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A generalization of Cauchy-Khinchin-van Dam inequality

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Abstract: We first give an alternative proof of a theorem originally presented by E. R. van Dam. Then we show a generalization of the van Dam matrix inequality.

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1. Introduction

D. de Caen (1998) gave an upper bound on the sum of squares of degrees in a graph by considering some positive semidefinite quadratic form related to the line graph of the complete graph. Following de Caen's idea, van Dam (1998) gave a matrix inequality, which generalizes the Cauchy-Schwarz inequality for vectors, and Khinchin's inequality for zero-one matrices. In Section 2, we first present a different proof of Theorem 1 of van Dam (1998). In Section 3, we give the main result of this paper, a generalization of the van Dam matrix inequality. Then, in Section 4, we compare with the result of Yan (2011).

2. An alternative proof of van Dam's theorem

Theorem 1 *(van Dam, 1998, Theorem 1) Let $A = (a_{ij})$ be a real $m \times n$ matrix. Then*

$$m \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} \right)^2 + n \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} \right)^2 \le \left(\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \right)^2 + mn \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2. \tag{1}$$

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PUBLIC INTEREST STATEMENT

D. de Caen had gave an upper bound on the sum of squares of degrees in a graph by considering some positive semidefinite quadratic form related to the line graph of the complete graph. Following de Caens idea, E. R. van Dam gave a matrix inequality, which generalizes the Cauchy-Schwarz inequality for vectors, and Khinchins inequality for zero-one matrices. In this paper, we present a different proof of van Dam's inequality and then give a generalization. Finally, we compare with the generalization of Zizong Yan (2011). We hope that the result can be used to the investigation of quantum entanglement.

The equality holds if and only if $a_{ij} = b_i + c_j$ for some real b_i and c_j, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$.

van Dam (cf. 1998) proved the theorem using the positivity of the matrix $nI_n - J_n$, where I_n is the identity matrix of order n and J_n is the square matrix of order n with all elements are equal to 1.

It is easy to see that the matrix $nI_n - J_n$ has eigenvalue 0 with multiplicity 1 and n with multiplicity $n-1$. By $\alpha_n = \frac{1}{\sqrt{n}}(1, 1, \dots, 1)^T$, we denote the eigenvector of $nI_n - J_n$ associated to the eigenvalue 0, and $\beta^{1n}, \beta^{2n}, \dots, \beta^{(n-1)n}$ denote the orthonormal basis of eigenspace associated to the eigenvalue n. For positive integers m, n we can obtain eigenvalues of the matrix $(mI_m - J_m) \otimes (nI_n - J_n)$ are 0 with multiplicity $m + n - 1$ and mn with multiplicity $(m-1)(n-1)$. Moreover, the vector set $\{\alpha_m \otimes \alpha_n, \alpha_m \otimes \beta^{1n}, \dots, \alpha_m \otimes \beta^{(n-1)n}, \beta^{1m} \otimes \alpha_n, \dots, \beta^{(m-1)m} \otimes \alpha_n\}$ is an orthonormal basis of eigenspace of $(mI_m - J_m) \otimes (nI_n - J_n)$ with eigenvalue 0. Given an arbitrary real $m \times n$ matrix $A = (a_{ij})$, we have an mn column vector defined as

$$\mathrm{Vec}(A) = (a_{11}, a_{12}, \dots, a_{1n}, a_{21}, \dots, a_{2n}, \dots, a_{m1}, \dots, a_{mn})^t.$$

From the positive-semidefinite property of $(nI_n - J_n) \otimes (mI_m - J_m)$ we have that

$$\mathrm{Vec}(A)^t (mI_m - J_m) \otimes (nI_n - J_n) \mathrm{Vec}(A) \geq 0.$$

Hence

$$\begin{aligned} & mn \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2 - m \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} \right)^2 - n \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} \right)^2 + \left(\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \right)^2 \\ &= \mathrm{Vec}(A)^t \left(mnI_m \otimes I_n - mI_m \otimes J_n - nJ_m \otimes I_n + J_m \otimes J_n \right) \mathrm{Vec}(A) \\ &= \mathrm{Vec}(A)^t \left[(nI_n - J_n) \otimes (mI_m - J_m) \right] \mathrm{Vec}(A) \geq 0. \end{aligned}$$

This proves (1).

Also, from the positive-semidefinite property of $(nI_n - J_n) \otimes (mI_m - J_m)$, it follows that $\mathrm{Vec}(A)^t [(mI_m - J_m) \otimes (nI_n - J_n)] \mathrm{Vec}(A) = 0$ if and only if $[(mI_m - J_m) \otimes (nI_n - J_n)] \mathrm{Vec}(A) = 0$, i.e.

