indeed define the following two-index distribution

$$P_n(x;m)=\frac{x^n}{n!(m+n)!C_m(-x)}, \tag{54}$$

where the generating function is given by the relation

$$\frac{C_m(-xt)}{C_m(-x)}=\sum_{n=0}^{+\infty}\frac{t^n}{n!}P_n(x;m). \tag{55}$$

The evaluation of the associated momenta can be easily simplified with the use of the well-known property, satisfied by the Tricomi functions:

$$(-1)^r\left(\frac{d}{dx}\right)^r C_n(x)=C_{n+r}(x). \tag{58}$$

Accordingly, we calculate the following average values

$$\begin{aligned}\langle n\rangle &= \frac{C_{m+1}(-x)}{C_m(-x)},\\ \left\langle n^2\right\rangle &= \frac{C_{m+2}(-x)}{C_m(-x)}+\langle n\rangle.\end{aligned} \tag{59}$$

The higher order moments are also given by similar closed relations.

It is remarkable about this probability distribution that, unlike the Poisson distribution, the variance:

$$\sigma = \sqrt{\overline{n^2} - \bar{n}^2}, \text{where } \bar{a} = \langle a \rangle \tag{60}$$

is smaller than $\sqrt{\bar{n}}$.

This type of distribution can be exploited in quantum mechanics within the context of bunching phenomena. This example show that the use of multi-index polynomials and Bessel-type functions with their associated formalism offers wide possibilities in the applications of pure and applied mathematics.

Funding
The author received no direct funding for this research.

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Citation information
Cite this article as: Generalizations on Humbert polynomials and functions, Clemente Cesarano, *Cogent Mathematics* (2017), 4: 1310354.

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