$$\begin{aligned} \mathrm{Vec}(A) = {} & k_0 \alpha_m \otimes \alpha_n + k_1 \alpha_m \otimes \beta^{1n} + \cdots + k_{n-1} \alpha_m \otimes \beta^{(n-1)n} \\ & + l_1 \beta^{1m} \otimes \alpha_n + \cdots + l_{m-1} \beta^{(m-1)m} \otimes \alpha_n, \end{aligned}$$

for some $k_0, k_1, \dots, k_{n-1}, l_1, \dots, l_{m-1}$. Thus

$$\mathrm{Vec}(A)^t [(mI_m - J_m) \otimes (nI_n - J_n)] \mathrm{Vec}(A) = 0$$

if and only if

$$a_{ij} = k_0 + k_1 \beta_j^{1n} + \cdots + k_{n-1} \beta_j^{(n-1)n} + l_1 \beta_i^{1m} + \cdots + l_{m-1} \beta_i^{(m-1)m} = b_i + c_j,$$

where

$$b_i = l_1 \beta_i^{1m} + \cdots + l_{m-1} \beta_i^{(m-1)m}, \quad c_j = k_0 + k_1 \beta_j^{1n} + \cdots + k_{n-1} \beta_j^{(n-1)n}.$$

3. Main result

We fix some notations which will be used in the following:

$$[s]=\{1,2,\dots,s\},$$
$$\Gamma_{k,n}=\{\alpha=(i_1,\dots,i_k),1\le i_1<i_2<\cdots<i_k\le n,\ i=1,2,\dots,k\},$$
$$n_\alpha=n_{i_1}n_{i_2}\dots n_{i_k},\quad \text{for } \{i_1,i_2,\dots,i_k\}=\alpha\in\Gamma_{k,n},\quad \text{if } k=0,\quad \text{fix } n_\alpha=1.$$

Since $(n_1I_{n_1}-J_{n_1})\otimes(n_2I_{n_2}-J_{n_2})\otimes\cdots\otimes(n_sI_{n_s}-J_{n_s})$, $s\in\mathcal{Z}^+$, $n_i\in\mathcal{Z}^+$, $i=1,2,\dots,s$, is positive, we have

THEOREM 2 *Let $A=(a_{i_1i_2\dots i_s})$ be a real $n_1\times n_2\times\cdots\times n_s$ matrix. Then*

$$\sum_{l=0}^{s}\sum_{\alpha\in\Gamma_{s-l,s}}(-1)^l n_\alpha\sum_{i_k,(k\in\alpha)}\left(\sum_{i_k,(k\in[s]\setminus\alpha)}a_{i_1i_2\dots i_s}\right)^2\ge 0. \tag{2}$$

The equality holds if and only if Vec(A) is in the kernel of $(n_1I_{n_1}-J_{n_1})\otimes(n_2I_{n_2}-J_{n_2})\otimes\cdots\otimes(n_sI_{n_s}-J_{n_s})$. Here, $a_{i_1i_2\dots i_s}$ is the $i_1(n_2\dots n_s-1)+i_2(n_3\dots n_s-1)+\cdots+i_{s-1}(n_s-1)+i_s$ component of Vec(A).

Notation: We have an orthonormal basis for the eigenspace with eigenvalue 0 (i.e. kernel) of $(n_1I_{n_1}-J_{n_1})\otimes(n_2I_{n_2}-J_{n_2})\otimes\cdots\otimes(n_sI_{n_s}-J_{n_s})$

$$\gamma_1\otimes\gamma_2\otimes\cdots\otimes\gamma_s,$$

where $\gamma_i\in\{\alpha_{n_i}\}\cup\{\beta^{1n_i},\beta^{2n_i},\dots,\beta^{(n_i-1)n_i}\}$ and at least one $\gamma_i\in\{\alpha_{n_i}\}$.

For example, when $s=3$, we have

$$\begin{aligned}\Omega={}&mnl\sum_{i,j,k}a_{ijk}^2+m\sum_i\left(\sum_{jk}a_{ijk}\right)^2+n\sum_j\left(\sum_{ik}a_{ijk}\right)^2+l\sum_k\left(\sum_{ij}a_{ijk}\right)^2\\&-mn\sum_{ij}\left(\sum_k a_{ijk}\right)^2-ml\sum_{ik}\left(\sum_j a_{ijk}\right)^2-nl\sum_{jk}\left(\sum_i a_{ijk}\right)^2-\left(\sum_{i,j,k}a_{ijk}\right)^2\ge 0,\end{aligned} \tag{3}$$

where $n_1=m$, $n_2=n$, $n_3=l$.

4. The case of s = 3

For the case of $s=3$, Yan (2011) present another inequality.

THEOREM 3 *(Yan, 2011, Theorem 1.1) Let $A=(a_{ijk})$ be a real $m\times n\times l$ and α,β,γ are real numbers. Then*

$$\begin{aligned}\Pi={}&mnl\sum_{i,j,k}a_{ijk}^2+2\beta\gamma m\sum_i\left(\sum_{jk}a_{ijk}\right)^2+2\alpha\gamma n\sum_j\left(\sum_{ik}a_{ijk}\right)^2\\&+2\alpha\beta l\sum_k\left(\sum_{ij}a_{ijk}\right)^2-\left(2\gamma-\gamma^2\right)mn\sum_{ij}\left(\sum_k a_{ijk}\right)^2\\&-(2\beta-\beta^2)ml\sum_{ik}\left(\sum_j a_{ijk}\right)^2-nl(2\alpha-\alpha^2)\sum_{jk}\left(\sum_i a_{ijk}\right)^2\\&-(\alpha+\beta+\gamma-1)^2\left(\sum_{i,j,k}a_{ijk}\right)^2\ge 0.\end{aligned} \tag{4}$$

Next, we will have a comparison between these two results.

$$
\begin{aligned}
\Pi-\Omega=&(2\beta\gamma-1)m\sum_i\left(\sum_{jk}a_{ijk}\right)^2+(2\alpha\gamma-1)n\sum_j\left(\sum_{ik}a_{ijk}\right)^2\\
&+(2\alpha\beta-1)l\sum_k\left(\sum_{ij}a_{ijk}\right)^2+(1-\gamma)^2mn\sum_{ij}\left(\sum_k a_{ijk}\right)^2\\
&+(1-\beta)^2ml\sum_{ik}\left(\sum_j a_{ijk}\right)^2+(1-\alpha)^2nl\sum_{jk}\left(\sum_i a_{ijk}\right)^2\\
&-(\alpha+\beta+\gamma)(\alpha+\beta+\gamma-2)\left(\sum_{i,j,k}a_{ijk}\right)^2.
\end{aligned}
$$

Due to

$$(\alpha+\beta+\gamma)(\alpha+\beta+\gamma-2)=(1-\alpha)^2+(1-\beta)^2+(1-\gamma)^2+(2\alpha\beta-1)+(2\alpha\gamma-1)+(2\beta\gamma-1),$$

we have that

$$
\begin{aligned}
\Pi-\Omega=&(2\beta\gamma-1)\left(m\sum_i\left(\sum_{jk}a_{ijk}\right)^2-\left(\sum_{i,j,k}a_{ijk}\right)^2\right)+(2\alpha\gamma-1)\left(n\sum_j\left(\sum_{ik}a_{ijk}\right)^2-\left(\sum_{i,j,k}a_{ijk}\right)^2\right)\\
&+(2\alpha\beta-1)\left(l\sum_k\left(\sum_{ij}a_{ijk}\right)^2-\left(\sum_{i,j,k}a_{ijk}\right)^2\right)+(1-\gamma)^2\left(mn\sum_{ij}\left(\sum_k a_{ijk}\right)^2-\left(\sum_{i,j,k}a_{ijk}\right)^2\right)\\
&+(1-\beta)^2\left(ml\sum_{ik}\left(\sum_j a_{ijk}\right)^2-\left(\sum_{i,j,k}a_{ijk}\right)^2\right)+(1-\alpha)^2\left(nl\sum_{jk}\left(\sum_i a_{ijk}\right)^2-\left(\sum_{i,j,k}a_{ijk}\right)^2\right)\\
=&\ \mathrm{Vec}(A)^tT\ \mathrm{Vec}(A),
\end{aligned}
$$

where

$$\mathrm{Vec}(A)^t=(a_{111},\ldots,a_{11l},a_{121},\ldots,a_{12l},\ldots,a_{mn1},\ldots,a_{mnl})$$

and

$$
\begin{aligned}
T=&(2\beta\gamma-1)(mI_m\otimes J_n\otimes J_l-J_m\otimes J_n\otimes J_l)+(2\alpha\gamma-1)(nJ_m\otimes I_n\otimes J_l-J_m\otimes J_n\otimes J_l)\\
&+(2\alpha\beta-1)(lJ_m\otimes J_n\otimes I_l-J_m\otimes J_n\otimes J_l)+(1-\gamma)^2(mnI_m\otimes I_n\otimes J_l-J_m\otimes J_n\otimes J_l)\\
&+(1-\beta)^2(mlI_m\otimes J_n\otimes I_l-J_m\otimes J_n\otimes J_l)+(1-\alpha)^2(nlJ_m\otimes I_n\otimes I_l-J_m\otimes J_n\otimes J_l).
\end{aligned}
$$

For the matrix T, we have mnl orthogonal eigenvectors as

$$\gamma_1\otimes\gamma_2\otimes\cdots\otimes\gamma_s,$$

where $\gamma_i\in\{\alpha_{n_i}\}\cup\{\beta^{1n_i},\beta^{2n_i},\ldots,\beta^{n_i-1n_i}\}$. By a direct calculation we can find all different eigenvalues for T are 0, $(1-\alpha)^2mnl$, $(1-\beta)^2mnl$, $(1-\gamma)^2mnl$, $(\alpha+\beta-1)^2mnl$, $(\alpha+\gamma-1)^2$, $(\beta+\gamma-1)^2$. So we can get that T is semipositive, i.e. $\Pi-\Omega\geq 0$ for all $A=(a_{ijk})\in\mathcal{R}^{m\times n\times l}$ and $\alpha,\beta,\gamma\in\mathcal{R}$. Then from (3) we can obtain the result (4). Or else, there is an advantage that (3) can be easily generalized.

Another comparison in the inequality (4) is that if $\alpha=\beta=l=1,\ \gamma=0$, then the inequality (1) can be obtained. But if $l=1$ in (3), we can obtain an equality.

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Author's contributions
Sun and Zhao involved in drafting the manuscript. Wang revised it critically and gave final approval of the version to be published.

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